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Laplace transform-adomian decomposition approach for solving random partial differential equations

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Abstract

Market share is a major indication of business success. Understanding the impact of numerous economic factors on market share is critical to a company's success. In this study, we examine the market shares of two manufacturers in a duopoly economy and present an optimal pricing approach for increasing a company's market share. We create two numerical models based on ordinary differential equations to investigate market success. The first model takes into account quantity demand and investment in R&D, whereas the second model investigates a more realistic relationship between quantity demand and pricing.

Subject Classification: *44A10.*

Keywords: *Gamma distribution, Normal distribution, Laplace transform, Random partial differential equations.*

1. Introduction

The majority of recent research focuses on the study of random partial differential equations. Numerous applications in the sciences, engineering, biology, and physics require an understanding of two-dimensional random differential equations. Nonlinear random differential equations with two variables are used to develop models that extend a wide range of scientific topics. Numerical techniques can be applied to analyze a number of these equations. It is required to obtain numerical approximations of the exact solutions to certain issues because there are no recognized analytical solutions.

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In order to solve problems based on the Euler's difference scheme, Cortés J. C. et al. presented a numerical solution approach. Later, they proposed an enhanced Euler's method to solve equations of this kind. Several authors solved some kinds of second order RODEs using the variational iteration method and the Adomian decomposition method. The numerical solution of ODEs with random coefficients that are Gaussian is provided by others. In this work, the Laplace Adomian Decomposition Method (LADM) is used to compute approximate solutions of nonlinear partial differential equations. [1-12].

2. Analysis of the Laplace Transform Adomian Decomposition Technique

Definition 1: *Let* $\mathcal{T}$ be defined as the Laplace transform. $\mathcal{H}(s) = \mathcal{L}\{\mathcal{T}(\mathrm{t})\} = \int_0^\infty \mathcal{T}(\mathrm{t})e^{-st}$. The Laplace of the t^nfunction given by: $\mathcal{L}\{\mathrm{t}^n\} = \frac{n!}{s^{n+1}}$

Definition 2: If $\mathcal{H}(s) =: \{\mathcal{T}(\mathrm{t})\}$ then $\mathcal{T}(t)$ is called inverse Laplace of $\mathcal{H}(s)$ and $\mathcal{T}(\mathrm{t}) = \mathcal{L}^{-1}\{\mathcal{H}(s)\}$. Laplace transform has derivative properties as $\mathcal{L}\{\mathcal{T}^{(n)}(\mathrm{t})\} = s^n\mathcal{L}\{\mathcal{T}(\mathrm{t})\} - \sum_{k=0}^{n-1} s^{n-1} - k\ \mathcal{T}^{(k)}(0)$

And $\mathcal{L}\{\mathrm{t}^n\mathcal{T}(\mathrm{t})\} = (-1)^n\mathcal{T}^{(n)}\,(s)$

Consider the general nonlinear partial differential equation as the formula:

$$L_t\Upsilon(\psi,t) + R\Upsilon(\psi,t) + N\Upsilon(\psi,t) = g(\psi,t) \qquad (1.1)$$

Getting Laplace transform to above equation

$$\mathcal{L}\,\{L_t\Upsilon(\psi,t)\} = \mathcal{L}\{g(\psi,t)\} - \mathcal{L}\{R\Upsilon(\psi,t)\} - \mathcal{L}\{N\Upsilon(\psi,t)\}$$

$$s\Upsilon(\psi,s) - \Upsilon(\psi,0) = \mathcal{L}\{g(\psi,t) - R\Upsilon(\psi,t) - N\Upsilon(\psi,t)\}$$

$$\Upsilon(\psi,s) = \frac{\mathcal{H}(\psi)}{s} - \frac{1}{s}\mathcal{L}\,\{-g(\psi,t) + R\Upsilon\,(\psi,t) + N\Upsilon(\psi,t)\} \qquad (1.2)$$

