Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 4, pp. 807–814
DOI : 10.47974/JIM-1875

Novel classes of bi-univalent functions of the Sălăgean type with a modified sigmoid unit action function

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Abstract

In this work, used (β, ζ)-Lucas polynomials that involve the modified sigmoid activation function $\psi(\delta)$ to develop a family of analytic functions along with Sălăgean-type bi-univalent functions. And obtain upper bounds for functions that belong to these subclasses. Furthermore, Fekete-Szegö inequality for these families is discussed.

Subject Classification: *30C45.*

Keywords: *Lucas polynomials, Bi-univalent functions, Modified sigmoid function, Analytic.*

This research was conducted at University of Anbar.

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1. Introduction

An activation function is crucial to mathematics for technicians as well as scientists. The activation function is the function that converts the total input to the value of the output signal. In neural networks made up of computers, specific functions like the uniqueness function, the sigmoid function, the hyperbolic tangent function, etc. are frequently utilized as activation functions. Certain training algorithms for artificial neural networks necessitate a continuous as well as differentiable activation function.

Assume that the open unit disk is $\pi = \{ \xi : \xi \in \mathbb{C} ;\ |\xi| < 1\}$. Let $\mathcal{A}$ stand for the shape's relatives of functions:

$$\vartheta(\xi) = \xi + \sum_{\sigma=2}^{\infty} t_\sigma \xi^\sigma, \tag{1}$$

which are analytical. Moreover, we will identify the class of every function in $\mathcal{A}$ that are univalent in π through $\mathcal{S}$. The image of π is correct, along with all univalent function $\vartheta \in \mathcal{S}$ contains a disk alongside a radius of $1/4$, according to the Koebe one Qoarter Theorem [1]. Consequently, the inverse of every univalent function ϑ is defined as ϑ^{-1} characterize $\xi = \vartheta^{-1}(\vartheta(\xi))$,

$$\chi = \vartheta^{-1}(\vartheta(\chi)), \left(|\chi| < r_0(\vartheta); r_0(\vartheta) \geq \tfrac{1}{4}\right), \tag{2}$$

whereas

$$h(\chi) = \vartheta^{-1}(\chi) = \chi - t_2\chi^2 + (2t_2^2 - s_3)\chi^3 - (5t_2^3 - 5t_2t_3 + t_4)\chi^4 + \cdots. \tag{3}$$

Additionally, obtain an additional equivalency if the function ϑ is univalent in π. If just if $\vartheta(0) = h(0)$ as well as $\vartheta(\pi) \subset h(\pi)$ are true (see[2]), then $\vartheta(\xi) \prec h(\xi)$. After looking at the class $\sum$ of bi-univalent functions, Lewien [3] obtained a coefficient bound for every function $\vartheta \in \sum$, which can be calculated by $|t_2| \leq 1.51$. After that, The efforts of Lewien [3], Clonie, and Branan [4] played a significant role in forming other assumptions $|t_2| \leq \sqrt{2}$ for all $\vartheta \in \sum$.

Indeed, the study of bi-univalent as well as analytic functions has been reenergized recently through Hameed et al. [5], with Bulot [6] following suit. Saloomi et al. [7, 8]. Adegaeni along with et al. [9], as well as [10, 11]). As of right now, functions $\vartheta \in \sum$ are still being worked on, so the coefficient estimation problem for every one of the Taylor-Maclaurin coefficients $|t_\sigma|, (\sigma \in \mathbb{N} = \{1,2,3,4, \ldots \ldots\}, \sigma \geq 3)$ has proven difficult (see [6]).

