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New generalizations of soft sets in soft topological spaces and their applications

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Abstract

The fundamental objective of the present this article is to display a complete description of novel ideas of generalized soft open and closed sets in soft topological spaces, which are defined as δ-ß-open and δ-ß-closed sets. Various essential properties and characterizations related to these ideas of generalized soft sets have been studied. In addition, the relationship between several well-known ideas of generalized soft open sets such as soft δ-pre-open, soft δ-semi-open, soft pre-open, soft E-open, soft ß-open sets, and soft δ-ß-open sets over the soft topological spaces have been investigated. Furthermore, several appropriate examples to support our main outcomes have been provided.

Subject Classification: *Primary (54C08), Secondary (54C10).*

Keywords: *Softδ-ß-open sets, Soft δ-ß-closed sets, Soft δ-ß-interior, Soft δ-ß-closure.*

1. Introduction

Soft set theory is an essential and powerful mathematical tool to study uncertainty and solve complex problems in real life". The notion of soft sets was initiated via Molodtsov [1]. The following articles [1, 2] discussed the essential results of the new theory. A rising number of manuscripts have been written about soft sets theory and their applications we refer the readers to see [3, 4]". Maji et al. [5] applied a soft set in multicriteria decision-making problems as well investigated various basic notions of soft set theory in [6]. A. Kharal & B. Ahmad [7] presented some characterizations of soft and inverse images of soft

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sets. In 2011, Shabir & Naz [8] initiated the study of soft topological spaces defined over an initial universe with a fixed set of parameters.

Furthermore, S. Hussain and B. Ahmad [9] further explored the characterizations of soft open sets and neighborhoods. Subsequently, Akdag and Ozkan [10, 11] introduced the concepts of soft b-open (b-closed) sets and soft α-open (soft α-closed) sets, respectively. Anjan and Debnath [12] elaborated on soft e-open and e-closed sets. More recently, Al-Jumaili and A. Farhan [13] delved into alternative forms of expanded compact and closed spaces. This article aims to contribute to the field by investigating various modern types, such as soft δ-ß-open and soft δ-ß-closed sets, within the context of soft topological spaces. Various characterizations related to these generalized soft open and closed sets are established [14,15].

2. Materials and Methods

Definition 2.1:[1] Assume $\mathcal{X}$ is an initial universe with E as a set of parameters. Presume $\mathcal{P}(\mathcal{X})$ indicates to the power of $\mathcal{X}$and $\emptyset \neq \mathcal{M} \subseteq E$ E. A pair$(\mathcal{D},\mathcal{M})$is said to a soft set(Shortly, S.S) over $\mathcal{X}$, wherever $\mathcal{F}$ is map defined as $\mathcal{D}: \mathcal{M} \longrightarrow \mathcal{P}(\mathcal{X})$ and described via $\mathcal{D}(\varepsilon) \in \mathcal{P}(\mathcal{X})\ \forall\ \varepsilon \in \mathcal{M}$.

Definition 2.2: [8] Let $\mathcal{T}$ be the family of (S.S) over$\mathcal{X}$, so $\mathcal{T}$ is said to have soft topology on $\mathcal{X}$ if it satisfies the next conditions:

(a) Φ & $\widetilde{\mathcal{X}}$, belong to $\mathcal{T}$.

(b) The union of any number of (S.S) in $\mathcal{T}$ belongs to $\mathcal{T}$.

(c) The intersection of any two (S.S) in $\mathcal{T}$ belongs to $\mathcal{T}$.

Remark 2.3: The members of $\mathcal{T}$ are called soft open sets (Shortly, (S.O.S)).

Remark 2.4: "We have the following implications; however, the converses of t*hes*e implications are not true, see [10] and [12] as explained in diagram 1.

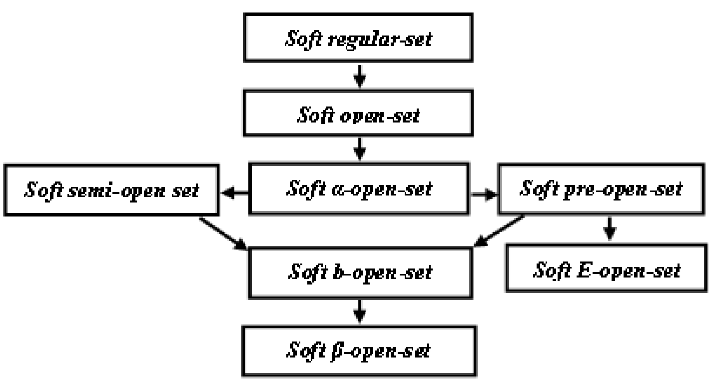

Diagram 1
The relations among some well-known generalized (S.O.S)

Definition 2.5: [12] A (S.S) $(\mathcal{D},\mathcal{M})$ in (S.$\mathcal{T}$.S) $\mathcal{X}$ is:

(a) S-δ-Semi-O iff $(\mathcal{D},\mathcal{M}) \tilde{\subseteq} Cl(Int_\delta(\mathcal{D},\mathcal{M}))$.

