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Semi-analytical iteration techniques for solving differential equations

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Abstract

The main objective of this work is to use and compare some techniques, namely the variational Iteration method (VIM), Decomposition Method (ADM), Differential Transform Method (DTM), and Temimi and Ansari Method (TAM) for differential equations of the first and second orders that appeared in physics and engineering. Notably, the results indicate that the VIM gives a solution in series form, which converges to an exact solution;. At the same time, DTM is easy to apply but requires transformation, ADM does not need any transformation except the calculation of Adomain polynomials. TAM solved nonlinear problems in much easier iteration steps, which converge to the true solution without applying any assumption. The results obtained converge rapidly to the exact solution.

Subject Classification: *12H20, 31B35.*

Keywords: *A domain decomposition method, Series solution, Differential equations, Temimi and Ansari method.*

1. Introduction

Differential equations are used in science and engineering to study various problems using mathematical models [8,9,10]. Some of the differential equations these models generate do not have an analytical solution. As a result, effective and efficient methods such as VIM, DTM, ADM, and TAM are

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required. In 1999, Ji-Huan He proposed the VIM in [1,2] to solve differential equations using an iteration scheme. The ADM is an efficient method for studying various differential equations [3,4,6]. The DTM quickly converges to exact solutions in polynomial series and requires no discretization [5,7].

In most cases, the method produces rapid convergence of the solution series. Main feature of the ATM is very effective, does not require any restrictive assumption to deal with nonlinear terms, and has a higher convergence and accuracy. In this research, we employ four semi-analytical methods to solve nonlinear differential equations these methods have not been compared before [11, 12].

2. Analysis of Methods, Applications, and Experimental Evaluation

If this section, some linear and nonlinear problems will be solved by the iterative methods ADM, VIM, DTM, and TAM. All of these examples have analytic solutions.

Example 1: Consider the linear differential equation of the first-order

$$y'(x) - 4y = 0, \quad y(0) = 1 \tag{1}$$

Solution by ADM: Writing the given equation in the operator formula

$$Ly = 4y \tag{2}$$

Applying the inverse operator L^{-1}in Eq.(2) and imposing the initial condition, we obtain $y(x) = y(0) + L^{-1}(4y)$ Using the initial condition, we get $y(x) = 1 + L^{-1}(4y)$, $y_0(x) = y(0) = 1$. The other components are determined using the recursive relation $y_{n+1}(x) = L^{-1}(4y_n)$, $y_1(x) = L^{-1}(4y_0) = 4x$, $y_2(x) = L^{-1}(4y_1) = 8x^2$, $y_3(x) = L^{-1}(4y_2) = 12x^3$. The series solution is given by $y(x) = 1 + 4x + 8x^2 + 12x^3 + \cdots \quad \rightarrow y(x) = e^{4x}$

2.1 *Solution by DTM*

Transformation of Eq. (1) with the initial condition, we get

$y(k+1) = \frac{4y(k)}{(k+1)}$, $\quad y(0) = 1, \quad k = 0,1,2,\ldots.$ $y(1) = \frac{4y(0)}{1} = 4$,

$y(2) = \frac{4y(1)}{2} = 8$, $y(3) = 12$, ...The series expansion becomes $y(x) = \sum_{n=0}^{\infty} x^k\, y(k)$

$$y(x) = 1 + 4x + 8x^2 + 12x^3 + \cdots..$$

2.2 *Solution by VTM*

The correction functional for Eq.(1), we have

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\,(Ly_n(s) - 4y_n(s))ds, \qquad n \geq 0$$

The first four iteration using the above equation gives $y(0) = y_0(x) = 1$, $y_1(x) = 1 + 4x$, y_2 (x) $= 1 + 4x + 8x^2$, y_3 (x) $= 1 + 4x + 8x^2 + 12x^3$. The solution of the problem becomes $y(x) = lim_{n\to\infty} y_n(x) = 1 + 4x + 8x^2 + 12x^3 + \cdots$

2.3 *Solution by TAM*

Writing Eq.(1) in the form gives $Ly - 4y = 0$ where $L(y) = y'$, $N(y) = -4y$, $f(x) = 0$. The first iterate becomes $L\big(y_0(x)\big) = 0$, y_0 (x)=1. Taking the inverse operator of Eq. (1), we get the second iterate $L(y_1(x)) + N(y_0(x)) + f(x) = 0$, $y_1(0) = 1$

$\int_0^x y_1$ (x)ds=4 $\int_0^x ds + \int_0^x 0\, ds$ that implies $y_1(x)$=$4x + 1$. The problems of the third and fourth iterates become $L\big(y_2(x)\big) + N\big(y_1(x)\big) + f(x) = 0, y_2(0) = 1$, $y_2(x)$=$1 + 4x + 8x^2$, $y_3(x)$=$1 + 4x + 8x^2 + 12x^3$. Finally, we get $y(x) = \lim_{n\to\infty} y_n$ $(x) = 1 + 4x + 8x^2 + 12x^3 + \cdots$