Lemma 1.1: [12] *Given a function* h *that is analytic in* π *and for which* $\Re\{h(\xi)\} > 0$, *if* $\vartheta \in H$, *subsequently* $|t_\sigma| \leq 2$ *for every value* ς.

$$h(\xi) = 1 + t_1\xi + t_2\xi^2 + t_3\xi^3 + \cdots.$$

Given two polynomials alongside real coefficients, $\beta(x)$ as well as $\zeta(x)$. According to [13], the recurrence relation beneath provides the $(\beta, \zeta) -$ polynomials $L_{\beta,\zeta,\sigma}(x)$ or more succinctly $L_\sigma(x)$:

$$L_\sigma(x) = \beta(x)L_{\sigma-1}(x) + \zeta(x)L_{\sigma-2}(x)\ (\sigma \in \mathbb{N} - \{1\}),$$

in addition to

$$L_0(x) = 2,$$
$$L_1(x) = \beta(x),$$
$$L_2(x) = \beta^2(x) + 2\zeta(x),$$
$$L_3(x) = \beta^3(x) + 3\beta(x)\zeta(x), \dots \tag{4}$$

This yields the generating function of the Lucas polynomials (see [14]):

$$M_{L_\sigma(x)}(\xi) = \sum_{\sigma=0}^{\infty} L_\sigma(x)\xi^\sigma = \frac{2-\beta(x)\xi}{1-\beta(x)\xi-\zeta(x)\xi^2}.$$

Let $\mathcal{A}_\psi$be a class of functions that has the following shape:

$$\vartheta_\psi(\xi) = \xi + \sum_{\sigma=2}^{\infty} \psi(\delta) t_\sigma \xi^\sigma = \xi + \sum_{\sigma=2}^{\infty} \frac{4}{2+2e^{-\delta}} t_\sigma \xi^\sigma, \tag{5}$$

The activation function of a sigmoid is represented by $\psi(\delta) = 2/1 + e^{-\delta}$ when $\delta \geq 0$. Moreover, $\mathcal{A}_1 = \mathcal{A}$.

$N^\varsigma, \varsigma \in N \cup \{0\}$ is the differential operator we are considering as $\mathcal{A}_\psi$ is the owner of ϑ_ψ, as specified by

$$N^0\vartheta_\psi(\xi) = \vartheta_\psi(\xi);\ N^1\varsigma_\psi(\xi) = N\vartheta_\psi(\xi) = \xi\vartheta_\psi'(\xi);\ N^\varsigma\vartheta_\psi(\xi) = N(N^{\varsigma-1}\vartheta_\psi(\xi)).$$

Observe that,

$$N^\varsigma\vartheta_\psi(\xi) = \xi + \sum_{\sigma=2}^{\infty} \sigma^\varsigma \psi(\delta) t_\sigma \xi^\sigma = \xi + \sum_{\sigma=2}^{\infty} \frac{4\sigma^\varsigma}{2+2e^{-\delta}} t_\sigma \xi^\sigma. \tag{6}$$

Presenting in this paper a few families of bi-univalent functions established by means of the (β, ζ) −Lucas polynomials. Furthermore, obtaining Fekete–Szegö inequality as well as coefficient bounds for the functions given in the above class.

2. Main results for the class $\mathrm{GH}_\Sigma(\boldsymbol{\mu}, \boldsymbol{\varpi}, \boldsymbol{\varepsilon}, \boldsymbol{\varsigma}, \boldsymbol{\Theta}(\boldsymbol{\delta};); \boldsymbol{x})$

Definition 2.1: If the conditions that follow for subordination are satisfied, a function $\vartheta \in \Sigma$ is declared to be in the class $GH_\Sigma(\mu, \varpi, \varepsilon, \varsigma, \Theta(\delta); x)$ for $0 \leq \mu \leq 1, \varpi \geq 0, \varepsilon \in \mathbb{C} \setminus \{0\}$ as well as $\Theta(\delta) = 2/1 + e^{-\delta}, \delta \geq 0$:

$$\left[1 + \frac{(1-\mu)}{\varepsilon} \frac{\xi\left(N^\varsigma\vartheta_\psi(\xi)\right)'}{N^\varsigma\vartheta_\psi(\xi)} \left(\frac{N^\varsigma\vartheta_\psi(\xi)}{\xi}\right)^\varpi \Big/ \frac{\mu}{\varepsilon}\left(2 + \frac{\xi\left(N^\varsigma\vartheta_\psi(\xi)\right)''}{\left(N^\varsigma\vartheta_\psi(\xi)\right)'}\right)\left(\left(N^\varsigma\vartheta_\psi(\xi)\right)'\right)^\varpi\right] \prec M_{L_\sigma(x)}(\xi)$$

