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On $H_3(1)$ Hankel determinant for starlike and convex functions associated with exponential function

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Abstract

This research aims to present and study the convex subclasses and starlike functions of the analytic functions defined in the open unit disk U and to investigate the third-order Hankel determinant $H_{3,1}$ for these subclasses related to the exponential function. As well as the upper bounds of $H_{2,2}(g)$ and $H_{3,1}(g)$ for the convex and starlike functions, and the estimates of the coefficients $|d_3|, |d_4|$, and $|d_5|$ were obtained.

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1. Introduction

Let Ω and A denoted by the families of univalent functions and normalized analytic, respectively, as follows

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$$\Omega = \{g \in A: g \text{ is univalent function in } U'\},$$

and

$$A = \{g \in HJ(A): g(0) = g'(0)\text{-}1 = 0, \quad (z \in U')\},$$

where *HJ(A)*: denotes the collection of analytical functions in the open unit disc $U' = \{z: z \in \not{c}, |z| < 1\}$.

For $g \in A$, hence it can be expressed in the series form of Taylor as follows:

$$g(z) = z + \sum_{\eta=2}^{\infty} \theta_\eta d_\eta z^\eta, \ (z \in U'), \tag{1}$$

where $\theta_\eta = [\frac{\mu^2+2}{\mu}]^\eta, such\ that\ (0 < \mu \leq 1)$, it's easy to find that $[\theta_2 = 9, \theta_3 = 27, \theta_4 = 81, \theta_5 = 243]$. For specified functions $h_1, h_2 \in A$, h_1 is denoted as subordinated of h_2 (written mathematically as $h_1 \prec h_2$), if there exist a Schwarz function v(z) analytic in U', with $v(0) = 0$ and $|v(z)| < 1, \ (z \in U')$ such that $h_1(z) = h_2(v(z))$. This subordination has been indicated by:

$$h_1(z) \prec h_2(z) \ or \ h_1 \prec h_2, \qquad (z \in U')$$

if h_1 is univalent in A, then

$$h_1(z) \prec h_2(z) \leftrightarrow h(0) = h_2(0) \ and \ h_1(U') \subset h_2(U').$$

Let L represent the family of analytic functions ℓ normalized by:

$$\ell(z) = 1 + \sum_{\eta=1}^{\infty} u_\eta z^\eta, \ (z \in U'). \tag{2}$$

and satisfy the condition (Re $\ell(z) > 0$, $z \in U'$). Introduced the sequence by Noonan and Thomas [1], see [2]. It's clear that, if $\ell(z) \in L$, then there's Schwarz's function v(z) with $v(0) = 0$ and $|v(z)| < 1, \ (z \in U')$ such that:

$$\ell(z) \prec \tfrac{1+z}{1-z}, \ (z \in U').$$

Many interesting subclasses of Ω associated with different image fields have been studied by different scientists. The most basic subfamilies are starlike (S_e^*) and convex (C_e^*) see [18], which are analytically defined as follows:

$$S_\alpha^* = \left\{g \in A: \text{Re}\left[\tfrac{zg'(z)}{g(z)}\right] > \alpha\right\} \ \& \ C_\alpha^* = \left\{g \in A: \text{Re}\left[\tfrac{(zg'(z))'}{g'(z)}\right] > \alpha\right\},$$

for $0 \leq \alpha < 1$, $(z \in U')$.

Recently, finding bounds on the coefficients of a function in a given set has become one of the most fundamental problems in geometric function theory because it affects geometric features. Finding the upper bounds on the modules of Hankel determinants for different subclasses of analytical functions has become an active area of research in the geometric domain, see [3,4]. Many researchers have studied and explored the Hankel determinant of the exponential function. For example, Shi et al. [9] have examined findings about the Hankel determinants to the inverse of several analytic functions related to the exponential function, and the third-order Hankel determinant for a particular family of analytic functions connected to the exponential function was discovered by Zhang et al. [5]. Some well-known analytical functions associated with the exponential function have been presented as special cases, as follows:

$$S_e^* = \left\{ \mathrm{g} \in \mathrm{A}: \mathrm{Re}\left[\frac{z\mathrm{g}'(z)}{\mathrm{g}(z)}\right] \prec e^z \right\} \ \& \ C_e^* = \left\{ \mathrm{g} \in \mathrm{A}: \mathrm{Re}\left[\frac{(z\mathrm{g}'(z))'}{\mathrm{g}'(z)}\right] \prec e^z \right\},$$

where ($z \in$ U').