(b) S-δ-Semi-C iff $Int\big(Cl_\delta(\mathcal{D},\mathcal{M})\big) \tilde{\subseteq} (\mathcal{D},\mathcal{M})$.

Definition 2.6: [12] A (S.S) $(\mathcal{F},\mathcal{A})$ in (S.$\mathcal{T}$.S) $\mathcal{X}$ is:

(a) S-δ-Pre-O iff $(\mathcal{D},\mathcal{M}) \tilde{\subseteq} Int\big(Cl_\delta(\mathcal{D},\mathcal{M})\big)$.

(b) S-δ-Pre-C iff $Cl(Int_\delta(\mathcal{D},\mathcal{M})) \tilde{\subseteq} (\mathcal{D},\mathcal{M})$.

3. Diverse Characterizations of Soft δ-ß-Open and δ-ß-Closed Sets

Definition 3.1: A (S.S) $(\mathcal{D},\mathcal{M})$ in a (S.$\mathcal{T}$.S)$(\mathcal{X},\mathcal{T},E)$ is called:

(a) *S-δ-ß*-O set ⟺ $(\mathcal{D},\mathcal{M}) \tilde{\subseteq} Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})))$.

(b) *S-δ-ß*-C set ⟺ $(\mathcal{D},\mathcal{M}) \tilde{\supseteq} Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})))$.

Example 3.2: Let $\mathcal{X} = \{\hbar_1,\hbar_2,\hbar_3,\hbar_4\}, E = \{y_1,y_2,y_3\}$ & $\mathcal{T} = \{\Phi,\tilde{\mathcal{X}},(\mathcal{G},E)\}$, wherever $(\mathcal{W},\mathcal{M}) = \{(y_1,\{\hbar_1\}),(y_2,\{\hbar_2,\hbar_4\}),(y_3,\{\hbar_2\})\}$.

In that case, $(\mathcal{X},\mathcal{T},E)$ is $(\mathcal{X},\mathcal{T},E)$ & $(\mathcal{W},\mathcal{M})$ is a *S-δ-ß*-O set.

Theorem 3.3: *Let*$(\mathcal{D},\mathcal{M})$ *be (S.S) in* $(S.\mathcal{T}.S)\mathcal{X}$, *so:*

(a) $(\mathcal{D},\mathcal{M})$*is S-δ-ß-O set* ⟺ $(\mathcal{D},\mathcal{M})^c$ *is S-δ-ß-C set.*

(b) $(\mathcal{D},\mathcal{M})$is S-δ-ß-C set ⟺ $(\mathcal{D},\mathcal{M})^c$ is S-δ-ß-O set.

Proof: It is follows immediately via Definition(3.1).

Definition 3.4: Let $(\mathcal{X},\mathcal{T},E)$ be a (S.$\mathcal{T}$.S) & $(\mathcal{D},\mathcal{M})$ be (S.S) over $\mathcal{X}$. So,

(a) *S-δ-ß*-Interior of (S.S)$(\mathcal{D},\mathcal{M})$ in $\mathcal{X}$ indicated via:
$S\delta - \text{ß}Int(\mathcal{D},\mathcal{M}) = \tilde{\bigcup}\{(\mathcal{W},\mathcal{M})\} \tilde{\subseteq} (\mathcal{D},\mathcal{M}): (\mathcal{W},\mathcal{M})\}$ is S-δ-ß-O set of $\mathcal{X}$.

(b) Soft *δ-ß*-Closure of $(\mathcal{D},\mathcal{M})$ in $\mathcal{X}$ is indicated via:
$S\delta - \text{ß}Cl(\mathcal{D},\mathcal{M}) = \tilde{\bigcap}\{(\mathcal{H},\mathcal{M})\} \tilde{\supseteq} (\mathcal{D},\mathcal{M}): (\mathcal{H},\mathcal{M})$ is S-δ-ß-C set of $\mathcal{X}$.

