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Results of third order differential subordination for holomorphic functions

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Abstract

In this article, we indicate novel findings via the new operator through the hadmard or convolutions product. research the class of accepted functions including the subordination derivative and the srivastava operator, which is organized in the complex plane. Finding the third differential degree of the operator and several theorems are proved to find the best subordinate and reaching numerous results in this regard.

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1. Introduction

The main topics of discussion are the superordination of univalent activities on an opaque unit disk. Recently, Srivastava used subordination to study the intriguing properties of a broader hypergeometric function, and this can be thought of as an expansion regarding differential differences in the public. Let $A \in V$ the category of analytical functions that need to be met $j(0) = j'(0) - 1 = 0$ and in the subsequent format:

$$j(t) = t + \sum_{i=1}^{\infty} e_i t^i, (t \in V), \tag{1}$$

and $k \in A$ are defined by $k(t) = t + \sum_{i=2}^{\infty} c_i t^i$, the Hadmard product of $j(t)$ and $k(t)$ is defined by

$$(j * k)(t) = t + \sum_{i=1}^{\infty} e_i c_i t^i = (k * j)(t), (t \in V). \tag{2}$$

If a schwarz function is discovered $\eta(0) = 0$ and $|\zeta(t)| < 1$, so $j(t) = k(\zeta\eta(t))$. Actually, it is widely acknowledged [1, 2]

$$j(t) < k(t), (t \in V) \Longrightarrow j(0) = k(t) \text{ and } j(V) \subset k(V).$$

In a particular instance, the converse implication is also true if k is a univalent function in open unit disk V [3- 5],

$$j(t) \prec k(t) \Leftrightarrow j(0) = k(0) \; so \; j(V) \subset k(V).$$

Definition 1.1 [6]: The differential operator $Y^{\lambda}_{\pi,\alpha,c}(\nu,\beta): A \to A$, presented by Amourah and Darus [6], where $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ $\nu, c \geq 0, \pi, \alpha, \beta > 0, \nu \neq \alpha$,

$$Y^{\lambda}_{\pi,\alpha,c}(\nu,\beta) j(t) = t + \sum_{i=2}^{\nu}\left[1 + \frac{(\lambda-1)(\alpha-\nu)\beta+\lambda c}{\pi+\alpha}\right]^{\lambda} e_i t^i. \tag{3}$$

$$M(Y^{\lambda}_{\pi,\alpha,c}(\nu,\beta) j(t))' = \frac{\pi+\alpha}{(\alpha-\nu)\beta+\lambda c} Y^{\lambda+1}_{\pi,\alpha,c}(\nu,\beta) j(t) - \left(1 - \frac{\pi+\alpha}{(\alpha-\nu)\beta+\lambda c}\right) Y^{\lambda}_{\pi,\alpha,c}(\nu,\beta) j(t). \tag{4}$$

Definition 1.2 [7, 8]: If $j \in A$, the formula determines the Srivastava and Attiya operator

$$N_{\mu,r} = t + \sum_{i=2}^{\infty}\left(\frac{r+1}{r+i}\right)^{\mu} e_i t^i, \;\; \mu \in \mathbb{C}, r \in \mathbb{C}/z_0 \tag{5}$$

$$M\left(N_{\mu+1}, r j(t)\right)' = (1+r) N_{\mu,r} j(t) - r N_{\mu+1,r} j(t). \tag{6}$$

Lemma 1.1 [9]: *Let* $p \in X[c,i]$, $i \geq 2$ *and* $q \in Q(c)$ *satisfying the next conditions* $\Re_e(\frac{\xi q''(\xi)}{q'(\xi)}) \geq 0$ *and* $\left|\frac{tq'(t)}{q'(\xi)}\right| \leq m$ *so,* $\xi \in \partial V \setminus X(q)$ *and* $m \geq i$. *If* $\mho \in \mathbb{C}$, $\Psi \in \Psi_i[\mho, q]$ *and* $\Psi(k(t), tk'(t), t''k(t), t^3 k'''(t); t) \subset \mho$, *then*

$$k(t) < q(t).$$

2. Main Results

Definition 2.1: Given $j \in A$, the operator $NY^{\lambda,\mu}_{\pi,\alpha,c,r}(\nu,\beta): A \to A$ as follows the operator $N_{\lambda,\mu}$ of Srivastava and Attiya and the operator $Y^{\lambda}_{\pi,\alpha,c,r}(\nu,\beta)$ of Amorah and Darus

$$NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) = t + \sum_{i=2}^{\nu} \left[\frac{r(\lambda-1)(\alpha-\nu)\beta+r\lambda c}{(r+i)(\pi+\alpha)}\right]^{\lambda+\mu} e_i^2 t^i. \quad (7)$$

The identity relations listed below can be derived from (7):

$$M\left(NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t)\right)' = NY_{\pi+1,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) + \alpha Y_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t), \quad (8)$$

$$NY_{\pi+1,\alpha,c,r}^{\lambda+1,\mu}\ (\nu,\beta)j(t) = NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) - \lambda t\left(NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t)\right)', \quad (9)$$

and

$$M\left(NY_{\pi,\alpha,c,r}^{\lambda,\mu+1}\ (\nu,\beta)j(t)\right)' = NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) + rNY_{\pi,\alpha,c,r}^{\lambda,\mu+1}\ (\nu,\beta)j(t). \quad (10)$$

The $NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)$ is involved in differential subordination resells. The class of allowable functions associated with the operators $N_{\mu,r}$ defined by Srivastava and $Y_{\pi,\alpha,c}^{\lambda}\ (\nu,\beta)$ defined by Amourah is examined in the current definition 2.1 [4, 10].

