

On bi-univalent functions of quasi-subordination associated with convolution operator

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Abstract

Within the open unit disk, the work presents a novel subclass of bi-univalent functions. The generalized derived operator and the differential operator are convolved to produce this subclass, which meets the quasi-subordination criterion. The paper also calculates the coefficient estimates for $|a_2|$ and $|a_3|$ this subclass. Furthermore, the two operators are applied to the subclass, resulting in new findings.

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1. Introduction

This Taylor expansion is defined by $\mathcal{A}$ and describes the set of normalized functions that meet the condition $\mathcal{J}(0) = \mathcal{J}'(0) - 1 = 0$.

$$\mathcal{J}(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j \zeta^j, \zeta \in \mathfrak{U} \quad (1)$$

From the open disk of the unit, where C is a complex number and $\mathfrak{U} = \{\zeta \in C \text{ and } |\zeta| < 1\}$, are analytical expressions. In order to provide additional clarification regarding univalent functions, let $\mathcal{H}$ represent the collection of all open functions in with a value on the open unit disk.

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The Koebe one-quarter theorem posits that a disk of radius exists in the image of $\mathcal{A}$. [1] for every univalent function $J \in \mathcal{H}$. Consequently, an inverse function is defined for every univalent function J , utilizing $J(J^{-1}(\zeta)) = \zeta$, where $\zeta \in \mathfrak{A}$ and

$$J(J^{-1}(\varpi)) = \varpi, \left(|\varpi| < \rho_0(J); \rho_0(J) \geq \frac{1}{4} \right)$$

where the function

$$g(\varpi) = J^{-1}(\varpi) = \varpi - a_2\varpi^2 + (2a_2^2 - a_3)\varpi^3 - (5a_2^2 - 5a_2a_3 + a_4)\varpi^4 + \dots \quad (2)$$

If J and J^{-1} are both univalent in $\mathfrak{A}$, then J is known to be a bi-univalent function in $\mathfrak{A}$. The normalization of the bi-univalent function class in $\mathfrak{A}$ is with regard to (1), and they are represented by Σ .

In $\mathfrak{A}$, $J(\zeta)$ and $r(\zeta)$ have investigated the overarching principles governing the coordination and subordination of the two holomorphic functions. If there is an analytic function $t(\zeta)$ in the open unit disk $\mathfrak{A}$ with $J(\zeta) = r(t(\zeta))$, then for each $\zeta \in \mathfrak{A}$, the function $J(\zeta)$ is subordinate to $r(\zeta)$, meaning $J(\zeta) < r(\zeta)$, $\zeta \in \mathfrak{A}$. Furthermore, for every r that is a univalent in $\mathfrak{A}$, the equation $J(\zeta) < r(\zeta)$ is equivalent to $J(0) = r(0)$ and $J(\mathfrak{A}) \subset r(\mathfrak{A})$, so r is always a univalent in $\mathfrak{A}$. If there exists an analytic function $h(\zeta)$ with $|h(\zeta)| \leq 1$, then $J(\zeta) = h(\zeta)r(\zeta)$, where ζ is an element of $\mathfrak{A}$, and $r(\zeta)$ is the principal component of $J(\zeta)$.

Lewin [2] investigated bi-univalent functions and identified a coefficient restriction that applies universally to these functions J , where $|a_2| \leq 1.51$. According to their hypothesis, which was based on the work of Lewin [2], Clunie, and Brannan [3-5], the absolute value of a_2 should be less than or equal to 2 for any J . Recent papers by Adegani and colleagues [1], Specifically Srivastava and Wanas [7], Guney et al. [6], and others (e.g., [8-19]) have reignited interest in the area of bi-univalent and analytic functions, building on the work of Srivastava et al. [4]. Here we have the definition of $\mathcal{A}$ for the type (1) function $J \in \mathcal{A}$ and the type (2) function $g \in \mathcal{A}$ of $\mathcal{A}$.

$$g(\zeta) = \zeta + \sum_{j=2}^{\infty} b_j \zeta^j, \quad (\zeta \in \mathfrak{A}) \quad (3)$$

Lewin [2] looked into bi-univalent functions and found a limit on the coefficients that works for all of them as follows:

$$(J * g)(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j b_j \zeta^j, \quad (\zeta \in \mathfrak{A}). \quad (4)$$