and

$$\left[1 + \frac{(1-\mu)}{\varepsilon} \frac{\chi\left(N^\varsigma h_\psi(\chi)\right)'}{N^\varsigma h_\psi(\chi)} \left(\frac{N^\varsigma h_\psi(\chi)}{\chi}\right)^\varpi \Big/ \frac{\mu}{\varepsilon}\left(2 + \frac{\chi\left(N^\varsigma h_\psi(\chi)\right)''}{\left(N^\varsigma h_\psi(\chi)\right)'}\right)\left(\left(N^\varsigma h_\psi(\chi)\right)'\right)^\varpi\right] \prec M_{L_\sigma(x)}(\chi)$$

in which (1) denotes the function $h = \vartheta^{-1}$.

Should we select $\mu = 0$ and $\mu = 1$ in Definition 2.1, we obtain the Remark that follows.

Remark 2.1: If the conditions that follow for subordination are satisfied, a function $\vartheta \in \Sigma$ is declared to be in the class $G_\Sigma(\varpi, \varepsilon, \varsigma, \Theta(\delta); x)$ for $\mu = 0, \varpi \geq 0, \varepsilon \in \mathbb{C} \setminus \{0\}$ as well as $\Theta(\delta) = 2/1 + e^{-\delta}, \delta \geq 0$:

$$\left[\frac{\xi\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'}{N^{\varsigma}\vartheta_{\psi}(\xi)*\varepsilon}\Big/\left(\frac{N^{\varsigma}\vartheta_{\psi}(\xi)}{\xi}\right)^{\varpi}-2\varepsilon\right]\prec M_{L_{\sigma}(x)}(\xi),\left[\frac{\chi\left(N^{\varsigma}h_{\psi}(\chi)\right)'}{N^{\varsigma}h_{\psi}(\chi)*\varepsilon}\Big/\left(\frac{N^{\varsigma}h_{\psi}(\chi)}{\chi}\right)^{\varpi}\right]\prec M_{L_{\sigma}(x)}(\chi).$$

And a function $\vartheta \in \Sigma$ is declared to be in the class $H_{\Sigma}(\varpi,\varepsilon,\varsigma,\Theta(\delta);x)$ for $\mu = 1$, $\varpi \geq 0, \varepsilon \in \mathbb{C} \setminus \{0\}$ as well as $\Theta(\delta) = 2/1 + e^{-\delta}, \delta \geq 0$:

$$\left[\left(\frac{\xi\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)''}{\varepsilon\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'}\right)\Big/\left(\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'\right)^{\varpi}\right]\prec$$

$$M_{L_{\sigma}(x)}(\xi),\left[\left(\frac{r\left(N^{\varsigma}h_{\psi}(\chi)\right)''}{\varepsilon\left(N^{\varsigma}h_{\psi}(\chi)\right)'}\right)\Big/\left(\left(N^{\varsigma}h_{\psi}(\chi)\right)'\right)^{\varpi}\right]\prec M_{L_{\sigma}(x)}(\chi)$$

in which (2) denotes the function $h = \vartheta^{-1}$.

Theorem 2.1: *Suppose $\vartheta \in \Sigma$ as defined by (1), belong to the class $GH_{\Sigma}(\mu,\varpi,\varepsilon,\varsigma,\psi(\delta);x)$. Next*

$$|t_2| \leq \varepsilon|\beta(x)|\sqrt{|\beta(x)|}\Big/2^{\varsigma}\psi(\delta)\sqrt{\left|\frac{\left[(\mu+1)^2-\frac{1}{2}[\varepsilon\varpi(\mu+\varpi(3\mu+1)+\varepsilon)]\right]}{\beta^2(x)+2\varpi^2(\mu+1)^2\zeta(x)}\right|}, \tag{7}$$

$$|t_3| \leq \frac{3|\varepsilon||\beta(x)|}{(2\mu+1)3^{\varsigma}\psi(\delta)}\Big/\frac{2|\varepsilon|^2\,\beta^2(x)}{(\mu+1)^2 3^{\varsigma}\psi(\delta)}, \tag{8}$$