The Hankel determinant was introduced by Pommerenke [12,13] and defined:

$$H_{n,m}(\mathrm{g}) = \begin{bmatrix} d_m & d_{m+1} & \cdots & d_{m+n-1} \\ d_{m+1} & d_{m+2} & \cdots & d_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m+n-1} & d_{m+n} & \cdots & d_{m+2(n-1)} \end{bmatrix} \text{ where } \mathrm{g} \in A \text{ and } m, n \geq 1.$$

By assigning different values to n and m, different Hankel determinants can be obtained:

$$\begin{bmatrix} 1 & d_2 \\ d_2 & d_3 \end{bmatrix} \ \& \ \begin{bmatrix} d_2 & d_3 \\ d_3 & d_4 \end{bmatrix} \ \& \ \begin{bmatrix} 1 & d_2 & d_3 \\ d_2 & d_3 & d_4 \\ d_3 & d_4 & d_5 \end{bmatrix} \text{ where } d_1 = 1.$$

It's worth noting that $H_{2,1}(\mathrm{g})$ is a function of the classical Fekete-Szego. For different subclasses of A, upper bounds for $\left|H_{2,1}(\mathrm{g})\right|$ have been obtained by different authors see [16,17,18,19], the upper bound for $H_{2,2}(\mathrm{g})$ has been found by many authors see [14,15]. It's easy to conclude that:

$$H_{2,1}(\mathrm{g}) = d_3 - d_2^2, \tag{3}$$

$$H_{2,2}(\mathrm{g}) = d_2 d_4 - d_3^2, \tag{4}$$

$$H_{3,1}(\mathrm{g}) = d_3(d_2 d_4 - d_3^2) - d_4(d_4 - d_2 d_3) + d_5(d_3 - d_2^2). \tag{5}$$

2. A set of lemmas

Several lemmas are presented to prove the first results

Lemma 1 [7, 8, 10]: *If $\ell \in L$ be given by (2), then:*

$$\left|u_\eta\right| \leq 2, \ for \ \eta \geq 1, \tag{6}$$

$$\left|u_{i+j} - \beta u_i u_j\right| \leq 2 \ max\{1; |1 - 2\beta|\}, for \ \beta \in ¢, \tag{7}$$

$$|u_2 - \gamma u_1^2| \leq 2 \ max\{1; |2\gamma - 1|\}, for \ \gamma \in ¢. \tag{8}$$

Lemma 2 [11]: *If $\ell \in L$ be given by (2), then:*

$$|Du_1^3 - Eu_1 u_2 + Fu_3| \leq 2|D| + 2|E - 2D| + 2|D - E + F|. \tag{9}$$

Lemma 3 [6]: *Let $\ell \in L$ given by (2). Then:*

$$|u_{m+2n} - \mu u_m u_n^2| \leq 2(1 + 2\mu), for \ \mu \in R. \tag{10}$$

3. Second and third-order Hankel determinant for S_e^*

The upper bounds are found for the 2nd and 3rd Hankel determinants. In addition to some initial coefficients. In combination with the class S_e^* [2].

Theorem 1: *For* $\mathrm{g} \in S_e^*$ *submitted by (1), then:*

$$\left|H_{2,2}(\mathrm{g})\right| \leq \frac{1}{54}.$$

Proof: Let $\mathrm{g} \in S_e^*$. Then by the Schwartz function (v), that means is v(z) analytic in U˙ and satisfies the conditions v(0) = 0 and $|v(z)| < 1$ for all $z \in$ U˙ so that:

$$\frac{z\mathrm{g}'(z)}{\mathrm{g}(z)} = e^{v(z)}.$$

Using (1), we obtain:

$$\frac{z\mathrm{g}'(z)}{\mathrm{g}(z)} = \frac{\mathrm{z} + \sum_{\eta=2}^{\infty} \eta\theta_\eta d_\eta z^\eta}{\mathrm{z} + \sum_{\eta=2}^{\infty} \theta_\eta d_\eta z^\eta}$$