Theorem 3.5: *Let* $(\mathcal{X},\mathcal{T},E)$ *be a* $(S.\mathcal{T}.S)$ *and* $(\mathcal{D},\mathcal{M})$ *be several soft set in* $(\mathcal{X},\mathcal{T},E)$. *So:*

(a) $S\delta - \text{ß}Cl(\mathcal{D},\mathcal{M})^c = \tilde{\mathcal{X}} - S\delta - \text{ß}Int(\mathcal{D},\mathcal{M})$.

(b) $S\delta - \text{ß}Int(\mathcal{D},\mathcal{M})^c = \tilde{\mathcal{X}} - S\delta - \text{ß}Cl(\mathcal{D},\mathcal{M})$.

Theorem 3.6: *In a* $(S.\mathcal{T}.S)(\mathcal{X},\mathcal{T},E)$, *we have the following statements:*

(a) *The arbitrary union of S-δ-ß-O sets is a S-δ-ß-O set.*

(b) *The arbitrary intersection of S-δ-ß-C sets is a S-δ-ß-C set.*

Proof:

(a) Let $\{(\mathcal{D},\mathcal{M})_\lambda : \lambda \in \Delta\}$ be a family of S-δ-ß-O sets. so $\forall \lambda, (\mathcal{D},\mathcal{M})_\lambda \tilde{\subseteq} [Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})_\lambda))]$. Now $\tilde{\bigcup}\{(\mathcal{D},\mathcal{M})_\lambda\} \tilde{\subseteq} \tilde{\bigcup}[Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})_\lambda))] \tilde{\subseteq} [Cl(Int(Cl_\delta(\tilde{\bigcup}(\mathcal{D},\mathcal{M})_\lambda)))]$. So, $\tilde{\bigcup}((\mathcal{D},\mathcal{M})_\lambda)$ is soft *δ-ß*-open.

(b) The proof follows directly from (a) by taking compliments.

Theorem 3.7: *Let* $(\mathcal{D},\mathcal{M})$ *be a (S.S) in* $(S.\mathcal{T}.S)(\mathcal{X},\mathcal{T},E)$*, so:*

(a) $(\mathcal{D},\mathcal{M})$ *is S-δ-ß-C set* $\Leftrightarrow (\mathcal{D},\mathcal{M}) = \mathcal{S}\delta - ß - Cl(\mathcal{D},\mathcal{M})$.

(b) $(\mathcal{D},\mathcal{M})$ *is S-δ-ß-O set* $\Leftrightarrow (\mathcal{D},\mathcal{M}) = \mathcal{S}\delta - ßInt(\mathcal{D},\mathcal{M})$.

Theorem 3.8: *Let* $(\mathcal{X},\mathcal{T},E)$ *be a* $(S.\mathcal{T}.S)$*. So, the next properties hold for S-δ-ß-Closure and S-δ-ß-Interiors:*

(a) $S\delta - ßCl(\Phi) = (\Phi) \& S\delta - ßCl(\tilde{\mathcal{X}}) = (\tilde{\mathcal{X}})$.

(b) $S\delta - ßInt(\Phi) = (\Phi)$ *and* $S\delta - ßInt(\tilde{\mathcal{X}}) = (\tilde{\mathcal{X}})$.

(c) $S\delta - ßCl(\mathcal{D},\mathcal{M})$ *is Sδ-ß-closed of* $\mathcal{X}$.

(d) $S\delta ßInt(\mathcal{D},\mathcal{M})$ is S*δ-ß*-open of $\mathcal{X}$.

(e) $S\delta - ßCl(\mathcal{D},\mathcal{M}) \tilde{\subseteq} S\delta - ßCl(\mathcal{W},\mathcal{M})\ if (\mathcal{D},\mathcal{M}) \tilde{\subseteq} (\mathcal{W},\mathcal{M})$.

(f) $S\delta - ßInt(\mathcal{D},\mathcal{M}) \tilde{\subseteq} S\delta - ßInt(\mathcal{W},\mathcal{M})\ if (\mathcal{D},\mathcal{M}) \tilde{\subseteq} (\mathcal{W},\mathcal{M})$.

(g) $S\delta - ßCl(S\delta - ßCl(\mathcal{D},\mathcal{M})) = S\delta - ßCl(\mathcal{D},\mathcal{M})$.

(h) $S\delta - ßInt(S\delta - ßInt(\mathcal{D},\mathcal{M})) = S\delta - ßCl(\mathcal{D},\mathcal{M})$.