Example 2: Consider the second order of the ODEs takes the form

$$u''(x) - 4u'(x) + 3u(x) = 0 \tag{3}$$

u $(0) = 1$, and $u'(0) = 1$

2.4 *Solution by ADM*

Writing Eq. (3) in an operator form as: $L_{xx}u = 4\, L_x u\,(x) - 3u(x)$ where the differential operator L is given by $L_x = \frac{d}{dx}$, and $L_{xx} = \frac{d^2}{dx^2}$, and there an inverse operator is: $L_x{}^{-1}(.) = \int_0^x (.)dx$, and $L_{xx}^{-1}(.) = \int_0^x \int_0^x (.)dxdx$. Applying L_{xx}^{-1} on both sides of Eq.(3) and imposing the initial conditions, we get $u(x) = 1 - 3L_{xx}^{-1}(u(x)) + 4L_x{}^{-1}(u\,(x))$. Then have the following: $u_0(x) = 1 + x$, $u(x) = 1 - 3L_{xx}^{-1}\big(u(x)\big) + 4L_x{}^{-1}\big(u\,(x)\big)$.

$u_{n+1} = 4L_x^{-1}(u_n) - 3\, L_{xx}^{-1}(u_n)$ $n \geq 0$, $u_1(x) = 4L_x{}^{-1}$ $(u_0) - 3L_{xx}^{-1}(u_0)$

$u_1(x) = \frac{x^2}{2} - \frac{x^3}{2}$, $u_2(x) = 4L_x{}^{-1}$ $(u_1) - 3L_{xx}^{-1}(u_1), u_2(x) = \frac{2}{3}x^3 - \frac{5x^4}{8} + \frac{3x^5}{40} +..$

and so on. The unknown function $u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

2.5 *Solution by TAM*

Writing Eq.(3) in the form gives $L(u) - 4u' + 3u = 0$, $L(u) = u''$, $L\big(u_0(x)\big) = 0$, $u(0) = 1$, $u'(0) = 1$, $N(u) = -4u' + 3u$. Integrate both sides of the above equation twice from 0 to x and substitute the initial condition. We get $u_0(x) = 1 + x$, $u_1(x)$, the second iteration can be carried as $L\,(u_1(x)) + N\,(u_0(x)) = 0$, $u_1(0) = 1$, and $u'_1(0) = 1$, By integrating both sides of the above equation twice from 0 to x, we obtain $u_1(x) = \frac{x^2}{2} - \frac{x^3}{2}$, Then $u_2(x) = \frac{2}{3}x^3 - \frac{5x^4}{8} + \frac{3x^5}{40} + \cdots$., and so on. The solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$, $u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

2.6 *Solution Using DTM*

Writing Eq. (3) in a general form and the differential transform DTM is obtained as follows: $u'' = 4u' - 3u$. By substituting the differential transformation in the above equation, we get

$$(K+1)\,(K+2)\ Y(K+2) = 4(K+1)Y(K+1) - 3Y(K)$$

$$Y(K+2) = \frac{4(K+1)Y(K+1)-3Y(K)}{(K+1)(K+2)} \tag{4}$$

By using initial condition $Y(0) = 1\,, Y(1) = 1$. Using the recursive relation in Eq. (4) with $k \geq 0$, we obtain values for $Y(2), Y(3)$ … …as shown below

For $k = 0\ Y(2) = \frac{1}{2}[4Y(1) - 3Y(0)] = \frac{1}{2}$, $k = 1\ Y(3) = \frac{1}{3.2}[8Y(2) - 3Y(1)] = \frac{1}{3!}$, $k = 2\ \ Y(4) = \frac{1}{12}[12Y(3) - 3Y(2)] = \frac{1}{4!}$. Then, the series expansion becomes $y(x) = \sum_{n=0}^{\infty} x^k\, y(k)$, $y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

2.7 *Solution by VIM*

By using Eq. (3), we get:

$$u_{n+1}(x) = u_n(x) + \int_0^t \left[\lambda(t)\left((u''_n - 4u'_n + 3u)(t)\right)\right] dt \tag{5}$$

Using the formula $\lambda = (t - x)$. Substituting the value of λ into the functional (5) gives the iteration formula: $u_{n+1}(x) = u_n(x) + \int_0^x (t - x)((u''_n - 4u'_n + 3u)(t))\, dt$, The initial approximation will be: $u_0(x) = u(0) + xu'(0) = 1 + x$, By using the above equation, the following successive approximations become

$$u_0 = 1 + x$$

$$u_1 = u_0 + \int_0^x (t - x)\left((u''_0 - 4u'_0 + 3u_0)(t)\right)dt = 1 + \mathrm{x} - \frac{x^2}{2}$$

$$u_2 = u_1 + \int_0^x (x - t)\left((u''_1 - 4u'_1 + 3u_1)(t)\right)dt = 1 + \mathrm{x} - \frac{x^2}{2} + \frac{x^3}{3!},$$

and so on. Then: $u(x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$

3. Conclusion

In this paper, we used four different iterative methods to solve differential equations. The results showed that the DTM is the simplest method, providing a rapidly convergent solution that requires less computational work as long as the order of the equation is not too high. The VIM elegantly provides a convergent solution in a series of steps. The ADM solves in the form of a decomposing infinite series. It is a highly effective method for solving differential equations; it does not require any form of transformation or linearization, and the calculation of Adomain polynomials is a requirement. TAM solved both linear and nonlinear problems in much easier iterative steps which converge to the true solution without applying any assumptions.

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