$= \left(1 + \sum_{\eta=2}^{\infty} \eta\theta_\eta d_\eta\, z^{\eta-1}\right)[1 - \theta_2 d_2 z + (\theta_2^2 d_2^2 - \theta_3 d_3)z^2 - (\theta_2^3 d_2^3 - 2\theta_2 d_2 \theta_3 d_3 + \theta_4 d_4)z^3 + (\theta_2^4 d_2^4 - 3\theta_2^2 d_2^2 \theta_3 d_3 + 2\theta_2 d_2 \theta_4 d_4 + \theta_3^2 d_3^2 - \theta_5 d_5)z^4 - \cdots]$

$= 1 + \theta_2 d_2 z + (2\theta_3 d_3 - \theta_2^2 d_2^2)z^2 + (\theta_2^3 d_2^3 - 3\theta_2 d_2 \theta_3 d_3 + 3\theta_4 d_4)z^3 + (4\theta_5 d_5 + 4\theta_2^2 d_2^2 \theta_3 d_3 - 4\theta_2 d_2 \theta_4 d_4 - 2\theta_3^2 d_3^2 - \theta_2^4 d_2^4)z^4 + \cdots.$ (11)

Using the Schwartz function v(z) in terms of $\ell \in L$, this found:

$$\ell(z) = \frac{1+v(z)}{1-v(z)} = 1 + u_1 z + u_2 z^2 + u_3 z^3 + \cdots,$$

it's equivalent to

$$v(z) = \frac{\ell(z)-1}{\ell(z)+1} = \frac{u_1 z + u_2 z^2 + \cdots}{2 + u_1 z + u_2 z^2 + \cdots}.$$

Using v(z)'s series expansion, we obtain:

$$\exp\big(v(z)\big) = 1 + \mathrm{v}(\mathrm{z}) + \frac{v(z)^2}{2!} + \frac{v(z)^3}{3!} + \frac{v(z)^4}{4!} + \cdots = 1 + \frac{u_1}{2}z + \left(\frac{u_2}{2} - \frac{u_1^2}{8}\right)z^2 + \left(\frac{u_3}{2} - \frac{u_1 u_2}{4} + \frac{u_1^3}{48}\right)z^3 + \left(\frac{u_4}{2} + \frac{u_1^2 u_2}{16} - \frac{u_2^2}{8} - \frac{u_1 u_3}{4} + \frac{u_1^4}{384}\right)z^4 + \cdots.$$ (6)

By equating the coefficients of equations (11) and (12), we get:

$$d_2 = \frac{u_1}{2\theta_2},$$ (13)

$$d_3 = \frac{1}{4\theta_3}\left(u_2 + \frac{1}{4}u_1^2\right),$$ (14)

$$d_4 = \frac{1}{6\theta_4}\left(u_3 + \frac{u_1 u_2}{4} - \frac{u_1^3}{48}\right),$$ (15)

$$d_5 = \frac{1}{8\theta_5}\left(u_4 + \frac{u_1 u_3}{6} - \frac{u_1^2 u_2}{12} + \frac{u_1^4}{144}\right).$$ (16)

By using (3), (13) and (14) we find:

$$\left|H_{2,2}(\mathrm{g})\right| = |d_3 - d_2^2| = \left|\frac{1}{4\theta_3}\left(u_2 + \frac{1}{4}u_1^2\right) - \left(\frac{u_1}{2\theta_2}\right)^2\right| = \left|\frac{1}{108}\left(u_2 - \frac{1}{12}u_1^2\right)\right|,$$

from (8), Theorem 1 has been proven.

Theorem 2: *For* $\mathrm{g} \in S_e^*$ *submitted by (1), then:*

$$|d_2 d_4 - d_3^2| \leq \frac{3865}{104976}.$$

Proof: As mentioned in (13)-(15), we get:

$$|d_2d_4 - d_3^2| = \left|\left(\frac{u_1}{2\theta_2}\right)\left[\frac{1}{6\theta_4}\left(u_3 + \frac{u_1u_2}{4} - \frac{u_1^3}{48}\right)\right] - \left(\frac{1}{4\theta_3}\left(u_2 + \frac{1}{4}u_1^2\right)\right)^2\right|$$

