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Numerical solution for solving nonlinear Volterra-Fredholm system by using iterative techniques

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Abstract

In this work, we shall provide numerical schemes for solving systems of nonlinear Volterra-Fredholm integral equations, by applying the quick and innovative methods to the problem under consideration. We attempt to talk about a few numerical topics, such examination of uniqueness. Lastly, the suggested approaches' applicability and correctness are evaluated, and comparisons are made with a few numerical instances.

Subject Classification: *39B12, 47J25.*

Keywords: *Numerical solution, Nonlinear volterra, Volterra-fredholm, Iterative techniques.*

1. Introduction

This study aims to address the non-homogeneous Volterra-Fredholm integral equation (V-FIE) of the second sort, which takes the following form

$$\Xi(\beta ,\Upsilon) = \Phi(\beta,\Upsilon) + \int_0^\beta \int_\Omega \vartheta\big(\beta,\Upsilon,\xi,t,\Xi(\xi,t)\big)d\xi dt, \tag{1.1}$$

where $\Xi(\beta ,\Upsilon)$ is an unknown function, the functions $\Phi(\beta, \Upsilon)$ and $\vartheta(\beta, \Upsilon, \xi, t, \Xi(\xi, t))$ are analytic on $\mathbf{D} = \Omega \times [0, T]$ and where $T > 0$, and Ω is a closed subset of R^n, $n = 1, 2, 3$. These kinds of equations come up in a number of physical and biological issues, the theory of parabolic boundary value problems, and the mathematical modeling of the spatiotemporal evolution of epidemic models. [28] contains in-depth analyses and descriptions of these models. The following researchers solve the two-dimensional V-FIEs: The V-FIEs and by

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[24] and [20], respectively, are solved in 1986 using the time collocation technique. [21] employ methods for solving V-FIEs that are based on the ADM. Two-dimensional V-FIE are solved by [4] using the homotopy perturbation technique, and by [3] using the two-dimensional Legendre Wavelets method. We direct the reader to the following publications for further information on numerical solutions to two-dimensional V-FIEs [17, 22]. Different kinds of differential and integral equations can be approximately solved using power series expansion and its characteristics [25].

Numerous methods, including matrix-based approach [16], HPM [8] and MHPM [8], spline collocation method [5], and iterative method [23, 24], have been used recently for the nonlinear computation of the two-dimensional V-FIEs. A HPM for solving nonlinear V-FIEs of the first sort was suggested by Dong [8].

The linear VFIEs were solved by Shekarabi et al. [26] using the two-dimensional Bernstein operational matrices technique, and the V-FIEs were solved by Dastjerdi et al. [9] using the radial basis function approximation. Additionally, [25] [28] suggests using the Taylor polynomial approach to approximate the solution of the V-FIEs. Conversely, Paripour and Kamyar [23] used novel basis functions to solve nonlinear V-FIEs directly, whilst Bhrawy et al. [6] employed the Legendre-Gauss-Lobatto collocation technique to approximate the multi-dimensional Fredholm integral equation solution.

In this work, we use four approximation techniques (ADM, MADM, HAM) to solve a broad type of nonlinear V-FIEs. Additionally, we shall examine a few fresh uniqueness findings for the solutions. Using a number of test issues as examples, a successful conclusion is presented, backed up by the tabulated results of the given instances, demonstrating the approaches versatility and competence

2. Description of the Methods

The development of more sophisticated and effective techniques for solving V-FIE has been the focus of certain potent techniques, such the VIM, [1], HAM [2,14,15,18,27], ADM [13,19, 25,26,28,29], MADM [7,12,25]. In this part, we will go over each of these techniques:

2.1 ADM Technique.

Consider the V-FIE (1.1). The ADM presents the subsequent expression:

$$\Xi(\beta, \Upsilon) = \sum_{n=0}^{\infty} \Xi n(\beta, \Upsilon) \tag{2.1}$$

The components Ξ_n (β, Υ) will be found recurrently for the solution $\Xi(\beta, \Upsilon)$ of (1.1).