additionally, certain people $\eta \in \mathbb{R}$,

$$|t_3-\eta t_2^2| \leq \begin{cases} \dfrac{|\varepsilon||\beta(x)|}{(\varpi+2)(2\mu+1)3^{\varsigma}\psi(\delta)}, if \left|1-\eta\dfrac{3^{\varsigma}}{2^{2\varsigma}\psi(\delta)}\right| \leq |\varepsilon|(2\mu+1)\beta^2(x) \\ \dfrac{|\varepsilon|^2|\beta(x)|^3}{\left|\dfrac{\left[(\mu+1)^2-\frac{\varepsilon}{2}[(\varpi+2)(\varpi(3\mu+1))]\right]}{\beta^2(x)+2(\varpi+1)^2(\mu+1)^2\zeta(x)}\right|3^{\varsigma}\psi(\delta)}, if \left|1-\eta\dfrac{3^{\varsigma}}{2^{2\varsigma}\psi(\delta)}\right| \geq |\varepsilon|(2\mu+1)\beta^2(x). \end{cases} \tag{9}$$

Proof: Using the Taylor-Maclaurin expansion (1), let $\vartheta \in GH_{\Sigma}$ $(\mu,\varpi,\varepsilon,\varsigma,\psi(\delta);x)$ be given. Then, there are two analytical functions (u as well as v) with the property that $u(0) = 0$ as well as $v(0) = 0$.

$$|u(\xi)| = |m_1\xi + m_2\xi + m_3\xi + \cdots| < 1, |v(\chi)| = |\sigma_1\chi + \sigma_2\chi + \sigma_3\chi + \cdots| < 1 \ (\forall\, \xi, \chi \in \pi).$$

Therefore, we are able to write

$$\left[1+\frac{(1-\mu)}{\varepsilon}\frac{\xi\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'}{N^{\varsigma}\vartheta_{\psi}(\xi)}\left(\frac{N^{\varsigma}\vartheta_{\psi}(\xi)}{\xi}\right)^{\varpi}\Big/\frac{\mu}{\varepsilon}\left(\frac{\xi\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)''}{\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'}\right)\left(\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'\right)^{\varpi}\right]=M_{L_{\sigma}(x)}(u(\xi)),$$

$$\left[1+\frac{(1-\mu)}{\varepsilon}\frac{\chi\left(N^{\varsigma}h_{\psi}(\chi)\right)'}{N^{\varsigma}h_{\psi}(\chi)}\left(\frac{N^{\varsigma}h_{\psi}(\chi)}{\chi}\right)^{\varpi}\Big/\frac{\mu}{\varepsilon}\left(\frac{\chi\left(N^{\varsigma}h_{\psi}(\chi)\right)''}{\left(N^{\varsigma}h_{\psi}(\chi)\right)'}\right)\left(\left(N^{\varsigma}h_{\psi}(\chi)\right)'\right)^{\varpi}\right]=M_{L_{\sigma}(x)}(v(\chi)).$$

Using the equalities previously determined, we can determine that

$$\left[\frac{1+\frac{(1-\mu)}{\varepsilon}\frac{\xi\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'}{N^{\varsigma}\vartheta_{\psi}(\xi)}\left(\frac{N^{\varsigma}\vartheta_{\psi}(\xi)}{\xi}\right)^{\varpi}}{\frac{\mu}{\varepsilon}\left(2+\frac{\xi\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)''}{\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'}\right)\left(\left(N^{\varsigma}\vartheta_{\psi}(\xi)\right)'\right)^{\varpi}}\right]=1+L_1(x)m_1\xi+[L_1(x)m_2+L_2(x)m_1^2]\xi^2+\cdots, \tag{10}$$

$$\left[\frac{1+\frac{(1-\mu)}{\varepsilon}\frac{\chi\left(N^{\varsigma}h_{\psi}(\chi)\right)'}{N^{\varsigma}h_{\psi}(\chi)}\left(\frac{N^{\varsigma}h_{\psi}(\chi)}{\chi}\right)^{\varpi}}{\frac{\mu}{\varepsilon}\left(2+\frac{\chi\left(N^{\varsigma}h_{\psi}(\chi)\right)''}{\left(N^{\varsigma}h_{\psi}(\chi)\right)'}\right)\left(\left(N^{\varsigma}h_{\psi}(\chi)\right)'\right)^{\varpi}}\right]=1+L_1(x)\sigma_1\chi+[L_1(x)\sigma_2+L_2(x)\sigma_1^2]\chi^2+\cdots \tag{11}$$