Define the following differential operator as below

$$G^0 J(\zeta) = J(\zeta) \quad (5)$$

$$G^1 J(\zeta) = (1 - \kappa)J(\zeta) + \kappa \zeta J'(\zeta), \quad \kappa \geq 0 \quad (6)$$

$$G^j J(\zeta) = G_{\kappa}(G^{j-1} J(\zeta)). \quad (7)$$

If J is given by (1), then from (7), we see that

$$G^j J(\zeta) = \zeta + \sum_{j=2}^{\infty} [1 + (j-1)\kappa]^m a_j \zeta^j, \quad (8)$$

where $m \in N_0 = \{0, 1, 2, 3, 4, \dots\}$ and $\kappa \geq 0$.

The generalized derived operator was introduced by the authors in [19].

$$\Gamma_{\lambda,\vartheta}^{\mu,\alpha} = \zeta + \sum_{j=2}^{\infty} \frac{(1+\lambda(j-1))^\alpha}{(1+\vartheta(j-1))^{\alpha-1}} C(\mu, j) a_j \zeta^j \quad (9)$$

where $\mu, \alpha \in N_0 = \{0, 1, 2, 3, \dots\}$, $\vartheta \geq \lambda \geq 0$ and $C(\mu, j) = \frac{(\mu+j-1)_{j-1}}{(1)_{j-1}}$.

The convolution operator between differential operator $G^j \mathcal{J}(\zeta)$ and generalized derived operator $\Gamma_{\lambda,\eta}^{\mu,\alpha}$

$$\begin{aligned} \mathcal{G}_{\lambda,\vartheta}^{\mu,\alpha,j} \mathcal{J}(\zeta) &= G^j \mathcal{J}(\zeta) * \Gamma_{\lambda,\vartheta}^{\mu,\alpha} \\ \mathcal{G}_{\lambda,\vartheta}^{\mu,\alpha,j} \mathcal{J}(\zeta) &= \zeta + \sum_{j=2}^{\infty} \Psi_{j,\mu} a_j \zeta^j \end{aligned} \quad (10)$$

where

$$\Psi_{j,\mu} = [1 + \kappa]^m \frac{(1+\lambda)^\alpha}{(1+\vartheta)^{\alpha-1}} C(\mu, j)$$

According to Robertson, quasi-subordination is a new concept that encompasses a wider understanding of subordination and majorization [18]. Let $\mathcal{J}(\zeta)$ and $r(\zeta)$ be two quasi-subordinate functions representing $r(\zeta)$, written as $\mathcal{J}(\zeta) \prec_q r(\zeta)$, $\zeta \in \mathfrak{A}$, where ζ belongs to set $\mathfrak{A}$. We may express $\mathcal{J}(\zeta)$ as $h(\zeta)r(t(\zeta))$, where h and t are analytic functions satisfying the conditions $t(0) = 0$, $|h(\zeta)| \leq 1$, and $|t(\zeta)| < 1$. Given that $\mathcal{J}(\zeta)$ is strictly less than $r(\zeta)$ in set $\mathfrak{A}$, and $h(\zeta)$ is equal to 1 for every ζ in $\mathfrak{A}$, we may conclude that $\mathcal{J}(\zeta)$ is equal to $r(t(\zeta))$. Furthermore, it is important to note that for each element ζ in set $\mathfrak{A}$, the function $\mathcal{J}(\zeta)$ is equal to the product of $h(\zeta)$ and $r(\zeta)$. Therefore, $\mathcal{J}(\zeta)$ is orthogonal to the summation of $r(\zeta)$ in set $\mathfrak{A}$. In 2017, A. Magish et al. [19] established the category of bi-univalents by use of quasi-subordination and determined the boundaries for the coefficients. Ma and Minda examine and analyze the unified classes in their study [13]:

$$\begin{aligned} D(\Omega) &= \left\{ \mathcal{J} \in \mathcal{A} : \frac{\zeta \mathcal{J}'(\zeta)}{\mathcal{J}(\zeta)} < \Omega(\zeta) : \zeta \in \mathfrak{A} \right\}, \\ L(\Omega) &= \left\{ \mathcal{J} \in \mathcal{A} : 1 + \frac{\zeta \mathcal{J}'(\zeta)}{\mathcal{J}(\zeta)} < \Omega(\zeta) : \zeta \in \mathfrak{A} \right\}. \end{aligned}$$