Additionally, an endless sequence of polynomials defines the nonlinear function $\vartheta(\beta, \Upsilon, \xi, t, \Xi(\xi, t))$.

$$\vartheta\big(\beta, \Upsilon, \xi, t\Xi(\xi, t)\big) = \sum_{n=0}^{\infty} A_n,$$

Where A_n are the so-called A domain polynomials, which can be calculated for different classes of nonlinear operators based on certain algorithms defined by A domain [7,19]. These polynomials represent the nonlinear term. In this reference [29] the author Wazwaz developed a novel algorithm for computing these polynomials. Using (2.1) and (2.2) as substitutes in (1.1) produces

$$\sum_{n=0}^{\infty} \Xi_n(\beta,\Upsilon) = \Phi(\beta,\Upsilon) + \int_0^{\beta}\int_{\Omega}(\sum_{n=0}^{\infty} A_n)d\xi dt \quad (2.3)$$

The components $\Xi_n(\beta\,,\Upsilon), n \geq 0$ will be determined in a recursive manner. The recursive connection is therefore introduced using the decomposition method:

$$\Xi_0(\beta,\Upsilon) = \Phi(\beta,\Upsilon),$$
$$\Xi_n(\beta,\Upsilon) = \int_0^{\beta}\int_{\Omega} A_n - 1 d\xi dt, \quad n \geq 1.$$

We may get the components of $\Xi_n(\beta,\Upsilon)\, n \geq 0$ recurrently by using relation (2.4). Consequently, we can easily acquire the series solution of $\Xi_0(\beta,\Upsilon)$. As previously noted, the convergence of the solution can be accelerated by combining the MADM with the noise terms phenomenon. More specially, this phenomenon can offer the answer without requiring laborious A domain polynomial computations. As will be seen in a moment, the noise terms phenomenon in conjunction with the modified decomposition approach provides a potentially useful tool for solving differential equations in general and integral equations in particular. Then, $\Xi(\beta,\Upsilon) = \sum_{i=0}^{n} \Xi_i\,(\beta,\Upsilon)$ as the approximate solution.

2.2 *MADM Technique*

This method is based on the assumption that the function $\Phi(\beta,\Upsilon)$ can be divided into two parts, namely $\Phi_1(\beta,\Upsilon)$ and $\Phi_2(\beta,\Upsilon)$. Under this assumption we set

$$\Phi(\beta,\Upsilon) = \Phi_1(\beta,\Upsilon) + \Phi_2(\beta,\Upsilon),$$

Wazwaz developed a novel algorithm for computing these polynomials. Substituting (2.1), (2.2) and (2.5) into (1.1) yields

$$\sum_{n=0}^{\infty} \Xi_n\,(\beta,\Upsilon) = \Phi_1(\beta,\Upsilon) + \Phi_2(\beta,\Upsilon) + \int_0^{\beta}\int_{\Omega}(\sum_{n=0}^{\infty} A_n)d\xi dt \quad (2.6)$$

Consequently, the recursive relation is introduced using the improved decomposition method:

$$\Xi_0(\beta,\Upsilon) = \Phi_1(\beta,\Upsilon),$$
$$\Xi_1(\beta,\Upsilon) = \Phi_2(\beta,\Upsilon) + \int_0^{\beta}\int_{\Omega} A_0 d\xi dt,$$
$$\Xi_0(\beta,\Upsilon) = \int_0^{\beta}\int_{\Omega} A_{n-1} d\xi dt, \;\; n \geq 2. \quad (2.7)$$

Relation (2.7) will enable us to determine the components $\Xi_n(\beta,\Upsilon) n \geq 0$ recurrently , and as a result, the series solution of $\Xi_0(\beta,\Upsilon)$ is readily obtained . Then, $\Xi(\beta,\Upsilon) = \sum_{i=0}^{n} \Xi_i(\beta,\Upsilon)$ as the approximate solution.

2.3 *HAM Technique*

The solution provided in HAM is primarily local. We split the interval / into sub-intervals in order to extend this solution over the interval /=[0,T] to I_j = [β_{j-1},β_j), j=1,2,3,…, p, where 0 $0 \leq \beta_0 < \beta_1 < \cdots < \beta_p$= T. We solve the equation (1.1) in each subinterval I_j. Let $\Xi_1(\beta)$ be solution of equation (1.1) in the subinterval I_1 . For, $2 \leq i \leq p$, $\Xi_i(\beta)$ is solution of equation (1.1) in the subinterval I_1. with initial conditions by obtaining the initial conditions from the interval I_{i-1}.