Definition 2.2: Given $q \in Q_0 \cap X[0,1]$, let $\mho$ be a set in the complex plane $\mathbb{C}$. Functions that satisfy the following admission requirements fall under the class $\emptyset_i[\mho, q]$ of admissible functions $\emptyset: \mathbb{C}^4 \times \mathrm{V} \to \mathrm{C}$, so $t \in \mathrm{V}, \xi \in \partial V/X(q), k \geq 2$.

$$\emptyset(o,x,z,y;t) \notin \mho,\ z = q(\xi), x = \frac{k\xi q'(t)+\pi q(t)}{(\pi+1)},$$

$$\Re_e\left(\frac{(\pi+1)^2 z-\pi^2 z}{(x(\pi+1)-xz)} - 2\pi\right) \geq k\Re_e\left(\frac{\xi^2 q''(\xi)}{q'(\xi)}\right)$$

and

$$\Re_e\left(\frac{(\pi+\xi)^2 y-2(\pi+1)z+\left(3\pi^3 z+4\pi^2 z\right)}{(x(\pi+1)-xz)} + 6\pi + 2\right) \geq k^2\Re_e\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right)$$

Theorem 2.1: *Suppose this class is $\emptyset \in \emptyset_i[\mho, q]$. If the following condition is met and the function j belongs to A and q belongs to Q_0:*

$$\Re_e\left(\frac{\alpha q''(\xi)}{q'(\xi)}\right) \geq 0 \text{ and } \left|\frac{NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t)}{q'(\xi)}\right| \leq k, \quad (11)$$

then

$$NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) < q(t).$$

Proof: From the relation between (7) and (8) gives

$$NY_{\pi+1,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) = \frac{t\left(NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t)\right)' + 2\pi NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t)}{3(\pi+1)+\pi}. \quad (12)$$

Then

$$NY_{\pi+1,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) = \frac{tg'(t)+2\pi g(t)}{3(\pi+1)+\pi},\ g(t) = NY_{\pi,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t). \quad (13)$$

Using a related reasoning, we obtain

$$NY_{\pi+2,\alpha,c,r}^{\lambda,\mu}\ (\nu,\beta)j(t) = \frac{t^2 g''(t)+(3\pi+1)tg'(t)+4\pi^2 g(t)}{6(\pi+1)^2+2\pi}, \quad (14)$$

and

$$NY_{\pi+3,\alpha,c,r}^{\lambda,\mu}\,(\nu,\beta)j(t) = \frac{t^3 g'''(t)+2\pi t^2 g''(t)+ 3\pi^2 t g'(t)+4\pi^3 k(t)}{3(\pi+1)^3+2\pi^2}. \tag{15}$$

Currently, the conversion from $\mathbb{C}^4$to $\mathbb{C}$ as defined through

$$z(\theta,8,\pi,\partial) = \theta,\;\; X(\theta,\pi) = \frac{7+\pi\mu}{2\pi+1}, 3(\theta,8,\pi,\partial) = \frac{7+(4\pi+1)+\pi^2\mu}{2(\pi+1)^2}, \tag{16}$$

$$y(\theta,8,\pi,\partial) = \frac{\partial+2(\pi+1)\pi+3\pi^2+4\pi^3\theta}{3(\pi+1)^3+2\pi^2}. \tag{17}$$

Let

$$\Psi(\theta,8,\pi,\partial) = \emptyset(\mu,x,\pi,y) = \emptyset\begin{pmatrix}\theta, \frac{7+\pi\mu}{2\pi+1}, \frac{7+(4\pi+1)+\pi^2\mu}{2(\pi+1)^2}, \\ \frac{\partial+2(\pi+1)+3\pi^2+4\pi^3\theta}{3(\pi+1)^3+2\pi^2}\end{pmatrix}. \tag{18}$$

Utilizing equation (18), lemma 1.1., and steps (12) through (15), we obtain

$$\Psi(k(t),tk'(t),t''k(t),t^3k'''(t);t) = \emptyset\begin{pmatrix} NY_{\pi,\alpha,c}^{\lambda,\mu}\,j(t), NY_{\pi+1,\alpha,c}^{\lambda,\mu}\,j(t), \\ NY_{\pi+2,\alpha,c}^{\lambda,\mu}\,j(t), NY_{\pi+3,\alpha,c}^{\lambda,\mu}j(t);t\end{pmatrix}. \tag{19}$$

Consequently, (11) provides

$$\Psi(g(t),tg'(t),t''g(t),t^3g'''(t);t) \in \mho,$$

$$1+\frac{\pi}{7} = \frac{(\pi+1)^2 z-\pi^2 z}{2(x(\pi+1)-x)} - 5, \frac{\pi}{7} = \frac{(\pi+1)^2[(\pi+1)y-3(\pi+1)z]+3\pi^2 z}{2(\pi(x-3z)-x)}.$$

By lemma 1.1 and equation (11) suggests that $NY_{\pi,\alpha,c,r}^{\lambda,\mu}\,(\nu,\beta)j(t) < q(t)$.