The unit disk $\mathfrak{A}$ encompasses an analytic and univalent function $\Omega(\zeta)$ with a positive real portion. This function is defined by $\Omega(0) = 1$ and $\Omega'(0) > 0$. The region of $\Omega(\mathfrak{A})$ exhibits symmetry about the real axis and possesses a star-shaped structure centered at 1. Additionally, the function is univalent. $\Omega(\mathfrak{A})$ exhibits both real axis symmetry and a star-like shape around the unit circle $\mathfrak{A}$ centered at 1. Furthermore, $\Omega'(0)$ is greater than zero. The function under consideration belongs to the classes $D(\Omega)$ and $S(\Omega)$, exhibits symmetry with respect to the real axis, and has a star-like shape with respect to 1. In [13] states that the function in the $D(\Omega)$ class precisely belongs to the Ma-Minda category, but in the $L(\Omega)$ class, it can be either convex or starlike. This analysis assumes that

$$\mathcal{r}(\zeta) = \mathcal{r}_0 + \mathcal{r}_1 \zeta + \mathcal{r}_2 \zeta^2 + \mathcal{r}_3 \zeta^3 + \dots \quad (11)$$

and

$$\Omega(\zeta) = 1 + \delta_1 \zeta + \delta_2 \zeta^2 + \delta_3 \zeta^3 + \dots \quad (12)$$

This article aims to define the estimate of the coefficients $|a_2|$ and $|a_3|$ for the bi-univalent function class Σ by using new subclasses and applying them to the new operator.

2. Coefficient bounds for the class $\Sigma_{\lambda, \vartheta}^{\mu, \alpha, j}(\gamma, \eta, \Omega)$

Definition 2.1: Let $\mathcal{J} \in \Sigma$ be a function given in (1) in the class $\Sigma_{\lambda, \vartheta}^{\mu, \alpha, j}(\gamma, \eta, \Omega)$ and fulfills the following quasi-subordination requirements:

$$\left\{ (1 - \gamma) \frac{\zeta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'} + \frac{\eta \zeta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)''}{(1 - \eta)(\zeta) + \eta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'} \right\} - 1 \prec_q \Omega(\zeta) - 1, \quad (13)$$

and

$$\left\{ (1 - \gamma) \frac{\varpi \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} + \frac{\eta \varpi \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)''}{(1 - \eta)(\varpi) + \eta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} \right\} - 1 \prec_q \Omega(\varpi) - 1, \quad (14)$$

where $(0 \leq \eta \leq 1 \text{ and } \gamma = 0, \zeta, \varpi \in \mathfrak{A})$.

For particular values to parameter γ, μ, α, j and η we get new and well-known the subclasses.

Remark 2.2: For $\gamma = 0$ a function $\mathcal{J} \in \Sigma$, defined in (1) is said to be in the class $\Sigma_{\lambda, \vartheta}^{\mu, \alpha, j}(\eta, \Omega)$

$$\left\{ \frac{\zeta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'} + \frac{\eta \zeta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)''}{(1 - \eta)(\zeta) + \eta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'} \right\} - 1 \prec_q \Omega(\zeta) - 1,$$

and

$$\left\{ \frac{\varpi \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} + \frac{\eta \varpi \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)''}{(1 - \eta)(\varpi) + \eta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} \right\} - 1 \prec_q \Omega(\varpi) - 1,$$

where $0 \leq \eta \leq 1, \zeta, \varpi \in \mathfrak{A}$ with g is inverse function of $\mathcal{J}$.

Remark 2.3: For $\eta = 0$, a function $\mathcal{J} \in \Sigma$, given in (1) is said to be in the class $\Sigma_{\lambda, \vartheta}^{\mu, \alpha, j}(\gamma, \Omega)$ then next condition are satisfied

$$\left\{ (1 - \gamma) \frac{\zeta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'} \right\} - 1 \prec_q \Omega(\zeta) - 1,$$

and

$$\left\{ (1 - \gamma) \frac{\varpi \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} \right\} - 1 \prec_q \Omega(\varpi) - 1$$

where $\gamma = 0$ and $\zeta, \varpi \in \mathfrak{A}$.