And taking inverse Laplace transform

$$\Upsilon(\psi,t) = \mathcal{H}(\psi) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{-g(\psi,t) + R\Upsilon(\psi,t) + N\Upsilon(\psi,t)\}\right] \qquad (1.3)$$

Furthermore, the nonlinear term can be decomposed as follows:

$$\Upsilon(\psi,t) = \sum_{n=0}^{\infty} \Upsilon_n(\psi,t) \qquad (1.4)$$

$$N\Upsilon(\psi,t) = \sum_{n=0}^{\infty} A_n(\Upsilon_0,\Upsilon_1,\dots,\Upsilon_n) \qquad (1.5)$$

Where, A_n are Adomian polynomials of $\Upsilon_0,\Upsilon_1,\Upsilon_2 \dots \Upsilon_n$ and it can be calculated by formula

$$A_0 = f(\Upsilon_0),\ A_1 = \Upsilon_1\frac{df(\Upsilon_0)}{d\,\Upsilon_0}$$

$$A_2 = \Upsilon_2\frac{df(\Upsilon_0)}{d\Upsilon_0} + \frac{y_1^2}{2!}\frac{d^2f(\Upsilon_0)}{d\Upsilon_0^2}$$

More general, we have the solution as: $\Upsilon_0(\psi,t) = G(\psi,t)$

$$\Upsilon_{j+1}(\psi,t) = -\mathcal{L}^{-1}\left[s^{-n}\left[R\Upsilon_j(\psi,t)\right] + s^{-n}\left[A_j\right]\right],\ j \geq 0$$

3. Numerical Results

The Laplace-Adomian Method is used in this work to get the analytical solution of random partial differential equations, with the goal of evaluating the solution behavior graphically.

Example 2.1: Consider the Nonlinear Random Differential Equation as follows:

$$\beta \mathcal{X}_t = \beta\, \mathcal{X}_{xx} + 6\beta\mathcal{X} - 6\mathcal{X}^2 \tag{1.6}$$

With initial condition $\mathcal{X}(x,0) = \frac{\beta}{(1+e^x)}$

Solution: *By* applying Laplace Transform Decomposition Method to both sides of Eq. (1.6), we get $\mathcal{X}_0(x,t) = \frac{\beta}{(1+e^x)^2}$ *when* $B\sim$ Normal Distribution

$$\mathcal{X}_1(x,t) = \mathcal{L}^{-1}\left[-\frac{1}{s}\mathcal{L}\left\{\mathcal{X}_{0xx} + 6\mathcal{X}_0 - \frac{6}{\beta}\mathcal{X}_0^2\right\}\right] = \frac{-36\,\beta t\,(e^{2x}+e^x)}{(1+e^x)^4}$$

$$\mathcal{X}_2(x,t) = \mathcal{L}^{-1}\left[-\frac{1}{s}\mathcal{L}\left\{\mathcal{X}_{1xx} + 6x_1 - \frac{A_1}{\beta}\right\}\right] = \frac{120\,\beta t^2\,(e^{2x}+e^x)}{(1+e^x)^4}$$

$$\mathcal{X}_3(x,t) = \mathcal{L}^{-1}\left[-\frac{1}{s}\mathcal{L}\left\{\mathcal{X}_{2xx} + 6\mathcal{X}_2 - \frac{A_2}{\beta}\right\}\right] = \frac{-12\,\beta t^3\,(4e^{3x}-7e^{2x}+e^x)}{3(1+e^x)^5}$$

By the same way, we can get $\mathcal{X}_4(x,t), \mathcal{X}_5(x,t), \ldots$ Then the approximate solution given by

$$\mathcal{X} = \frac{\beta}{(1+e^x)^2} - \frac{10\beta t(e^x+e^{2x})}{(1+e^x)^2} + \frac{25\,\beta t^2(2ex^{2x}-e^x)}{(1+e^x)^4} - \frac{125\beta t^3(4e^{3x}-7e^{2x}+e^x)}{3(1+e^x)^5} + \ldots$$

