

Inverse approximation theorem using Bernstein's Durrmeyer operator in terms of 2nd modulus of smoothness

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Abstract

Many inverse theorems have been proven using the ordinary modulus of smoothness but no one proved the inverse theorems for approximation by Bernstein's Durrmeyer operator in terms of 2nd modulus of smoothness.

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1. Introduction

Zhang, Z. studied Bernstein-Durrmeyer operator with usual weights in [4], Bernstein-Durrmeyer with Jacobi weights have been studied in [3], in [9] better approximation of functions by Bernstein-Durrmeyer type operators is introduced, Bernstein-Durrmeyer operator was also modified in [10]. In this paper we used Bernstein-Durrmeyer operator to approximate functions using 2nd modulus of smoothness.

We need to define the symmetric difference of function $g \in L_p(I)$, $0 < p < 1$ as follows [5, 11]:

$$\Delta_h^r(g, y, I) = \begin{cases} \sum_{\tau=0}^r \binom{r}{\tau} (-1)^{r-\tau} g\left(y + \left(\tau - \frac{rh}{2}\right)\right), & y \pm \frac{rh}{2} \in I \\ 0 & \text{o.w} \end{cases} \quad (1)$$

we can find all types of modulus of smoothness with properties it in [2, 11, 12]. The definition of the 2nd modulus of smoothness is

$$\omega_2(g^m, t)_{p, \omega} = \sup_{0 \leq h < t} \|g(y+h) - 2g(y) + g(y-h)\|_{p, \omega}, \quad t \geq 0. \quad (2)$$

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when $g \in Lp(I), I = [0,1], 0 < p < 1$, equipped with norm g and 2nd weighed function w defined as follow

$$\|g\|_{p,\omega} \leq \left(\int_I |g(y)w(y)|^p dy \right)^{\frac{1}{p}} = \|wg\|_{p,\omega},$$

where $w(y)$ bounded mapping then there exist $c(p) \geq 1$ s.t $\|w\|_{p,w} \leq c(p)$.

Now define Bernstein–Durmeyer operator in Lp , $0 < p < 1$, with 2nd modulus of smothness

$$\begin{aligned} D_Y(g_i) = & \int_I \sum_{j=0}^{\frac{r}{2}-1} g_i \left(y + \left(\frac{kh}{n} - j \right) \right) - 2g_i(y) + g_i \left(y - \left(\frac{kh}{n} - j \right) \right) \\ & + \int_I \sum_{j=\frac{r}{2}+1}^r g_i \left(y + \left(\frac{kh}{n} - j \right) \right) - 2g_i(y) + g_i \left(y - \left(\frac{kh}{n} - j \right) \right) \\ & + (n+1) \sum_{k=0}^n p_{n,k}(y) p_{n,k}(t) dt w(y) \\ & + \int_I c_{r(-1)} g_i(y) (n+1) \sum_{k=0}^n p_{n,k}(y) p_{n,k}(t) w(y) dt \end{aligned} \quad (3)$$

The definition of Bernstein kernel in [7, 8, 13] is

$$k_n(y, t) = (n+1) \sum_{k=0}^n p_{n,k}(y) p_{n,k}(t) \quad (4)$$

We get the form of final operator by substituted (4) in (3) as follow

$$\begin{aligned} D_Y(g_i) = & \int_I \sum_{j=0}^{\frac{r}{2}-1} g_i \left(y + \left(\frac{kh}{n} - j \right) \right) - 2g_i(y) + g_i \left(y - \left(\frac{kh}{n} - j \right) \right) w(y) k_Y(t) dt \\ & + \int_I \sum_{j=\frac{r}{2}+1}^r g_i \left(y + \left(\frac{kh}{n} - j \right) \right) - 2g_i(y) + g_i \left(y - \left(\frac{kh}{n} - j \right) \right) w(y) k_Y(t) dt \\ & + \int_I c_{r(-1)} g_i(y) w(y) k_Y(t) dt. \end{aligned}$$