Remark 2.4: For $\eta = 1$ a function $J \in \Sigma$, defined in (1) is called be in the class $\Sigma_{\lambda, \vartheta}^{\mu, \alpha, j}(\gamma, \eta, \Omega)$ then next condition are satisfied

$$\left\{ (1 - \gamma) \frac{\zeta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} J(\zeta) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} J(\zeta) \right)'} + \frac{\zeta \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} J(\zeta) \right)''}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} J(\zeta) \right)'} \right\} - 1 <_q \Omega(\zeta) - 1,$$

and

$$\left\{ (1 - \gamma) \frac{\varpi \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} + \frac{\varpi \left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)''}{\left(\mathcal{G}_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} \right\} - 1 <_q \Omega(\varpi) - 1$$

where $\gamma = 0$ and $\zeta, \varpi \in \mathfrak{U}$.

Remark 2.5: For $\alpha = 0, \lambda = 0, \vartheta = 0$ and $\mu = 0$ a function $J \in \Sigma$, given in (1) is called $\Sigma(\gamma, \eta, \Omega)$, if satisfying the following quasi-subordination

$$\left\{ (1 - \gamma) \frac{\zeta J'(\zeta)}{J(\zeta)} + \frac{\zeta \eta J''(\zeta)}{(1 - \eta)(\zeta) + \eta J'(\zeta)} \right\} - 1 <_q \Omega(\zeta) - 1,$$

and

$$\left\{ \left\{ (1 - \gamma) \frac{\varpi g'(\varpi)}{g(\varpi)} + \frac{\varpi \eta g''(\varpi)}{(1 - \gamma)(\varpi) + \eta g'(\varpi)} \right\} \right\} - 1 <_q \Omega(\varpi) - 1,$$

To determine the estimates coefficients $|a_2|$ and $|a_3|$, the following Lemma must be investigated.

Lemma 2.6: [12] If $\mathfrak{t} \in P$, then $|\mathfrak{t}_i| \leq 2$ for all i , where P is the family of all function $\mathfrak{t}$, analytic in $\mathfrak{U}$, for which $\Re\{\mathfrak{t}(\zeta)\} > 0$, $(\zeta \in \mathfrak{U})$, where $\mathfrak{t}(\zeta) = 1 + \mathfrak{t}_1 \zeta + \mathfrak{t}_2 \zeta^2 + \mathfrak{t}_3 \zeta^3 + \dots$, $(\zeta \in U)$.

Theorem 2.7: Let $J(\zeta)$ defined in (1) be in class $\Sigma_{\lambda, \vartheta}^{\mu, \alpha, n}(\gamma, \eta, \Omega)$. Then

$$|a_2| \leq \min \left\{ \frac{|r_0| \delta_1}{|(1 - \gamma + 2\eta) \Psi_{2, \mu}|}, \sqrt{\frac{|4r_0|(\delta_1 + |(\delta_2 - \delta_1)|)}{|2(1 - \gamma + 3\eta) \Psi_{3, \mu} - (1 - \gamma + 4\eta) \Psi_{2, \mu}^2|}} \right\}, \quad (15)$$

$$|a_3| \leq \min \left\{ \frac{\delta_1 |r_0| + |r_1|}{|(1 - \gamma + 3\eta) \Psi_{3, \mu}|} - \frac{|r_0^2| \delta_1^2}{|4(1 - \gamma + 2\eta)^2 \Psi_{2, \mu}^2|}, \frac{\delta_1 |r_0 + r_1|}{|(1 - \gamma + 3\eta) \Psi_{3, \mu}|} + \frac{|4r_0|(\delta_1 + |(\delta_2 - \delta_1)|)}{|2(1 - \gamma + 3\eta) \Psi_{3, \mu} - (1 - \gamma + 4\eta) \Psi_{2, \mu}^2|} \right\}. \quad (16)$$

where

$$\Psi_{2, \mu} = [1 + \kappa]^m \frac{(1 + \lambda)^\alpha}{(1 + \vartheta)^{\alpha - 1}} C(\mu, 2)$$

and

$$\Psi_{3, \mu} = [1 + 2\kappa]^m \frac{(1 + \lambda)^\alpha}{(1 + 2\vartheta)^{\alpha - 1}} C(\mu, 3).$$