Theorem3.9: *Let*$(\mathcal{X},\mathcal{T},E)$ *be a* $(S.\mathcal{T}.S)$*. So, the following statements hold:*

(a) $S\delta - ßCl(\mathcal{D},\mathcal{M})\tilde{\bigcup}(\mathcal{W},\mathcal{M}) \tilde{\supseteq} S\delta - ßCl(\mathcal{D},\mathcal{M})\tilde{\bigcup} S\delta - ßCl(\mathcal{W},\mathcal{M})$.

(b) $S\delta - ßCl(\mathcal{D},\mathcal{M}) \tilde{\bigcap} (\mathcal{W},\mathcal{M}) \tilde{\subseteq} S\delta - ßCl(\mathcal{D},\mathcal{M}) \tilde{\bigcap} S\delta - ßCl(\mathcal{W},\mathcal{M})$.

Proof:

(a) As$(\mathcal{D},\mathcal{M}) \tilde{\subseteq} ((\mathcal{D},\mathcal{M})\tilde{\bigcup}(\mathcal{W},\mathcal{M}))$ or$(\mathcal{W},\mathcal{M}) \tilde{\subseteq} ((\mathcal{D},\mathcal{M})\tilde{\bigcup}(\mathcal{W},\mathcal{M}))$
$\Rightarrow S\delta - ßCl(\mathcal{D},\mathcal{M}) \tilde{\subseteq} S\delta - ßCl((\mathcal{D},\mathcal{M})\tilde{\bigcup}(\mathcal{W},\mathcal{M}))$
or $S\delta - ßCl(\mathcal{W},\mathcal{M}) \tilde{\subseteq}$
$S\delta - ßCl((\mathcal{D},\mathcal{M})\tilde{\bigcup}(\mathcal{W},\mathcal{M}))$,
so, $S\delta ßCl(\mathcal{D},\mathcal{M})\tilde{\bigcup}(\mathcal{W},\mathcal{M}) \tilde{\supseteq} S\delta ßCl(\mathcal{D},\mathcal{M})\tilde{\bigcup}$
$S\delta - ßCl(\mathcal{W},\mathcal{M})$

(b) The proof of this part is similar to part (a) thus omitted.

Theorem 3.10: *Let* $(\mathcal{X},\mathcal{T},E)$ *be a* $(S.\mathcal{T}.S)$*. So, the following statements hold:*

(a) $S\delta - ß\, Int(\mathcal{D},\mathcal{M})\tilde{\bigcup}(\mathcal{W},\mathcal{M}) \tilde{\supseteq} S\delta - ßInt(\mathcal{D},\mathcal{M})\tilde{\bigcup} S\delta - ßInt(\mathcal{W},\mathcal{M})$.

(b) $S\delta - ßInt(\mathcal{D},\mathcal{M})\tilde{\bigcap}(\mathcal{W},\mathcal{M}) \tilde{\subseteq} S\delta - ßInt(\mathcal{D},\mathcal{M})\tilde{\bigcap} S\delta - ßInt(\mathcal{W},\mathcal{M})$.

Theorem 3.11: *Let*$(\mathcal{F},\mathcal{A})$ *be S-δ-ß-O set in* $(S.\mathcal{T}.S)(\mathcal{X},\mathcal{T},E)$*, so the following statements hold:*

(a) If $\delta\mathrm{Int}(\mathcal{D},\mathcal{M})=\Phi$, so $(\mathcal{D},\mathcal{M})$is S-δ-pre-open set.
(b) *If $\delta Cl(\mathcal{D},\mathcal{M})=\Phi$, so $(\mathcal{D},\mathcal{M})$ is S-δ-semi-open set.*

Proof: Apparent it is follows from Definition (3.1).

Proposition 3.12 [12]: Let$(\mathcal{D},\mathcal{M})$ be soft subset in $\mathcal{X}$, so

(a) $SSCl_\delta(\mathcal{D},\mathcal{M})=(\mathcal{D},\mathcal{M})\tilde{\cup}Int\big(Cl_\delta(\mathcal{D},\mathcal{M})\big)$ & $SSInt_\delta(\mathcal{D},\mathcal{M})=(\mathcal{D},\mathcal{M})$ $\tilde{\cap}Cl(Int_\delta(\mathcal{D},\mathcal{M}))$.
(b) $SPCl_\delta(\mathcal{D},\mathcal{M})=(\mathcal{D},\mathcal{M})\tilde{\cup}Cl(Int_\delta(\mathcal{D},\mathcal{M}))$ and $SPInt_\delta(\mathcal{D},\mathcal{M})=(\mathcal{D},\mathcal{M})$ $\tilde{\cap}Int(Cl_\delta(\mathcal{D},\mathcal{M}))$.