Furthermore, it is widely known that

$$|m_i|\le 1 \text{ and } |\sigma_i|\le 1, (i\in\mathbb{N}). \tag{12}$$

The equivalent coefficients in (10) and (11), can be compared to obtain

$$[(\varpi+1)(\mu+1)]\frac{2^{\varsigma}}{\varepsilon}\psi(\delta)t_2=L_1(x)m_1, \tag{13}$$

$$\left[\varpi(2\mu+1)\frac{3^{\varsigma}}{\varepsilon}\psi(\delta)t_3+\frac{1}{2\varepsilon}(\varpi-1)(\varpi+2)(3\mu+1)2^{2\varsigma}\psi^2(\delta)t_2^2\right]= L_1(x)m_2+L_2(x)m_1^2, \tag{14}$$

$$[(\varpi+1)(\mu+1)]\frac{-2^{\varsigma}}{\varpi}\psi(\delta)t_2=L_1(x)\sigma_1 \tag{15}$$

and

$$\left[\varpi(2\mu+1)\left(\frac{2^{2\varsigma+1}}{\varepsilon}\psi^2(\delta)t_2^2-\frac{3^{\varsigma}}{\varepsilon}\psi(\delta)t_3\right)+\frac{1}{2}\varpi(\varpi-1)(3\mu+1)\frac{2^{2\varsigma}}{\varepsilon}\psi^2(\delta)t_2^2\right] =L_1(x)\sigma_2+L_2(x)\sigma_1^2. \tag{16}$$

It is evident from equations (13) as well as (15) that

$$m_1=-\sigma_1, \tag{17}$$

$$[(\varpi+1)^2(\mu+1)^2]\frac{2^{2\varsigma+1}}{\varepsilon^2}\psi^2(\delta)t_2^2=L_1^2(x)(m_1^2+\sigma_1^2). \tag{18}$$

Adding (14) to (16) yields

$$(\varpi+2)[\mu+\varpi(3\mu+1)]\frac{2^{2\varsigma}}{\varepsilon}\psi^2(\delta)t_2^2 = L_1(x)(m_2+\sigma_2)+L_2(x)(m_1^2+\sigma_1^2). \tag{19}$$

Using (18) in equation (19) makes it clear that the result is

$$t_2^2=\frac{\varepsilon^2L_1^3(x)(m_2+\sigma_2)}{[\varepsilon(\varpi+2)[\mu+\varpi(3\mu+1)]L_1^2(x)-2[(\mu+1)^2]L_2(x)]2^{2\varsigma}\psi^2(\delta)}. \tag{20}$$

Subtracting (16) from (14) as well as factoring in (17), we obtain

$$t_3 = \frac{\varepsilon L_1(x)[m_2-\sigma_2]}{2\varpi(2\mu+1)3^{\varsigma}\psi(\delta)} \Big/ \frac{2^{2\varsigma}\psi(\delta)}{3^{\varsigma}}\, t_2^2. \tag{21}$$

Then, considering (18), (21) turns into and from (3) as well as (12), it is evident that

$$|t_3| \leq \frac{|\varepsilon||\beta(x)|}{(\varpi+2)(2\mu+1)3^{\varsigma}\psi(\delta)} \Big/ \frac{|\varepsilon|^2\,\beta^2(x)}{(\varpi+1)^2(\mu+1)^2 3^{\varsigma}\psi(\delta)}.$$

Form (21), where $\eta \in \mathbb{R}$ is written

$$t_3 - \eta t_2^2 = \frac{\varepsilon L_1(x)(m_2-\sigma_2)}{2(\varpi+2)(2\mu+1)3^{\varsigma}\,\psi(\delta)} \Big/ \left(\frac{2^{2\varsigma}\psi(\delta)}{3^{\varsigma}} - \eta\right) t_2^2. \tag{22}$$

