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Negative theorem for r-monotone approximation

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Abstract

Recently, many theorems were introduced for monotone, convex, 3-monotone approximation, r-monotone for functions in L_p Quasi-normed space. Now, there is a question: can we strengthen these results in terms of moduli of smoothness of higher order? The answer is no, which we see in our work here. Here, we introduce the negative theorem for the r-monotone approximation of function in L_p Quasi-normed space for $0 < p < 1$.

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1. Introduction

between the function's structural properties and its degree of approximation. One of the main tasks is to study the relationship between the rate of decrease of the degree of approximation to zero and the function's smoothness. The evolution of approximation theory comes from a long history of approaches for the direct, inverse (or negative) theorems. The initial and most conventional approaches were concepts in 1885 Weierstrass, in1898 Lebesgue,

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in 1911-1919 Vallee-Poussin, in 1911-1930 Jackson, in 1937-1952 Bernstein, in 1945- 1969- 1974 Nikols'kij, in 1947 Chebyshev, and in 1985 Kolmogorov and many of their followers[6].

Better development of the direct theorem, the inverse (or negative) theorem, and the positive theorem via r-monotone approximation has been proposed in 2018 by Bhaya & introduced a new modulus of smoothness for Uniform approximation[1], in 2018-2019, Kopotun Leviatan Shevchuk studied Interpolatory pointwise estimates for monotone polynomial approximation and Uniform and pointwise shape preserving approximation (SPA) by algebraic polynomials[4], [5], and in 2022, AL-Saidy, Maktoof & Mazed introduced On best multiplier approximation of k-monotone of ELP$[-\pi,\pi]$[7].

In the sequel, in 2020Almurieb introduced a second-order modulus of smoothness in the degree obtained by monotone approximation[2]. She demonstrated that an approximation of f can be derived through the use of negative theorems, such as the fact that an approximation neural network is not monotone for the partition Y_s, (see [Almurieb 2021], Theorem 3.3.4).

There is a natural question about applying a smooth function with higher orders. In particular, we give a negative answer to the mentioned question and prove a negative theorem for the best r-monotone approximation of function in $L_{p[0,1]}$ Space, as in the theorem 4.1, will be proven at the end of the paper. In this work, we shall estimate the degree of best approximation in terms of the modulus of smoothness. For this aspect, we have the direct theorem (estimate) or Jackson theorem, first introduced and proved by D. Jackson. Then Bernstein introduced a converse theorem for Jackson theorem called the converse theorem, inverse theorem, or Bernstein theorem see example[1], [2], [3][6]. The counterexample of the Jackson theorem or inverse theorem is called the negative theorem [8]. In many applications, it is desired to get the best approximation that has the same geometrical properties of the target function; that's what we call shape-preserving approximation or constrained approximation, or we sometimes call it r-monotone approximation see for example, [5]. For the recent application, we need the constrained approximation. But there is a problem that these constraints restrict very much the degree of the best approximation to $r+1$ modulus of smoothness degree [9,10]. That's what we shall prove here. We shall prove that this constraint restricts the degree of the smoothness.

2. Some Basic Concepts and Notations

We mean by $L^M_{p[0,1]}$, $0<p<1$ the class of all functions with M derivatives in space $L_{p[0,1]}$ and π_n the class of algebraic polynomials of degree $\leq n$.

$$L_{p[0,1]} = \left\{ f\colon [0,1] \to R, with \left(\int_0^1 |f(x)|^p\, dx\right)^{\frac{1}{p}} < \infty \right\}.$$

$$\|f(x)\|_p = (\int_0^1 |f(x)|^p\, dx)^{\frac{1}{p}}, \quad 0<p<1.$$

The r^{th} divided difference [1] of functions defined on the interval [0,1] is

$$f(x_0, x_1, x_2, \cdots\cdots, x_r) = \sum_{i=0}^{r} \frac{f(x_i)}{\omega'(x_i)},$$

$$\omega(x) = \prod_{i=0}^{r}(x - x_i).$$

The r-monotone function [1] defined on the interval [0, 1] are those functions which the r^{th}divided difference of them is non-negative for any r + 1 points $x_0, x_1, x_2, \cdots\cdots, x_r$ in the interval [0, 1].

Denoted by the space of all r-monotone functions defined on [0,1],

$$\Delta^r = \{f : [0,1] \to IR : \Delta_h^r f(x) \geq 0, \qquad 0 \leq x + rh \leq 1\}. [7]$$

The degree of best r-monotone approximation is

$$E_n^{(r)}(f)_p = inf\left\{\left\| f - p_n \right\|_p : p_n \in \pi_n \cap \Delta^r\right\},$$

for $f \in L_{p[0,1]}$.

Let us now define the r^{th} order modulus of smoothness for functions $f \in L_{p[0,1]}$, as

$$\omega_r(f, \delta)_p = \sup_{0<|h|\leq\delta} \left\| \Delta_h^r f(x) \right\|_{L_{p[0,1]}}, \quad \text{for } r \in N,\ \delta > 0.$$

Where

$$\Delta_h^r(f) = \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} f(x + ih).$$

Is the r^{th}symmetric difference.