Theorem 3.13: *For every soft subset $(\mathcal{D},\mathcal{M})$ of $(S.\mathcal{T}.S)(\mathcal{X},\mathcal{T},E)$, $(\mathcal{D},\mathcal{M})$ is S-δ-ß-*O set $\Leftrightarrow (\mathcal{D},\mathcal{M})=SPInt_\delta(\mathcal{D},\mathcal{M})\tilde{\cup}SSInt_\delta(\mathcal{D},\mathcal{M})$.

Proof: Let $(\mathcal{D},\mathcal{M})$ be S-δ-ß-O, so, $(\mathcal{D},\mathcal{M})\tilde{\subset}Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})))$ by Proposition (3.12), we get,

$$SPInt_\delta(\mathcal{D},\mathcal{M})\tilde{\cup}SSInt_\delta(\mathcal{D},\mathcal{M})=[(\mathcal{D},\mathcal{M})\tilde{\cap}Int(Cl_\delta(\mathcal{D},\mathcal{M}))]\tilde{\cup}$$
$$[(\mathcal{D},\mathcal{M})\tilde{\cap}Cl(Int_\delta(\mathcal{D},\mathcal{M}))]=(\mathcal{D},\mathcal{M})\tilde{\cap}[Int(Cl_\delta(\mathcal{D},\mathcal{M}))\tilde{\cup}Cl(Int_\delta(\mathcal{D},\mathcal{M}))]=$$
$$(\mathcal{D},\mathcal{M})\tilde{\cap}Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})))=(\mathcal{D},\mathcal{M}).$$

Conversely, Assume, $(\mathcal{D},\mathcal{M})=SPInt_\delta(\mathcal{D},\mathcal{M})\tilde{\cup}SSInt_\delta(\mathcal{D},\mathcal{M})$, then via Proposition (3.12), obtain

$$(\mathcal{D},\mathcal{M})=SPInt_\delta(\mathcal{D},\mathcal{M})\tilde{\cup}\,SSInt_\delta(\mathcal{D},\mathcal{M})=[(\mathcal{D},\mathcal{M})\tilde{\cap}Int(Cl_\delta(\mathcal{D},\mathcal{M}))]\tilde{\cup}$$
$$[(\mathcal{D},\mathcal{M})\tilde{\cap}Cl(Int_\delta(\mathcal{D},\mathcal{M}))]))]$$
$$=(\mathcal{D},\mathcal{M})\tilde{\cap}[Int(Cl_\delta(\mathcal{D},\mathcal{M}))\tilde{\cup}Cl(Int_\delta(\mathcal{D},\mathcal{M}))]$$
$\tilde{\subseteq}\,Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})))=(\mathcal{D},\mathcal{M})$ and thus$(\mathcal{D},\mathcal{M})$ is S-δ-ß-O set.

Theorem 3.14: *Let $(\mathcal{D},\mathcal{M})$ be soft subset in $(S.\mathcal{T}.S)(\mathcal{X},\mathcal{T},E)$, so* $S\delta-ßCl(\mathcal{D},\mathcal{M})=SPCl_\delta(\mathcal{D},\mathcal{M})\tilde{\cap}SSCl_\delta(\mathcal{D},\mathcal{M})$.

Proof: Clear, $S\delta-ßCl(\mathcal{D},\mathcal{M})\tilde{\subseteq}SPCl_\delta(\mathcal{D},\mathcal{M})\tilde{\cap}\,SSCl_\delta(\mathcal{D},\mathcal{M})$. via definition, $S\delta ßCl(\mathcal{D},\mathcal{M})\tilde{\supseteq}Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})))$, as, $S\delta ßCl(\mathcal{D},\mathcal{M})$ is *S-δ-ß*-C set of $\mathcal{X}$, via Prop-(3.12),

$$SPCl_\delta(\mathcal{D},\mathcal{M})\tilde{\cap}\,SSCl_\delta(\mathcal{D},\mathcal{M})=[(\mathcal{D},\mathcal{M})\tilde{\cup}Cl(Int_\delta(\mathcal{D},\mathcal{M}))]))]\tilde{\cap}$$
$$[(\mathcal{D},\mathcal{M})\tilde{\cup}Int(Cl_\delta(\mathcal{D},\mathcal{M}))]=[(\mathcal{D},\mathcal{M})\tilde{\cup}(Cl(Int_\delta(\mathcal{D},\mathcal{M}))\tilde{\cap}Int(Cl_\delta(\mathcal{D},\mathcal{M})))]$$
$$=[(\mathcal{D},\mathcal{M})\tilde{\cup}Cl(Int(Cl_\delta(\mathcal{D},\mathcal{M})))]=(\mathcal{D},\mathcal{M})\,\tilde{\subseteq}\,S\delta-ßCl(\mathcal{D},\mathcal{M}).$$