Proof: Since $\mathcal{J} \in \Sigma_{\lambda, \vartheta}^{\mu, \alpha, j}(\gamma, \eta, \Omega)$ and $g = \mathcal{J}^{-1}$. Then there is analytic functions $u, v \in \mathcal{A}$ such that $u : \mathfrak{A} \rightarrow \mathfrak{A}$ and $v : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $v(0) = u(0) = 0$, satisfying

$$\left\{ (1 - \gamma) \frac{\zeta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'}{\left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)} + \frac{\eta \zeta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)''}{(1 - \eta)(\zeta) + \eta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'} \right\} - 1 <_q r(\zeta)(\Omega(v(\zeta)) - 1), \quad (17)$$

$$\left\{ (1 - \gamma) \frac{\varpi \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'}{\left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)} + \frac{\eta \varpi \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)''}{(1 - \eta)(\varpi) + \eta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} \right\} - 1 <_q r(\varpi)(\Omega(u(\varpi)) - 1) \quad (18)$$

Define the functions τ and t by

$$\tau(\zeta) = \frac{1 + v(\zeta)}{1 - v(\zeta)} = 1 + \tau_1(\zeta) + \tau_2 \zeta^2 + \tau_3 \zeta^3 + \dots \quad (19)$$

$$t(\varpi) = \frac{1 + u(\varpi)}{1 - u(\varpi)} = 1 + t_1(\varpi) + t_2 \varpi^2 + t_3 \varpi^3 + \dots \quad (20)$$

or equivalently:

$$v(\zeta) = \frac{\tau(\zeta) - 1}{\tau(\zeta) + 1} = \frac{1}{2} \left(\tau_1 \zeta + \left(\tau_2 - \frac{\tau_1^2}{2} \right) \zeta^2 \dots \right) \quad (21)$$

$$u(\varpi) = \frac{t(\varpi) - 1}{t(\varpi) + 1} = \frac{1}{2} \left(t_1 \varpi + \left(t_2 - \frac{t_1^2}{2} \right) \varpi^2 + \dots \right) \quad (22)$$

Then $\tau(\zeta)$ and $t(\varpi)$ analytic in $\mathfrak{A}$, with $\tau(\zeta) = t(\varpi) = 1$. Since $\tau(\zeta)$ and $t(\varpi)$ have positive real part in unit disk $\mathfrak{A}$, $v_i(\zeta) \leq 2$ and $u_i(\varpi) \leq 2$, $i = 1, 2, \dots$

In view (21), (22) in (17) and (18) clearly we have

$$\left\{ (1 - \gamma) \frac{\zeta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'}{\left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)} + \frac{\eta \zeta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)''}{(1 - \eta)(\zeta) + \eta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} \mathcal{J}(\zeta) \right)'} \right\} - 1 <_q r(\zeta) \left(\Omega \left(\frac{\tau(\zeta) - 1}{\tau(\zeta) + 1} \right) - 1 \right), \quad (23)$$

$$\left\{ (1 - \gamma) \frac{\varpi \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'}{\left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)} + \frac{\eta \varpi \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)''}{(1 - \eta)(\varpi) + \eta \left(g_{\lambda, \vartheta}^{\mu, \alpha, j} g(\varpi) \right)'} \right\} - 1 <_q r(\varpi) \left(\Omega \left(\frac{t(\varpi) - 1}{t(\varpi) + 1} \right) - 1 \right) \quad (24)$$

Using (21) and (22) together with (11) and (12), it is evident that

$$r(\zeta) \left(\Omega \left(\frac{\tau(\zeta) - 1}{\tau(\zeta) + 1} \right) - 1 \right) = \frac{1}{2} r_0 \tau_1 \delta_1 \zeta + \left(\frac{1}{2} r_1 \tau_1 \delta_1 + \frac{1}{2} r_0 \delta_1 \left(\tau_2 - \frac{\tau_1^2}{2} \right) + \frac{1}{4} r_0 \delta_2 \tau_1^2 \right) \zeta^2 + \dots, \quad (25)$$

$$r(\varpi) \left(\Omega \left(\frac{t(\varpi) - 1}{t(\varpi) + 1} \right) - 1 \right) = \frac{1}{2} r_0 t_1 \delta_1 \varpi + \left(\frac{1}{2} r_1 t_1 \delta_1 + \frac{1}{2} r_0 \delta_1 \left(t_2 - \frac{t_1^2}{2} \right) + \frac{1}{4} r_0 \delta_2 t_1^2 \right) \varpi^2 + \dots \quad (26)$$