2. Auxiliary Results

Proposition 2.1: If $g \in Lp, 0 < p < 1$, then $\omega_2(g, t)_{p,\omega} \leq 2^{\frac{1}{p}-1} \|g\|_{p,w}$

Proof: By (2) we get

$$\omega_2(g, t)_{p,\omega} = \sup_{0 \leq h < t} \|g(y+h) - 2g(y) + g(y-h)\|_{p,\omega} \leq 2^{\frac{1}{p}-1} \|g\|_{p,w}.$$

$$K_r(g, t) = \min_{g \in Lp} \{ \|g - f\|_{p,w} + t^r \|D^B g\|_{p,w} \}, \text{ where } D^B = \frac{\partial |\beta|}{\partial x^{\beta_i}}, i = 1, \dots, d, |\beta| = \beta_1 + \dots + \beta_d, \beta = (\beta_1, \dots, \beta_d) \in N^d, r > 0.$$

Proposition 2.2: Let $f \in Lp, 0 < p < 1, K_r(f, t) \approx \omega_2(f, t)_{p,\omega}$.

Proof: By proof $\omega_k(f, \delta)_{p,\omega} \sim K(f, t^r)_{p,\omega}$ we can see the proof it in [1, 6]. By properties of modulus of smoothness in [6, 12] we get $\omega_k(f, \delta)_{p,\omega} \leq \omega_2(f, \delta)_{p,\omega}$. Then $\omega_2(f, \delta)_{p,\omega} \sim K(f, t^r)_{p,\omega}$.

3. The Main Results

Theorem 3.1: Let $g_i \in Lp, 0 < p < 1$ then

$$\|ED\gamma(g_i)\|_{p,w} \leq c(p)(1 + 2 \left(2^{\frac{1}{p}-1}\right)) \|g_i\|_{p,w}$$

$$\begin{aligned} \textbf{Proof: } \|ED\gamma(g_i)\|_{p,w} = & \left\| \int_I \sum_{j=0}^{\frac{r}{2}-1} (g_i \left(y + \left(\frac{kh}{n} - j\right)\right) - 2g_i(y) \right. \\ & + g_i \left(y - \left(\frac{kh}{n} - j\right)\right) w(y) k_\gamma(t) dt + \int \sum_{\frac{r}{2}+1}^r (g_i \left(y + \left(\frac{kh}{n} - j\right)\right) - 2g_i(y) \\ & \left. + g_i \left(y - \left(\frac{kh}{n} - j\right)\right) w(y) k_\gamma(t) dt + g_i(y) w(y) k_\gamma(t) dt \right\|_{p,w} \end{aligned} \quad (5)$$

$$\begin{aligned} \|ED\gamma(g_i)\|_{p,w} \leq & 2^{\frac{1}{p}-1} \left\| \int_I \left(\sum_{j=0}^{\frac{r}{2}-1} (g_i \left(y + \left(\frac{kh}{n} - j\right)\right) - 2g_i(y) \right. \right. \\ & \left. \left. + g_i \left(y - \left(\frac{kh}{n} - j\right)\right) w(y) k_\gamma(t) dt \right) \right\|_{p,w} \\ & + \left\| \int_I \left(\sum_{j=\frac{r}{2}+1}^r (g_i \left(y + \left(\frac{kh}{n} - j\right)\right) - 2g_i(y) + g_i(y - \right. \right. \\ & \left. \left. \left(\frac{kh}{n} - j\right)\right) w(y) k_\gamma(t) dt \right) \right\|_{p,w} + \left\| \int_I c_{r(-1)} g_i(y) w(y) k_\gamma(t) dt \right\|_{p,w}. \end{aligned} \quad (6)$$