The following Figure 1 represented the error for the approximate solutions of Eq. (1.1) using LTADM when $B\sim$ Normal Distribution

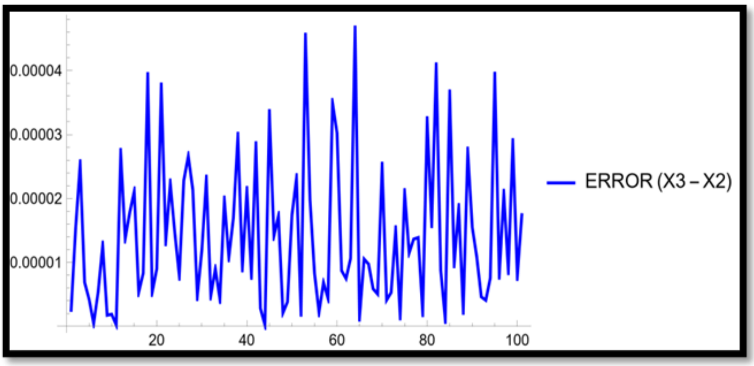

Figure 1
The error for the approximate solutions of Eq. (1.1) using LTADM

Example 2.2: Here, we solve the previous example when B~ Gamma Distribution and comparison the results. By applying Laplace Transform Decomposition Method to both sides of Eq. (1.6), we get the results of approximate solutions and absolute error. The following Figure 2 shows the error using LTADM.

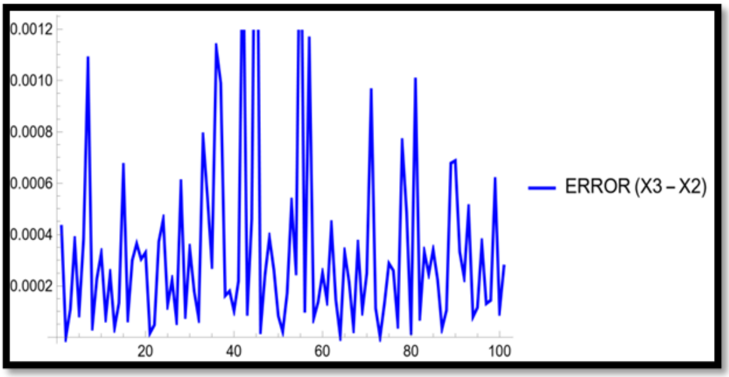

Figure 2
The error using LTADM

Example 2.3: Consider the Nonlinear Random Differential Equation as follows:

$$\chi_t + \chi_{xx} - \chi^2 - \chi\chi_{xx} = 0 \qquad (1.7)$$

With initial conditions $\chi(x,0) = \mathrm{B}\,\mathrm{Sin}\,x$ Where $\mathrm{B} \sim \mathrm{N}\,(0\,,1)$ is parameter with Normal Distribution.

Solution: By applying Laplace Transform Decomposition Method to both sides of Eq. (1.7), we get $\chi_0(x,t) = \mathrm{B}\,\mathrm{Sin}\,x$

And $\chi_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{R\chi_0(x,t) + A_0\}\right] = \mathrm{B}\,t\,\mathrm{Sin}\,x$

$$\chi_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\{R\chi_1(x,t) + A_1\}\right] = \frac{Bt^2}{6}\mathrm{Sin}\,x\;(3 - 2B(t-3).\,\mathrm{Sin}\,x)$$

$$\chi\_3\,(x,t) = (2B\text{^}4\,t\text{^}5)/315\,(\mathrm{Sin}[x])\text{^}4\,(5t\text{^}2 - 35t + 63) + (Bt\text{^}3)/1260(210\mathrm{Sin}[x] + 21B\text{^}2\,(\mathrm{Sin}[x])\text{^}3\,(5t\text{^}3 - 18t\text{^}2 - 10t + 40) + B(2(\mathrm{Sin}[x])\text{^}2\,(-420 + 105t - Bt\text{^}2\,\mathrm{Sin}[x](35t - 126 - 4B\mathrm{Sin}[x](5t\text{^}2 - 35t + 36))) - 21(\mathrm{Sin}[x])\text{^}2\,(3t\text{^}2 + 10t - 80).$$