From (23), (24), (25) and (26), we get it as follows

$$(1 - \gamma + 2\eta)\Psi_{2,\mu}a_2 = \frac{1}{2}r_0\delta_1\tau_1, \quad (27)$$

$$2(1 - \gamma + 3\eta)\Psi_{3,\mu}a_3 - (1 - \gamma + 4\eta)\Psi_{2,\mu}^2a_2^2 = \frac{1}{2}r_1\tau_1\delta_1 + \frac{1}{2}r_0\delta_1\left(\tau_2 - \frac{\tau_1^2}{2}\right) + \frac{1}{4}r_0\delta_2\tau_1^2, \quad (28)$$

$$-(1 - \gamma + 2\eta)\Psi_{2,\mu}a_2 = \frac{1}{2}r_0\delta_1\tau_1, \quad (29)$$

and

$$-2(1 - \gamma + 3\eta)\Psi_{3,\mu}a_3 - (1 - \gamma + 4\eta)\Psi_{2,\mu}^2a_2^2 + 4(1 - \gamma + 3\eta)\Psi_{3,\mu}a_2^2 = \frac{1}{2}r_1\tau_1\delta_1 + \frac{1}{2}r_0\delta_1\left(\tau_2 - \frac{\tau_1^2}{2}\right) + \frac{1}{4}r_0\delta_2\tau_1^2, \quad (30)$$

From (27) and (29), we obtain that

$$a_2 = \frac{r_0\delta_1\tau_1}{2(1 - \gamma + 2\eta)\Psi_{2,\mu}} = -\frac{r_0\delta_1\tau_1}{2(1 - \gamma + 2\eta)\Psi_{2,\mu}}, \quad (31)$$

it follows that

$$\tau = -\tau \quad (32)$$

and

$$(1 - \gamma + 2\eta)^2\Psi_{2,\mu}^2a_2^2 = \frac{1}{4}r_0^2\delta_1^2(\tau_1^2 + \tau_1^2). \quad (33)$$

Adding (28) and (30), we get

$$[4(1 - \gamma + 3\eta)\Psi_{3,\mu} - 2(1 - \gamma + 4\eta)\Psi_{2,\mu}^2]a_2^2 = 2r_0\delta_1(\tau_2 + \tau_2) + r_0(\delta_2 - \delta_1(\tau_1^2 + \tau_1^2)). \quad (34)$$

Applying Lemma 2.6, for the coefficients $\tau_1\tau_2, \tau_1$ and τ_2 it follows from (32) and (33), getting

$$|a_2| \leq \frac{|r_0|\delta_1}{|(1 - \gamma + 2\eta)\Psi_{2,\mu}|}, \quad (35)$$

and

$$|a_2| \leq \sqrt{\frac{|4r_0|(\delta_1 + |(\delta_2 - \delta_1)|)}{|2(1 - \gamma + 3\eta)\Psi_{3,\mu} - (1 - \gamma + 4\eta)\Psi_{2,\mu}^2|}} \quad (36)$$

which yield coefficient a_2 in (15).

Subtracting (30) from (28), we obtain

$$4(1 - \gamma + 3\eta)(a_3 + a_2^2)\Psi_{3,\mu} = 2r_0\delta_1\tau_1 + r_0\delta_1(\tau_2 - \tau_2) \quad (37)$$

Upon substituting from (33) and (34) putting (37) and applying Lemma 2.6, we find that

$$|a_3| \leq \frac{\delta_1(|r_0| + |r_1|)}{|(1 - \gamma + 3\eta)\Psi_{3,\mu}|} - \frac{|r_0^2|\delta_1^2}{|4(1 - \gamma + 2\eta)^2\Psi_{2,\mu}^2|}, \quad (38)$$

$$|a_3| \leq \frac{\delta_1|r_0 + r_1|}{|2(1 - \gamma + 3\eta)\Psi_{3,\mu}|} + \frac{|4r_0|(\delta_1 + |(\delta_2 - \delta_1)|)}{|2(1 - \gamma + 3\eta)\Psi_{3,\mu} - (1 - \gamma + 4\eta)\Psi_{2,\mu}^2|} \quad (39)$$

The estimate in (16) is obtained by combining (38) and (39). At this point, the proof is finished.

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