3. The Auxiliary Results

We shall introduce some lemmas we need in our proof of the main theorem.

Lemma 3.1: [8] *Let* $b > 0$, $\alpha(x) = \frac{b^2}{(x^2 - b^2)}$, $g_m(x, b) = x^m e^{\alpha(x)+1}$, *and* $x \in (-b, b)$, *then*

$$\left| g_m^{(m)}(x, b) - m! \right| \leq c(m) b^{-2} x^2, \qquad |x| < b,$$

Where c(x) always indicates a positive constant depending upon x only.

Lemma 3.2:[8] *Let* $\delta_n \geq 0, \delta_n \to 0, n \to \infty,\ \sigma > 0$ *and*

$$\bar{g}_r(x, \delta_n, \sigma) = \delta_n^{2+2\sigma} g_r(x, \delta_n) + x^{r+2} - \delta_n^{2+2\sigma} x^r, \qquad x \in (-\delta_n, \delta_n);$$

then for sufficiently large n *and* $x \in (-\delta_n,\ \delta_n)$, *we have*

$$\bar{g}_r^{(r)}(x, \delta_n, \sigma) \geq 0.$$

In what follows, we set

$$F_l(x) = \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x),$$

$$Q_l(x) = q_l(x) + n_l^{-\frac{1}{4}}\left(x^{r+2} - n_l^{-\frac{5}{2}} x^r\right),$$

$$f_n(x) = \begin{cases} x^{r+2} - n^{-\frac{5}{2}}x^r, & |x| \geq n^{-\frac{9}{8}}, \\ \overline{g_r}\left(x, n^{-\frac{9}{8}}, \frac{1}{9}\right), & |x| < n^{-\frac{9}{8}}, \end{cases}$$

where $q_l(x)$ is the algebraic polynomial of the best approximation of the degree n_l to $F_{l-1}(x)$, and $\{n_l\}$ is a subsequence in N. Choose inductively:

Set n_l, to be part natural number N,

$$n_l \geq \left\| F_{l-1}^{(2r+8)} \right\|_p.$$

Lemma 3.3: *For F_l and $Q_l \in L_{p[0,1]}$, we have*

$$\left\| F_l - Q_l \right\|_p \sim n_l^{-\frac{1}{4}} \left\| f_{n_l}(x) - x^{r+2} + n_l^{-\frac{5}{2}} x^r \right\|_p \sim n_l^{-\frac{9r}{8} - \frac{11}{4}},$$

$$Q_l^{(r)}(0) \leq -c(p) n_l^{-\frac{11}{4}}.$$

Proof: Let $\left\| F_l - q_l \right\|_p = \left\| \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x) - q_l(x) \right\|_p$.

Using the Definition of $F_{l-1}(x)$ and q_l we get

$$\left\| F_l - q_l \right\|_p \leq c(p) \left\| F_{l-1}^{(2r+8)} \right\|_p n_l^{\frac{1}{2r+8}}. \tag{1}$$

We have

$$\begin{aligned} &F_l(x) - Q_l(x) \\ &= F_{l-1}(x) - q_l(x) + n_l^{-\frac{1}{4}}\left(x^{r+2} - n_l^{-\frac{5}{2}} x^r\right), \\ &= F_{l-1}(x) - q_l(x) + n_l^{-\frac{1}{4}}\left(f_{n_l}(x) - x^{r+2} - n_l^{-\frac{5}{2}} x^r\right), \\ &\leq c(p) \left\| F_{l-1}(x) - q_l(x) \right\|_p + n_l^{-\frac{1}{4}} \left\| f_{n_l}(x) - x^{r+2} - n_l^{-\frac{5}{2}} x^r \right\|_p. \end{aligned}$$

Then

$$\left\| f_{n_l}(x) - x^{r+2} - n_l^{-\frac{5}{2}} x^r \right\|_p = \left(\int_{-n_l^{-\frac{9}{8}}}^{n_l^{-\frac{9}{8}}} \left| e\, n_l^{-\frac{5}{2}} x^r e^{\left(\frac{n_l^{-\frac{9}{4}}}{x^2 - n_l^{-\frac{9}{4}}}\right)} \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq c(p) n_l^{\frac{1}{\frac{9}{8}r - \frac{5}{2}}} \tag{2}$$

$$\left\| F_l - Q_l \right\|_p \leq c(p) \left(\left\| F_{l-1}(x) - q_l(x) \right\|_p + n_l^{-\frac{1}{4}} \left\| f_{n_l}(x) - x^{r+2} - n_l^{-\frac{5}{2}} x^r \right\|_p \right).$$

From (1) and (2), we get

$$\| F_l - Q_l \|_p \leq c(p)\left(\| F_{l-1}^{(2r+8)} \|_p n_l^{\frac{1}{2r+8}} + n_l^{-\frac{1}{2}} n_l^{\frac{1}{\frac{9}{8}r-\frac{5}{2}}} \right).$$