$$\Xi_i(\beta_{i-1}) = \Xi_{i-1}), \text{ for } i = 2,\cdots, p.$$

The solution of equation (1.1) is given by

$$\Xi = \sum_{i=1}^{p} x I_i \Xi_i \,, \tag{2.8}$$

Where

$$x_m = \begin{cases} 1 & m > 1 \\ 0 & m \leq 1 \end{cases}$$

For equation (1.1), first we choose

$$L[\phi_i(\beta;p)] = \frac{\partial^n \phi(\beta;p)}{\partial \beta^n}, n = 0,1,2,\ldots$$

And

$$H_i(\beta,\Upsilon) = 1$$

$$\Xi_{i0}(\beta,\Upsilon) = \sum_{j=0}^{m} \Xi_{(i-1)_j}(\beta,\Upsilon),\ for\ i = 2,\ldots.$$

We use HAM to equation eqrefem2 to confirm its applicability and promise in solving Volterra-Fredholm integral equations. Initially, we select

$$L[(\beta,\Upsilon;p)] = \phi(\beta,\Upsilon;p)$$

And

$$H(\beta,\Upsilon) = 1$$

A nonlinear operator is defined as follows:

$$N[\phi(\beta,\Upsilon;\mathrm{p})] = \phi(\beta,\Upsilon;p) + \Phi(\beta,\Upsilon) + \int_0^\beta \int_\Omega \vartheta\big(\beta,\Upsilon,\xi,t,\phi(\xi,t;p)\big) d\xi dt, \tag{2.9}$$

The zero order deformation equation is:

$$(1-p)L[(\beta,\Upsilon;p) - \Xi_0(\beta,\Upsilon)] = phN[\phi(\beta,\Upsilon;p)]. \tag{2.10}$$

Differentiating equation (2.10), we have the so-called m^{th}-order deformation equation for $m \geq 1$ after dividing them by m!, setting p = 0, and rearranging m-times with regard to the embedding parameter p:

$$\Xi_m(\beta,\Upsilon) = x_m \Xi_{m-1}(\beta,\Upsilon) + hR_m(\vec{\Xi}_{m-1}), \tag{2.11}$$

Where

$$R_m(\vec{\Xi}_{m-1} = \Xi_{m-1}(\beta,\Upsilon) - \int_0^\beta \int_\Omega \frac{\partial^{m-1}\vartheta(\beta,\Upsilon,\xi,t,\phi(\xi,t;p))}{\partial p^{m-1}} |p = 0^{d\xi dt} - (1 - x_m)\Phi(\beta,\Upsilon). \tag{2.12}$$

Now we have :

$$\Xi(\beta,\Upsilon) = \Xi_0(\beta,\Upsilon) + \Xi_1(\beta,\Upsilon) + \Xi_2(\beta,\Upsilon) + \cdots$$

3. Uniqueness Results

Assume that the space of all continuous functions is $\mathbb{s}$. In D,$\phi: D \longrightarrow \mathbb{R}^n$ satisfied

$$\|\phi(\beta,\Upsilon)\| = O\big(\exp\big(\mu(\beta+\|\Upsilon\|)\big)\big), (\beta,\Upsilon) \in D, \mu > 0 \tag{3.1}$$

In this space $\mathbb{S}$ we define

$$|\phi| = \sup_{D}[\|\phi(\beta,\Upsilon)\|\exp(-\mu(\beta+\|\Upsilon\|))] . \tag{3.2}$$

With the aforementioned norm, it is obvious that $\mathbb{S}$ is a Banachspace. We see that a constant M>0 exists from (3.1) such that:

$$\|\phi(\beta,\Upsilon)\| = M(\exp\big(-\mu(\beta+\|\Upsilon\|)\big)),$$

We get

$$|\phi| \le M. \tag{3.3}$$

Our principal finding is the following theorem, which provides adequate conditions for the uniqueness and existence of equation (1.1) solutions based on the fixed point theorems (see for example [11]). Before starting and proving the main results, we provide the following theories:

(H1) : A continuous nonnegative function exists $h(\beta,\Upsilon,\xi,t)$defined on D^2 such that

$$\big\|\vartheta(\beta,\Upsilon,\xi,t,\Xi_1) - \vartheta(\beta,\Upsilon,\xi,t,\Xi_{2)}\big\| \le h(\beta,\Upsilon,\xi,t)\|\Xi_1 - \Xi_2\|$$

and

$$\int_0^\beta \int_\Omega h(\beta,\Upsilon,\xi,t)\exp(\mu(\xi+\|t\|))d\xi dt \le Q\exp\big(\mu(\beta+\|\Upsilon\|)\big),$$

Where

where $(\beta,\Upsilon,\xi,t,\Xi_i) \in D^2 \times \mathbb{R}^n, i = 1,2$ and $Q > 0$.