When we replace (22) with (20), we obtain

$$t_3 - \eta t_2^2 == \varepsilon L_1(x)\left[\frac{\left(\Omega(\eta,x)+\frac{1}{2(\varpi+2)(2\mu+1)3^{\varsigma}\psi(\delta)}\right)m_2}{\left(\Omega(\eta,x)-\frac{1}{2(\varpi+2)(2\mu+1)3^{\varsigma}\psi(\delta)}\right)\sigma_2}\right],$$

$$\Omega(\eta,x) = \frac{\varepsilon L_1^2(x)(1-\mu)}{\frac{[\varepsilon(\varpi+2)[\mu+\varpi(3\mu+1)]]}{\left([L_1(x)]^2-2(\varpi+1)^2(\mu+1)^2 L_2(x)\right)}2^{2\varsigma}\psi^2(\delta)}.$$

Consequently, we draw the conclusion that

$$|t_3 - \eta t_2^2| \leq \begin{cases} \dfrac{|\varepsilon||L_1(x)|}{\varpi(2\mu+1)3^{\varsigma}\psi(\delta)} \; if \; |\Omega(\eta,x)| \leq \dfrac{1}{\varpi(2\mu+1)3^{\varsigma}\,\psi(\delta)} \\ |\varepsilon||L_1(x)||\Omega(\eta,x)| \; if \; |\Omega(\eta,x)| \geq \dfrac{1}{\varpi(2\mu+1)3^{\varsigma}\psi(\delta)}, \end{cases}$$

which obviously finishes Theorem 2.1's proof**.**

Using Theorem 2.1 with $\mu = 0$, $\mu = 1$ yields the following result:

Corollary 2.1: *Assume the function $\vartheta(\xi)$ in class $G_{\Sigma}(\varpi,\varepsilon,\varsigma,\psi(\delta);x)$ become the one denoted by (1). Then for $\mu = 0$, obtain*

$$|t_2| \leq \varepsilon|\beta(x)|\sqrt{|\beta(x)|}\Big/ 2^{\varsigma}\psi(\delta)\sqrt{\left|\frac{\left[(\varpi+1)^2-\frac{\varepsilon}{2}[(\varpi+2)(\varpi+1)]\right]}{\left(\beta^2(x)+2(\varpi+1)^2\zeta(x)\right)}\right|},$$

$$|t_3| \leq \frac{|\varepsilon||\beta(x)|}{(\varpi+2)3^{\varsigma}\psi(\delta)} \Big/ \frac{|\varepsilon|^2\,\beta^2(x)}{(\varpi+1)^2 3^{\varsigma}\psi(\delta)}.$$

And $\mu = 1$, by $H_{\Sigma}(\varpi,\varepsilon,\varsigma,\Theta(\delta);x)$, obtain

$$|t_2| \leq \varepsilon|\beta(x)|\sqrt{|\beta(x)|}/2^{\varsigma}\psi(\delta)\sqrt{\left|\frac{[4(\varpi+1)^2-\varepsilon[(\varpi+2)(2\varpi+1)]]}{\beta^2(x)+8(\varpi+1)^2\zeta(x)}\right|},$$

$$|t_3| \leq \frac{|\varepsilon||\beta(x)|}{(\varpi+2)3^{\varsigma+1}\psi(\delta)} \Big/ \frac{|\varepsilon|^2\,\beta^2(x)}{4(\varpi+1)^2 3^{\varsigma}\psi(\delta)}.$$

Remark 2.2:

i. The results suggested by AL-Ameedee et al. [15] are obtained if we enter $\mu = 0, \varpi = 1$, and $\varepsilon = 1$, in Theorem 2.1.
ii. The results suggested by Swamy et al. [16] are obtained if we enter $\mu = 0, \varpi = 0$, and $\varepsilon = 1$, in Theorem 2.1.

iii. The results suggested by Altinkaya and Yalçin [17] are obtained if we enter $\mu = 0, \varpi = 1$, $\varepsilon = 1$, and $\psi(\delta) = 1$ in Theorem 2.1.

iv. The results suggested by Altinkaya [18] is obtained if we enter $\mu = 1, \varpi = 0$, $\varepsilon = 1$, and $\psi(\delta) = 1$ in Theorem 2.1.

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Received December, 2023