$$= \left|\frac{u_1u_3}{8748} - \frac{161u_1^2u_2}{34992} - \frac{13u_1^4}{1679616} - \frac{u_2^2}{11664}\right|,$$

using the triangle inequality means that:

$$|d_2d_4 - d_3^2| \leq \left|\frac{u_1}{6}\right|\left|\frac{-13u_1^3}{279936} - \frac{161u_1u_2}{5832} + \frac{u_3}{1458}\right| + \left|\frac{u_2^2}{11664}\right|,$$

from (6) and lemma 2 to impose values $D = \frac{-13}{279936}, E = \frac{161}{5832}$ and $F = \frac{1}{1458}$ then:

$$|d_2d_4 - d_3^2| = \frac{1}{3} * \frac{3829}{34992} + \frac{1}{2916} = \frac{3865}{104976}.$$

Thus, Theorem 2 was proven.

Theorem 3: *Assume that* $\mathrm{g} \in S_e^*$ *has the form (1), then:*

$$|d_4 - d_2d_3| \leq \frac{11}{729}.$$

Proof: Through the use of equations (13)-(15), it's concluded that:

$$|d_4 - d_2d_3| = \left|\left(\frac{1}{6\theta_4}\left(u_3 + \frac{u_1u_2}{4} - \frac{u_1^3}{48}\right)\right) - \left(\frac{u_1}{2\theta_2}\right)\left(\frac{1}{4\theta_3}\left(u_2 + \frac{1}{4}u_1^2\right)\right)\right|,$$

by using (6) such that $|u_1| \leq 2$, so:

$$|d_4 - d_2d_3| \leq \frac{11}{5832}|u_1|^3 = \frac{11}{729}.$$

Now, the proof has been completed.

Theorem 4: *If* $\mathrm{g} \in S_e^*$ *is obtained by (1), then:*

$$\left|H_{3,1}(\mathrm{g})\right| \leq \frac{1}{486}.$$

Proof: From (5), and using tringle inequality, we get:

$$\left|H_{3,1}(\mathrm{g})\right| \leq |d_3||d_2d_4 - d_3^2| + |d_4||d_4 - d_2d_3| + |d_5||d_3 - d_2^2|.$$

Now, the upper bounds of d_3, d_4 and d_5 will be found:

$$|d_3| = \left|\frac{1}{108}\left(u_2 + \frac{1}{4}u_1^2\right)\right| = \frac{1}{108}\left|u_2 + \frac{1}{4}u_1^2\right|,$$

using (8) we find:

$$|d_3| \leq \frac{1}{108} * 3 = \frac{1}{36}.$$

$$|d_4| = \left|\frac{1}{486}\left(u_3 + \frac{u_1u_2}{4} - \frac{u_1^3}{48}\right)\right| = \frac{1}{486}\left|\left(u_3 - \frac{u_1u_2}{12}\right) + \frac{u_1}{3}\left(u_2 - \frac{u_1^2}{16}\right)\right|,$$

by tringle inequality and using (7) and (8), we get:

$$|d_4| \leq \frac{1}{486} * 2 + \frac{2}{1458} * 2 = \frac{5}{729}.$$

$$|d_5| = \frac{1}{1944}\left|\left(u_4 - \frac{u_1u_3}{3}\right) + \frac{u_1}{2}\left(\frac{u_1^3}{72} - \frac{u_1u_2}{6} + u_3\right)\right|,$$

by using tringle inequality and from (6) and (7), and from lemma 2 and assuming the following values $D = \frac{1}{72}$, $E = \frac{1}{6}$ and $F = 1$, we obtained [18]:

$$|d_5| \leq \frac{1}{1944} * 2 + \frac{2}{3888} * 2 = \frac{1}{486}.$$

Finally, using the above inequality and the results obtained and previous theories, we concluded that:

$$\left|H_{3,1}(\mathrm{g})\right| \leq \frac{1}{36} * \frac{3865}{104976} + \frac{5}{729} * \frac{11}{729} + \frac{1}{486} * \frac{1}{54} = \frac{39601}{34012224}.$$

So, the proof of this theorem is complete.

4. Second and third-order Hankel determinant for C_e^*

In this part, the 2nd and 3rd Hankel determinants will be found respectively, in association with the subclass C_e^*.