Definition 2.3: If $\Re > 0$ and $\mho \in \mathbb{C}$. The functions $\emptyset\colon \mathbb{C}^4 \times V \to \mathbb{C}$ that fall under the admissible function class $\emptyset_{i,1}[\mho,\Re]$ are those that

$$\emptyset\begin{pmatrix} \Re e^{i\theta}, \frac{(k+1+\pi)\emptyset\Re e^{i\theta}}{2\pi+1}, \\ \frac{F+[(4\pi+1)k+(\pi+1)^2]\Re e^{i\theta}}{2(\pi+1)^2}, \\ \frac{g+(2\pi+3)F+3\pi^2 k+4\pi^3\Re e^{i\theta}}{3(\pi+1)^3+2\pi^2};t\end{pmatrix} \notin \mho, \tag{20}$$

so, $\Re_e(ge^{-i\theta}) \geq (\mu-1)up_{1m}$and $\Re_e(de^{-i\theta}) \geq 0\ (\forall\,\theta \in \Re, k \geq 2)$.

Corollary 2.1: *Given $\emptyset_{i,1}[\mho,\Re]$and $j \in A$, the following conditions are met:*

$$\emptyset\begin{pmatrix} \frac{NY_{\pi,\alpha,c,r}^{\lambda,\mu}\,(\nu,\beta)j(t)}{t}, \frac{NY_{\pi+1,\alpha,c,r}^{\lambda,\mu}\,(\nu,\beta)j(t)}{t}, \\ \frac{NY_{\pi+2,\alpha,c,r}^{\lambda,\mu}\,(\nu,\beta)j(t)}{t}, \frac{NY_{\pi+3,\alpha,c,r}^{\lambda,\mu}\,(\nu,\beta)j(t)}{t};t\end{pmatrix} \in \mho$$

Then

$$\left|\frac{NY_{\pi,\alpha,c,r}^{\lambda,\mu}\,(\nu,\beta)j(t)}{t}\right| < \Re.$$

Proof: By theorem 2.1 and definition 2.3, we obtain

$\Re_e(ge^{-i\theta}) \geq (\mu-1)up_{1m}$. The following subordination leads to the conclusion stated by the corollary.

$$NY_{\pi,\alpha,c,r}^{\lambda,\mu}(\nu,\beta)j(t) \prec q(t).$$

Definition 2.4: Let $\mho \in \mathbb{C}, L > 0$. The functions $\emptyset: \mathbb{C}^4 \times V \to \mathbb{C}$ that fall under the class $\emptyset_i[\mho, L]$ of admissible functions are those that

$$\emptyset\left(\begin{array}{c} Le^{i\theta}, \dfrac{\left(V+\left(\frac{1-\alpha}{2\alpha}\right)\right)L\Re e^{i\theta}}{\frac{2}{\alpha}}, \dfrac{i+\left[\left(\frac{1-\alpha}{2\alpha}\right)+1)u+\left(\frac{1-\alpha}{3\alpha}\right)^2\right]Le^{i\theta}}{\left(\frac{2}{\alpha}\right)^2}, \\ \dfrac{d+\left(\left(\frac{1-\alpha}{2\alpha}\right)+3\right)g+\left[\left(\frac{1-\alpha}{3\alpha}\right)^2+\left(\frac{1-\alpha}{\alpha}\right)\right)u\left(\frac{1-\alpha}{\alpha}\right)^2]Le^{i\theta}}{3(\pi+1)^3+2\pi^2}; t \end{array}\right) \notin \mho$$

Corollary 2.2: *In the event j is a function that satisfies the following conditions and $\emptyset \in \emptyset_i[\mho, L]$, then $\left|NY_{\pi,\alpha,c,r}^{\lambda,\mu}(\nu,\beta)j(t)\right| \leq uL, (t \in V, u \geq 2; L > z)$ and*

$$\emptyset\left(\begin{array}{c} NY_{\pi,\alpha,c,r}^{\lambda,\mu}(\nu,\beta)j(t), NY_{\pi,\alpha,c,r}^{\lambda+1,\mu}(\nu,\beta)j(t), \\ NY_{\pi,\alpha,c,r}^{\lambda+2,\mu}(\nu,\beta)j(t), NY_{\pi,\alpha,c,r}^{\lambda+3,\mu}(\nu,\beta)j(t); t \end{array}\right) \in \mho.$$

Then

$$\left|NY_{\pi,\alpha,c,r}^{\lambda,\mu}(\nu,\beta)j(t)\right| < L.$$

3. Conclusion

This study found that they maintain their geometric properties when the new operator is applied to the geometric function, as explored by the operators. New results are obtained in the unit disk if certain restrictions are imposed on the functions that are part of these subclasses.

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