Since $w(y)$ is bounded

$$\begin{aligned} \|ED\gamma(g_i)\|_{p,w} \leq & 2^{\frac{1}{p}-1} [c(p) \omega_2(g_i, t)_{p,\omega} + c(p) \omega_2(g_i, t)_{p,\omega} \\ & + c(p) \|g_i\|_{p,w}] \end{aligned}$$

$$\|ED\gamma(g_i)\|_{p,w} \leq c(p) 2(2^{\frac{1}{p}-1}) \omega_2(g_i, t)_{p,\omega} + c(p) (2^{\frac{1}{p}-1}) \|g_i\|_{p,w}$$

By Proposition 2.1 then

$$\|ED\gamma(g_i)\|_{p,w} \leq c(p)(1 + 2 \left(2^{\frac{1}{p}-1}\right)) \|g_i\|_{p,w}$$

Theorem 3.2: For $g \in Lp, 0 < p < 1$, we have

$$\|ED\gamma(gi)\|_{p,w} \leq 2^{\frac{1}{p}-1} \|ED\gamma(gi) - gi\|_{p,w}.$$

Proof: This theorem it is satisfy for $1 \leq q < \infty$ for $\gamma \in \mathbb{N}$ if $y_1, y_2, \dots, y_m$ is partition $(0,1)$, $m \in \mathbb{N}$

$$\begin{aligned} \|ED\gamma(gi)\|_{q,w} &\leq \|ED\gamma(gi) - gi\|_{q,w} \leq \left(\int_0^1 |ED\gamma(gi) - gi|^q w(y) dy\right)^{1/q} \\ &\leq 2^{\frac{1}{p}-1} \sum_{j=1}^m |(ED\gamma)(gi(yj)) - (gi(yj))|^q |y_{j+1} - y_j|^{\frac{1}{q}} w(yj) \\ &\leq 2^{\frac{1}{p}-1} \sum_{j=1}^m |(ED\gamma)(gi(yj)) - (gi(yj))|^p |y_{j+1} - y_j|^{\frac{1}{p}} w(yj) \leq \\ &2^{\frac{1}{p}-1} \|ED\gamma(gi) - gi\|_{p,w} \end{aligned}$$

Theorem 3.3: (Invers theorem) If $g \in Lp, 0 < p < 1, n \in \mathbb{N}$ and $\gamma \in \mathbb{N}$ then $\omega_2\left(gi, \frac{1}{1+\gamma}\right)_{p,\omega} \leq 3(2^{\frac{1}{p}-1}) \sum_{\gamma=1}^n \|ED\gamma(gi) - gi\|_{p,w}.$

Proof: For $n > 2$ there exist $m, \gamma \in \mathbb{N}$ s.t $n/2 \leq m \leq n$

$$\|ED\gamma(gi) - gi\|_{p,w} = \inf_{n/2 \leq m \leq n} \|ED\gamma(gi) - gi\|_{p,w}$$

$\|ED\gamma(gi) - gi\|_{p,w} \leq 2^{\frac{1}{p}-1} \sum_{\gamma=1}^n \|ED\gamma(gi - gi)\|_{p,w}$ by prpotion2.2 and by theorm3.2 we get

$$\begin{aligned} K_r(gi, \gamma^r) &\leq 2^{\frac{1}{p}-1} \|ED\gamma(gi) - gi\|_{p,w} + 2^{\frac{1}{p}-1} \|ED\gamma(gi)\|_{p,w} \\ &\quad + 2^{\frac{1}{p}-1} \sum_{\gamma=1}^n \|ED\gamma(gi - gi)\|_{p,w} \\ \omega_2(g, t)_{p,\omega} &\leq 3(2^{\frac{1}{p}-1}) \sum_{\gamma=1}^n \|ED\gamma(gi - gi)\|_{p,w}. \end{aligned}$$

5. Discussion and Conclusion

In this work, we used the Bernstein-Durmeyer operator with 2nd modules of smoothness to prove the invers theory in quasi- normed space.

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