Theorem 3.15: *Let $(\mathcal{D},\mathcal{M})$ be a soft subset in $(S.\mathcal{T}.S)(\mathcal{X},\mathcal{T},E)$, so* $S\delta-ßInt(\mathcal{D},\mathcal{M})=SPInt_\delta(\mathcal{D},\mathcal{M})\tilde{\cap}\,SSInt_\delta(\mathcal{D},\mathcal{M})$.

Proof: It is similar to evidence of Theorem (3.14) thus deleted.

Theorem 3.16: In a $(S.\mathcal{T}.S)(\mathcal{X},\mathcal{T},E)$, the following implications hold:

(a) *Each S-δ-pre-O is S-δ-ß-O set.*

(b) *Each S-δ-semi-O is S-δ-ß-O set.*

Proof: It is apparent and follows their respective definitions of thud omitted.

Remark 3.17: "The relationships between soft *δ*-*ß*-open sets and other corresponding types of generalized soft open are shown in the following diagram 2.

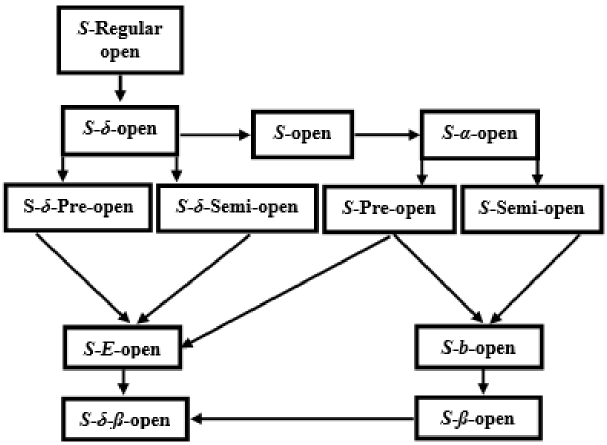

Diagram 2
The relations between S-δ-ß-O sets and other well-known kinds of generalized S-O-sets

The converses of these implications are not true in general; see [10],[12] and the following examples":

Example 3.18: Suppose $\mathcal{X} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}, E = \{y_1, y_2, y_3\}$ and $\mathcal{T} = \{\Phi, \widetilde{\mathcal{X}}, (\mathcal{F}_1, E), (\mathcal{F}_2, E), (\mathcal{F}_3, E) \dots \dots \dots (\mathcal{F}_{13}, E)\}$, are (S.S) over $\mathcal{X}$, defined as:

$(\mathcal{F}_1, E) = \{(y_1, \{\hbar_1\}), (y_2, \{\hbar_2, \hbar_3\}), (y_3, \{\hbar_1, \hbar_3\})\}$;

$(\mathcal{F}_2, E) = \{(y_1, \{\hbar_2, \hbar_4\}), (y_2, \{\hbar_1, \hbar_3, \hbar_4\}), (y_3, \{\hbar_1, \hbar_2, \hbar_4\})\}$;

$(\mathcal{F}_3, E) = \{(y_1, \Phi), (y_2, \{\hbar_3\}), (y_3, \{\hbar_1\})\}$;

$(\mathcal{F}_4, E) = \{(y_1, \{\hbar_1, \hbar_2, \hbar_4\}), (y_2, \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}), (y_3, \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\})\}$;

$(\mathcal{F}_5, E) = \{(y_1, \{\hbar_1, \hbar_3\}), (y_2, \{\hbar_2, \hbar_4\}), (y_3, \{\hbar_2\})\}$;

$(\mathcal{F}_6, E) = \{(y_1, \Phi), (y_2, \{\hbar_2\}), (y_3, \Phi)\}$;

$(\mathcal{F}_7, E) = \{(y_1, \{\hbar_1, \hbar_3\}), (y_2, \{\hbar_2, \hbar_3, \hbar_4\}), (y_3, \{\hbar_1, \hbar_2, \hbar_3\})\}$;

$(\mathcal{F}_8, E) = \{(y_1, \{\hbar_3\}), (y_2, \{\hbar_4\}), (y_3, \{\hbar_2\})\}$;