From our hypothesis

$$\| F_l - Q_l \|_p \leq c(p)\left(n_l n_l^{\frac{1}{2r+8}} + n_l^{-\frac{1}{2}} n_l^{\frac{1}{\frac{9}{8}r-\frac{5}{2}}} \right)$$

$$\leq c(p) n^{-\frac{9r}{8}-\frac{1}{4}}. \qquad (3)$$

We have

$$\| q_l \|_p \leq c(p) \left| F_l \right| n_l^{-r-8} .$$

$$\| F_l - Q_l \|_p = \| \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x) - q_l(x) + n_l^{-\frac{1}{4}}\left(x^{r+2} - n_l^{-\frac{5}{2}} x^r \right) \|_p$$

$$\leq c(p) \sum_{i=1}^{l} \| f_{n_l}(x) - x^{r+2} + n_l^{-\frac{5}{2}} x^r \|_p n_l^{-\frac{1}{4}}$$

$$\leq c(p) n_l^{-\frac{1}{4}} \| f_{n_l}(x) - x^{r+2} + n_l^{-\frac{5}{2}} x^r \|_p .$$

Lemma 3.4: *For any* $p(x) \in \pi_{n_l} \cap \Delta^r$ *and large enough* l, *we have*

$$\| F_l - p(x) \|_p \geq c(p) n_l^{-r-\frac{11}{4}}.$$

Proof: Let $\| Q_l^{(r)} - p^{(r)}(x) \|_p \leq c(p)\, n_l^r \| Q_l - p(x) \|_p$

$$= c(p)\, n_l^r \| Q_l - F_l + F_l - p(x) \|_p$$

$$\leq c(p)\, n_l^r \left(\| Q_l - F_l \|_p + \| F_l - p(x) \|_p \right).$$

Thus,

$$\| Q_l^{(r)} - p^{(r)}(x) \|_p \leq c(p)\, n_l^r \left(\| Q_l - F_l \|_p + \| F_l - p(x) \|_p \right).$$

From (3), follows

$$\| F_l - Q_l \|_p \leq c(p) n^{-\frac{9}{8}r-\frac{1}{4}}$$

$$\| Q_l^{(r)} - p^{(r)}(x) \|_p \leq c(p)\, n_l^r \left(n^{-\frac{9}{8}r-\frac{1}{4}} + \| F_l - p(x) \|_p \right)$$

$$\| F_l - p(x) \|_p \geq n_l^{-r} \| Q_l^{(r)} - p^{(r)}(x) \|_p - n^{-r-\frac{9}{8}r-\frac{1}{4}} .$$

4. The Main Result

Here, we shall answer the question that we asked in section one. That the direct r-monotone one is strict. As we see in the following theorem:

Theorem 4.1: *Let* $r \geq 1$. *Then there exists a function* $f \in L^r_{p[0,1]} \cap \Delta^r$ *such that*

$$E^{(r)}_{n_l}(f)_p > \omega_{r+3}\left(f, \frac{1}{n}\right)_p.$$

Proof: From Lemma3.2, we see that there exists $M > 0$ such that $n \geq M$,

$$(\Delta^r_h f_n) \geq 0.$$

Now select $\{n_l\}$by induction. Set $n_l = M$,

$$n_{l+1} = 2\left(n_l^{4(r+3)} + [\|F_l^{(2r+8)}\|] + [\|F_l^{(r+3)}\|]^5 + 1\right) \text{for } l = 1,2,3, \dots .$$

Define

$$f(x) = \sum_{i=1}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x).$$

It is clear $f \in L^r_{p[0,1]} \cap \Delta^r$.For any $p(x) \in \pi_{n_l} \cap \Delta^r$,

$$\|f(x) - p(x)\|_p = \|f(x) - F_l + F_l - p(x)\|_p$$

$$\geq -\|p(x) - F_l\|_p + \|F_l - f\|_p$$

$$= \left| \|F_l - f\|_p - \|p(x) - F_l\|_p \right|$$

$$= \left| \left\| \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x) - \sum_{i=1}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x) \right\|_p - \|p(x) - F_l\|_p \right|$$

$$= \left| \left\| \sum_{i=1}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x) \right\|_p - \|p(x) - F_l\|_p \right|.$$

$$> \|F_l - p(x)\|_p - \left\| \sum_{i=l+1}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x) \right\|_p.$$

Then, using Lemma3. 4, we get

$$\|f(x) - p(x)\|_p > c(p) n_l^{-r-\frac{11}{4}} - n_{n+1}^{-\frac{1}{4}}$$

$$> c(p) n_l^{-r-\frac{11}{4}} - n_n^{-r-3}.$$

Since p is the r-monotone best approximation to f. So

$$E^{(r)}_{n_j}(f)_p > c(p) n_l^{-r-\frac{11}{4}} > c(p) n_l^{-r}.$$

On the other hand, by Lemma3.3, we got

$$E^{(r)}_{n_l}(f)_p \geq c(p)\omega_{r+3}\left(f, \frac{1}{n}\right)_p.$$

5. Conclusion

If f is r-monotone function in $L_{p[0,1]}$ spaces where $0 < p < 1$. Then, the direct theorem of r-monotone approximation is strict regarding the modulus of smoothness of function f of order $r + 1$. In particular, for the best r-monotone approximation of function in $L_{p[0,1]}$ Spaces, we obtained a negative result.

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