Theorem 5: *If* $\mathrm{g} \in C_e^*$ *submitted by (1). Then:*

$$\left|H_{2,1}(\mathrm{g})\right| \leq \frac{1}{162}.$$

Proof: Let $\mathrm{g} \in C_e^*$. Then (1) can be written as a Schwartz function as follows:

$$\frac{(z\mathrm{g}'(z))'}{\mathrm{g}'(z)} = e^{v(z)}.$$

From (1) we obtain:

$$\frac{(z\mathrm{g}'(z))'}{\mathrm{g}'(z)} = 1 + 2\theta_2 d_2 z + (6\theta_3 d_3 - 4\theta_2^2 d_2^2)z^2 + (8\theta_2^3 d_2^3 - 18\theta_2 d_2 \theta_3 d_3 + 12\theta_4 d_4)z^3 + (20\theta_5 d_5 + 48\theta_2^2 d_2^2 \theta_3 d_3 - 32\theta_2 d_2 \theta_4 d_4 - 18\theta_3^2 d_3^2 - 16\theta_2^4 d_2^4)z^4 + \cdots. \quad (17)$$

By equating equations (12) and (17) we obtain:

$$d_2 = \frac{u_1}{36}, \quad (18)$$

$$d_3 = \frac{1}{324}\left(u_2 + \frac{1}{4}u_1^2\right), \quad (19)$$

$$d_4 = \frac{1}{972}\left(\frac{u_3}{2} + \frac{u_1 u_2}{8} - \frac{u_1^3}{96}\right), \quad (20)$$

$$d_5 = \frac{1}{4860}\left(\frac{u_4}{2} + \frac{u_1 u_3}{12} - \frac{u_1^2 u_2}{24} + \frac{u_1^4}{288}\right). \quad (21)$$

By using (3), (18) and (19) we get:

$$\left|H_{2,2}(\mathrm{g})\right| = |d_3 - d_2^2| = \left|\frac{1}{324}\left(u_2 + \frac{1}{4}u_1^2\right) - \left(\frac{u_1}{36}\right)^2\right| = \left|\frac{u_2}{324}\right|,$$

Therefore, from (6), Theorem 5 has been proven.

Theorem 6: *Let* $\mathrm{g} \in C_e^*$ *submitted by (1). Then:*

$$|d_2 d_4 - d_3^2| \leq \frac{47}{419904}.$$

Proof: From (4) and (18)-(20) we get:

$$|d_2d_4 - d_3^2| = \frac{1}{144}\left|\frac{u_1}{486}\left(u_3 - \frac{u_1^3}{16}\right) - \frac{u_2}{729}\left(u_2 - \frac{i^2}{8}u_1^2\right)\right|,$$

using the triangle inequality is obtained:

$$|d_2d_4 - d_3^2| \leq \left|\frac{u_1}{69984}\right|\left|u_3 - \frac{u_1^3}{16}\right| + \left|\frac{u_2}{104976}\right|\left|u_2 - \frac{i^2}{8}u_1^2\right|.$$

By taking (8) and (10), the required proof is obtained.

Theorem 7: *Suppose that* $\mathrm{g} \in C_e^*$ *submitted by (1), so that:*

$$|d_4 - d_2d_3| \leq \frac{1}{648}.$$

Proof: Through (18)-(20), it's obtained:

$$|d_4 - d_2d_3| = \frac{1}{1944}\left|\left(u_3 - \frac{u_1u_2}{6}\right) + \frac{u_1}{4}\left(u_2 - \frac{1}{4}u_1^2\right)\right|,$$

using the tringle inequality is found:

$$|d_4 - d_2d_3| \leq \left|\frac{1}{1944}\right|\left|u_3 - \frac{u_1u_2}{6}\right| + \left|\frac{u_1}{7776}\right|\left|u_2 - \frac{1}{4}u_1^2\right|,$$

and from Lemma 1, the proof of the seventh Theorem has ended.

Theorem 8: *Assume that* $\mathrm{g} \in C_e^*$ *it was given by (1), so:*

$$\left|H_{3,1}(\mathrm{g})\right| \leq \frac{1411}{226748160}.$$

***Proof*:** First, the following coefficients d_3, d_4 *and* d_5 will be found, and as shown in Theorem 4 the desired results will be found in the same way.

$$|d_3| \leq \frac{1}{108}, \quad |d_4| \leq \frac{5}{2916}, \quad |d_5| \leq \frac{1}{2430}.$$

Using all subclass C_e^* theorems and the above results, the proof is complete.