$(\mathcal{F}_9, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_2, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3\})\}$;
$(\mathcal{F}_{10}, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1, \mathscr{h}_3\}), (\mathscr{y}_2, \{\mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_2\})\}$;
$(\mathcal{F}_{11}, E) = \{(\mathscr{y}_1, \{\mathscr{h}_3\}), (\mathscr{y}_2, \{\mathscr{h}_2, \mathscr{h}_4\}), (\mathscr{y}_3, \{\mathscr{h}_2\})\}$;
$(\mathcal{F}_{12}, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1\}), (\mathscr{y}_2, \{\mathscr{h}_2\}), (\mathscr{y}_3, \Phi)\}$;
$(\mathcal{F}_{13}, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1\}), (\mathscr{y}_2, \{\mathscr{h}_2, \mathscr{h}_3\}), (\mathscr{y}_3, \{\mathscr{h}_1\})\}$;

In that case, $\mathcal{T}$is defines a soft topology on $\mathcal{X}$,hence $(\mathcal{X}, \mathcal{T}, E)$ is $(\mathrm{S}.\mathcal{T}.\mathrm{S})$ over $\mathcal{X}$. Evidently, $\mathcal{T}^c = \{\Phi, \widetilde{\mathcal{X}}, (\mathcal{F}_1, E)^c, (\mathcal{F}_2, E)^c, (\mathcal{F}_3, E)^c \dots\dots\dots (\mathcal{F}_{13}, E)^c\}$ are *S*-C sets. Now, take $(\mathcal{W}, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1, \mathscr{h}_2\}), (\mathscr{y}_2, \{\mathscr{h}_2, \mathscr{h}_3\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_3\})\}$; consequently $(\mathcal{W}, E) \widetilde{\subseteq} (\mathcal{F}_8, E)^c = Cl(Int(Cl_\delta(\mathcal{W}, E)))$.

Therefore,$(\mathcal{W}, E) \widetilde{\subseteq} Cl(Int(Cl_\delta(\mathcal{W}, E)))$.As a result, $(\mathcal{W}, E)$is *S*-*δ*-*ß*-open. As, $Int(Cl_\delta(\mathcal{W}, E)) = \{(\mathscr{y}_1, \{\mathscr{h}_1\}), (\mathscr{y}_2, \{\mathscr{h}_2, \mathscr{h}_3\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_3\})\}$ which doesn't contain$(\mathcal{W}, E)$, consequently $(\mathcal{W}, E)$ isn't *S*-*δ*-pre-O set, also obvious $(\mathcal{W}, E)$ is neither *S*-b-O nor *S*-*ß*-O nor *S*-semi-O nor *S*-O nor *S*-*δ*-O and nor *S*-r-O. Take $(\mathcal{D}, E) = \{(\mathscr{y}_1, \{\mathscr{h}_4\}), (\mathscr{y}_2, \{\mathscr{h}_1, \mathscr{h}_3\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_2\})\}$, then $Cl(Int(Cl_\delta (\mathcal{D}, E))) = \{(\mathscr{y}_1, \{\mathscr{h}_2, \mathscr{h}_4\}), (\mathscr{y}_2, \{\mathscr{h}_1, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\})\}$, therefore. Since $(\mathcal{D}, E) \widetilde{\subseteq} Cl(Int(Cl_\delta(\mathcal{D}, E)))$. Hence,$(\mathcal{D}, E)$ is *S*-*δ*-*ß*-O set.

$Cl(Int_\delta(\mathcal{D}, E)) = (\mathcal{F}_5, E)^c = \{(\mathscr{y}_1, \{\mathscr{h}_2, \mathscr{h}_4\}), (\mathscr{y}_2, \{\mathscr{h}_1, \mathscr{h}_3\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_3, \mathscr{h}_4\})\}$, which doesn't contain$(\mathcal{D}, E)$, consequently $(\mathcal{D}, E)$ isn't *S*-*δ*-semi, as well clear $(\mathcal{D}, E)$ is neither *S*-*b* nor *S*-*ß* nor *S*-semi nor *S*-open nor *S*-*δ* and nor *S*-r-open.