(H2): A constant N > 0 exists such that:

$$\|\Phi(\beta,\Upsilon)\| + \int_0^\beta \int_\Omega \|\vartheta(\beta,\Upsilon,\xi,t,0)\|\, d\xi dt \le N\exp\big(\ \mu(\beta+\|\Upsilon\|)\big).$$

Theorem 3.1: *Assuming the validity of (H1) and (H2), a unique solution to Eq. (1.1) exists if* $0 < Q < 1$.

Proof: Let the right half of equation (1.1) define the operator T: **S**→**S.** it appears that T(u) is continuous in **D** and T($\Xi(\beta,\Upsilon)$) $\in \mathbb{R}^n$ for $\Xi \in$ S and (β, t) ∈ **D.**

Using assumptions (H1) and (H2), we first demonstrate the satisfaction of (3.1).

$$\begin{aligned}
\|T(\Xi(\beta,\Upsilon))\| \le\ & \int_0^\beta \int_\Omega \big\|\vartheta\big(\beta,\Upsilon,\xi,t,\Xi(\xi,\Upsilon)\big) - \vartheta(\beta,\Upsilon,\xi,t,0)\big\| d\xi dt \\
& + \|\Phi(\beta,\Upsilon))\| + \int_0^\beta \int_\Omega \|\vartheta(\beta,\Upsilon,\xi,t,0)\| d\xi dt \\
& \le |\Xi| \int_0^\beta \int_\Omega h(\beta,\Upsilon,\xi,t,)\exp(\mu(\xi+\|t\|))d\xi dt \\
& \quad + \mathrm{N}\exp(\mu(\xi+\|\Upsilon\|)) \\
& \le [\,\mathrm{MQ}+\mathrm{N}\,]\exp(\mu(\xi+\|\Upsilon\|)).
\end{aligned}$$

$\therefore T(\Xi) \in$ S.

We shall also show that $T(\Xi)$ is a contraction map in the second place. Assuming Ξ_1, $\Xi_2 \in$ S, the following may be deduced from (H1) as shown in Table 1.

$$\|T(\Xi_1(\beta,\Upsilon)) - T(\Xi_2(\beta,\Upsilon))\|$$
$$\leq \int_0^\beta \int_\Omega \|\vartheta(\beta,\Upsilon,\xi,t,\Xi_1(\xi,t)) - \vartheta(\beta,\Upsilon,\xi,t,\Xi_2(\xi,t))\| d\xi dt$$
$$\leq |\Xi_1 - \Xi_2| \int_0^\beta \int_\Omega h(\beta,\Upsilon,\xi,t,) \exp(\mu(\xi + \|t\|)) d\xi dt$$
$$\leq Q\,|\Xi_1 - \Xi_2| \exp(\mu(\xi + \|t\|)).$$

Consequently, we have

$$|T(\Xi_1) - T(\Xi_2)| \leq Q\,|\Xi_1 - \Xi_1|.$$

In other words, T is a contraction map. According to the Banach contraction principle [10], **S** contains a single fixed point Ξ for T.

4. Numerical Examples

The semi-analytical methods for solving nonlinear V-FIEs based on ADM, MADM, and HAM are presented in this section.

Example 1: Consider the nonlinear V-FIE (1.1) with:

$$\Phi(\beta, t) = e^{-\Upsilon} [\cos(\beta) + \Upsilon\cos(\beta) + \frac{1}{2} t \cos(\beta - 2)\sin(2)]$$
$$\vartheta(\beta, \Upsilon, \xi, t, \Xi(\xi, t)) = -\cos(\beta - \xi)\, e^{-(\Upsilon - t)}\Xi(\xi, t),\ (\beta, \Upsilon) \in [0, 2] \times \Omega.$$

The exact solution is $\Xi(\beta, \Upsilon) = \cos(\beta)\, e^{-\Upsilon}$, with $\Omega = [0, 2]$. We applied the methods presented in this paper as shown in Figure 1.