5. Conclusion

The current research addresses convex and starlike functions. These two subfamilies of univalent functions are linked to the exponential function. Upper bounds $\left|H_{2,2}(\mathrm{g})\right|$ and $\left|H_{3,1}(g)\right|$ for subfamilies C^* and S^* were obtained. The 3rd Hankel determinant for some other subfamilies of analytical functions with linear and nonlinear coefficients, aside from the 4th Henkel determinant for this family of functions, can be discussed in the future.

References

[1] S. S. Miller and P. T. Mocanu, Differential subordinations: Theory and Application, Series on Monographs and Text-books in Pure and Applied Math., no. 225, Marcel Dekker Incorporated, New York and Basel, (2000).

[2] A. R. S. Juma, A. Al-Fayadh, S. P. Vijayalakshmi and T. V. Sudharsan, Upper bound on the third Hankel determinant for the class of univalent functions using an integral operator. Afr. Mat. 33 (2), 56, (2022).

[3] R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* 38, 365–386, (2015). [CrossRef]

[4] A. R. S. Juma, M. H. Saloomi, Coefficient estimates for subclasses of regular functions. *Iraqi Journal of Science,* 59(3c), pp. 1710-1716, (2018).

[5] H-Y Zhang, H. Tang, X-M Niu. Third-Order Hankel Determinant for Certain Class of Analytic Functions Related with Exponential Function. Symmetry; 10(10):501, (2018).

[6] C. Pommerenke, G. Jensen, Univalent Functions; Vandenhoeck and Ruprecht: Gottingen, Germany, (1975).

[7] I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, *J. Math. Anal. Appl.* 435, 369-379, (2016).

[8] V. Ravichandran and S. Verma, Borne pour le cinquieme coefficient des function etoiles, *Comptes Rendus Mathematique,* (6)353, pp, 505-510, (2015).

[9] L. Shi, H. M. Srivastava, A. Rafiq, M. Arif and M. Ihsan, Results on Hankel determinants for the inverse of certain analytic functions subordinated to the exponential function, *Mathematics,* 10, 3429, (2022).

[10] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain subclasses of analytic functions, *Proc. Amer. Math. Soc.* 20, 8-12, (1969). [CrossRef]

[11] M. Arif, M. Raza, H. Tang, S. Hussain and H. Khan, Hankel determinant of order three for familiar subsets of analytic functions related with sine function, *Open Math*. 17 (1), 1615-1630, (2019).

[12] C. Pommerenke, On the coefficients and Hankel determinants of univalent functions. *J. Land. Math. Sec.*1, 111-122, (1966). [CrossRef]

[13] C. Pommerenke, On the Hankel determinants of univalent functions. *Mathematika.* 14, 108-112, (1967). [CrossRef]

[14] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Ineq. Pure Appl. Math.* 7, 1–5, (2006).

[15] A. Janteng, S.A. Halim and M. Darus, Hankel determinant for starlike and convex functions. *Int. J. Math. Anal.* 1, 619–625, (2007).

[16] M. Fekete and G. Szegö, Eine benberkung uber ungerada schlichte funktionen. *J. Lond. Math. Soc.* 8, 85–89, (1933). [CrossRef]

[17] H. Tang, H. M. Srivastava, S. Sivasubramanian and P. Gurusamy, The Fekete-Szego functional problems for some classes of m-fold symmetric bi-univalent functions. *J. Math. Inequal.* 10, 1063–1092, (2016). [CrossRef].

[18] A. R. S. Juma and O. N. Kassar, Certain Properties for Analytic Functions Associated with q-Ruscheweyh Differential Operator. *Iraqi Journal of Science,* 61(9), 2350-2360, (2020).

[19] Abbood, Mohaimen M., Ebrahim, Hassan H., Al-Fayadh, Ali & Obaid, Ahmed J.(2023) Measure defined on Γ–algebra and some of their generalizations, *Journal of Interdisciplinary Mathematics*, 26:5, 821–827, DOI: 10.47974/JIM-1502.

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