Then the approximate solution given by:

$$\chi = B\,\mathrm{Sin}\,x + Bt\,\mathrm{Sin}\,x + (Bt\text{^}2)/6\;\mathrm{Sin}\,x\;(3 - 2B(t-3)\;\mathrm{Sin}\,x) + \frac{2B^4t^5}{315}(\mathrm{Sin}[x])^4(5t^2 - 35t + 63) + \frac{Bt^3}{1260}(210\mathrm{Sin}[x] + 21B^2(\mathrm{Sin}[x])^3(5t^3 - 18t^2 - 10t + 40) + B(2(\mathrm{Sin}[x])^2(-420 + 105t - Bt^2\mathrm{Sin}[x](35t - 126 - 4B\mathrm{Sin}[x](5t^2 - 35t + 36))) - 21(\mathrm{Sin}[x])^2(3t^2 + 10t - 80) + \ldots$$

The following figure represented the error for the approximate solutions of Eq. (1.7) using LTADM when $\mathrm{B} \sim$ Normal Distribution as shown in Figure 3.

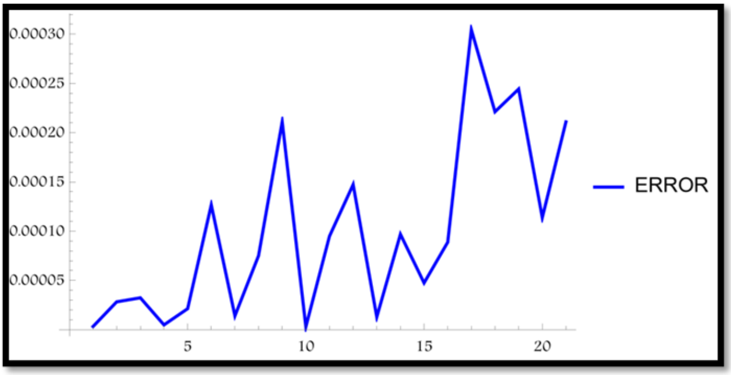

Figure 3
The approximate solutions of Eq. (1.7) using LTADM

Now, by applying ETADM for the previous example when B is parameter with Gamma Distribution. Taking Laplace Transform Decomposition Method to both sides of Eq. (1.7), we get the results of the error for solutions. In summary, the outcomes generated by the normal distribution are superior to those generated by the gamma distribution and thus of high quality as shown in Figure 4.

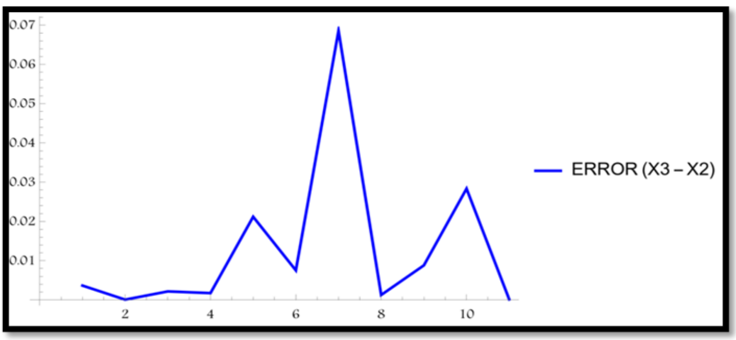

Figure 4
The results of the error for solutions

4. Conclusion

A combination of techniques LTADM is applied to solve various essential PDEs that are randomized with Normal and Gamma distributions. We are able to determine that the LTADM is a powerful and effective approach for finding approximate analytical solutions to nonlinear random partial differential equations.

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