Example 2: Consider the nonlinear V-FIE (1.1) with

$$\Phi(\beta, t) = \frac{\beta\Upsilon^2}{8(1+\Upsilon^2)(1+\Upsilon)} - \log\left(1 + \frac{\beta\Upsilon}{1+\Upsilon^2}\right),$$

$\vartheta(\beta,\Upsilon,\xi,t,\Xi(\xi,t)) = \frac{\beta(1-\xi^2)}{(1+\Upsilon)(1+t^2)}\left(1 - e^{(-\Xi(\xi,t))}\right),\quad (\beta, \Upsilon) \in [0, 1] \times \Omega$ as shown in Figure 2.

Table 1
Numerical Results of the Example 1.

(β,ϒ)	Exact	ADM	MADM	HAM
(0.03125,0.03125)	-0.000975134	-0.000975125	-0.000975129	-0.000975134
(0.0625,0.0625)	-0.003883492	-0.003883354	-0.003883368	-0.003883457
(0.125,0.125)	-0.015267453	-0.015265342	-0.015266544	-0.015266645
(0.25,0.25)	-0.057158421	-0.057130440	-0.057146352	-0.057157346
(0.5,0.5)	-0.182321014	-0.182036813	-0.182057834	-0.182214013
(1,1)	-0.405465006	-0.403802840	-0.404406365	-0.404802854

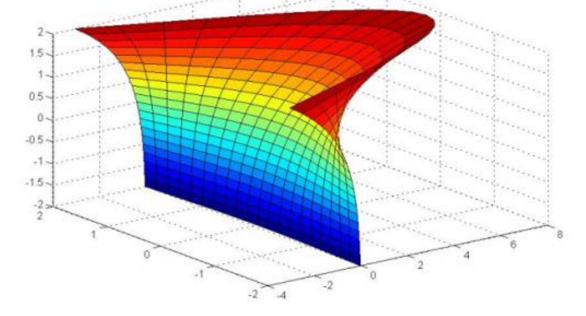

Figure 1
Numerical Results of the Example 1.

The exact solution is $\Xi(\beta, \Upsilon) = -\log\frac{1+\beta\Upsilon}{(1+\Upsilon^2)}$, with $\Omega = [0, 1]$. We applied the methods presented in this paper as shown in Table 2.

Table 2
Numerical Results of the Example 2.

(β,ϒ)	**Exact**	**ADM**	**MADM**	**HAM**
(0.03125,0.03125)	-0.000975134	-0.000975125	-0.000975129	-0.000975134
(0.0625,0.0625)	-0.003883492	-0.003883354	-0.003883368	-0.003883457
(0.125,0.125)	-0.015267453	-0.015265342	-0.015266544	-0.015266645
(0.25,0.25)	-0.057158421	-0.057130440	-0.057146352	-0.057157346
(0.5,0.5)	-0.182321014	-0.182036813	-0.182057834	-0.182214013
(1,1)	-0.405465006	-0.403802840	-0.404406365	-0.404802854

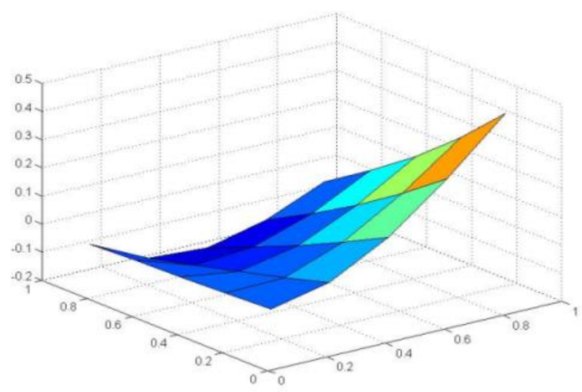

Figure 2
Numerical Results of the Example 2.

5. Discussion and Conclusion

In this study, the V-FIEs are solved using the ADM, MADM, and HAM. To show the method's validity and usefulness, we explained the procedures, applied them to two test problems, and compared the outcomes with the precise solutions. Furthermore, a minimal amount of iterations are required to achieve a desirable outcome. This assertion is supported by the provided numerical examples. The precise answer and the estimated solutions of the illustrated examples-which used ADM, MADM, and HAM with varying iterations and termsare contrasted in the above tables

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