Example 3.19: Let $\mathcal{X} = \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}, E = \{\mathscr{y}_1, \mathscr{y}_2,\}$ and $\mathcal{T} = \{\Phi, \widetilde{\mathcal{X}}, (\mathcal{F}_1, E), (\mathcal{F}_2, E), (\mathcal{F}_3, E), (\mathcal{F}_4, E), (\mathcal{F}_5, E), (\mathcal{F}_6, E)\}$, are soft sets over $\mathcal{X}$, defined as: $(\mathcal{F}_1, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1\}), (\mathscr{y}_2, \Phi)\}$; $(\mathcal{F}_2, E) = \{(\mathscr{y}_1, \{\mathscr{h}_3\}), (\mathscr{y}_2, \Phi)\}$;
$(\mathcal{F}_3, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1, \mathscr{h}_2\}), (\mathscr{y}_2, \Phi)\}$; $(\mathcal{F}_4, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1, \mathscr{h}_3\}), (\mathscr{y}_2, \Phi)\}$;
$(\mathcal{F}_5, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3\}), (\mathscr{y}_2, \Phi)\}$; $(\mathcal{F}_6, E) = \{(\mathscr{y}_1, \{\mathscr{h}_1, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_2, \Phi)\}$.

In that case the soft set $(\mathcal{W}, E) = \{(\mathscr{y}_1, \{\mathscr{h}_2, \mathscr{h}_4\}), (\mathscr{y}_2, \Phi)\}$ is *S*-*δ*-*ß*-open, but it's neither *S*-*E*-O nor *S*-*ß*-O nor *S*-*b*-O nor *S*-pre-O nor *S*-semi-O set.

Remark 3.20: In a $(\mathrm{S}.\mathcal{T}.\mathrm{S})(\mathcal{X}, \mathcal{T}, E)$, the union of two *S*-*δ*-*ß*-closed sets needs not to be *S*-*δ*-*ß*-closed set as shown in the example below.

Example 3.21: Suppose that $(\mathcal{X}, \mathcal{T}, E)$ is a $(\mathrm{S}.\mathcal{T}.\mathrm{S})$ described in Example(3-18).

Let us take the next two (S.S)$(\mathcal{M}, E)$&$(\mathcal{B}, E)$ in $(\mathcal{X}, \mathcal{T}, E)$ which are described as:

$(\mathcal{M}, E)$=$\{(\mathscr{y}_1, \{\mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_2, \{\mathscr{h}_1, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\})\}$;

$(\mathcal{B}, E)$=$\{(\mathscr{y}_1, \{\mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_2, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_4\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_4\})\}$.

So, $(\mathcal{M}, E)$ &$(\mathcal{B}, E)$ are *S*-*δ*-*ß*-C sets over $\mathcal{X}$, consequently

$(\mathcal{M}, E)\widetilde{\cup}(\mathcal{B}, E) =$
$\{(\mathscr{y}_1, \{\mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_2, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\}), (\mathscr{y}_3, \{\mathscr{h}_1, \mathscr{h}_2, \mathscr{h}_3, \mathscr{h}_4\})\}$ =$(\mathcal{C}, E)$
and since $Cl\left(Int\left(Cl_\delta(\mathcal{C}, E)\right)\right) = \mathcal{X} \widetilde{\not\subseteq} (\mathcal{C}, E)$. Thus,$(\mathcal{C}, E)$ isn't *S*-*δ*-*ß*-C set.

Remark 3.22: In a $(\mathrm{S}.\mathcal{T}.\mathrm{S})(\mathcal{X},\mathcal{T},E)$, the intersection of two *S-δ-ß*-O sets needs not to be *S-δ-ß*-O set as explain in the next example.

Example 3.23: Suppose $(\mathcal{X},\mathcal{T},E)$ is $(\mathrm{S}.\mathcal{T}.\mathrm{S})$ defined in Example (3-18). Take, (S.S) $(\mathcal{W},E)\&(\mathcal{K},E)$ in $(\mathcal{X},\mathcal{T},E)$ defined: $(\mathcal{W},E)=\{(y_1,\{\hbar_1\}),(y_2,\{\hbar_2\})\}$; $(\mathcal{K},E)=\{(y_1,\{\hbar_1\}),(y_2,\{\hbar_3\}),(y_3,\{\hbar_3\})\}$. So,$(\mathcal{W},E)$ & $(\mathcal{K},E)$ are *S-δ-ß*-O sets over $\mathcal{X}$, but $(\mathcal{W},E)\tilde{\sqcap}(\mathcal{K},E)=\{(y_1,\{\hbar_1\})\}(\mathcal{H},E)$ is not *S-δ-ß*-O set.

4. Conclusion

The main goal of this paper is to introduce and study new concepts of generalized soft open and soft closed sets, which are called soft δ-ß-open and soft δ-ß-closed sets over the soft topological spaces, and several of their characterizations have been investigated. Also, the relations between soft δ-ß-open sets and other corresponding types of well-known extended soft open sets have been discussed. “We hope that the findings in this article are just the beginning of a new structure and will not only form the theoretical basis for further applications of topology on soft sets but also lead to the development of information systems and various fields.

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