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Generalized RKM methods for solving classes of fourth-order fractional partial differential equation

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Abstract

This paper introduces new techniques for solving some fractional partial differential equations of fourth order (FPDEs). Combining numerical RKM techniques with the method of lines (MOL) led to the development of innovative methods; we apply the developed approach to some cases involving fourth-order FPDEs and compare the numerical and analytical answers.

Subject Classification: *35R11, 35R01.*
Keywords: *Method of lines, System of fourth-order ODEs, RKM method, Fourth-order PDEs.*

1. Introduction

The study of fractional integrals and derivatives is crucial in numerous fields of pure and practical mathematics. Numerous fields of applied science and mathematics rely on fractional differential equations' interpretation and modeling capabilities. These areas include fluid mechanics, diffusion, electrical circuits, damping laws, mathematical biology, relaxation processes, and many

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more, see ([1], [2], [3]); this highlights the significance of the quest for solutions to fractional differential equations in the scope of scientific inquiry. Due to the difficulty of finding exact solutions to these equations, different numerical methods were used to solve these equations numerically like [4] Changpin Li and An Chen (2018) focused on finite difference, Galerkin, and spectral approaches for (FPDEs), and [5] Mehmet Yavuz (2018) created a conformable fractional derivative formulation for approximate-analytical fractional PDE solutions. Moreover, [6] Shumaila Javeed et al. (2019) introduced the Homotopy Perturbation Method and discussed its application to the solution of FPDEs, [7] Jun Jiang et al. (2020) used a modified fractional approach to find exact solutions to nonlinear FPDEs.

Furthermore, [8] Laique Zada and Imran Aziz (2020) used the Haar wavelet collocation technique to solve FPDEs numerically. Also, [9] Yusuf Gurefe (2020) solved nonlinear FPDEs using the general Kudryashov technique. So, in this study, we will focus on developing a numerical method for solving FPDEs of order four by Combining numerical RKM techniques with the method of lines (MOL) [10]. We apply the developed approach to some cases involving fourth-order FPDEs and compare the numerical and analytical answers. This paper is organized as follows: The first part contains the introductory information. In part 2, the proposed RKM method is given in Specifics. To prove the reliability of the suggested method, we conduct numerical experiments in part 3, and part 4 contains some final ideas about the topic.

2. Preliminary

In this section, we presented the fundamental definitions and background information to the main issue of the study.

Definition 1 [**10**]**:** For a function $\mathrm{y}(\mathrm{x},\mathrm{T})$ with multiple variables, the Caputo derivative is defined as

$$D_{\mathrm{T}}^{\alpha}\mathrm{y}(\mathrm{x},\mathrm{T})=\frac{\partial^{\alpha}\mathrm{y}(\mathrm{x},\mathrm{T})}{\partial\mathrm{T}^{\alpha}}=\begin{cases}J_{\mathrm{T}}^{\mathrm{n}-\alpha}\left(\frac{\partial^{\mathrm{n}}\mathrm{y}(\mathrm{x},\mathrm{T})}{\partial\mathrm{T}^{\mathrm{n}}}\right), & \mathrm{n}-1<\alpha<\mathrm{n},\\ \frac{\partial^{\mathrm{n}}\mathrm{y}(\mathrm{x},\mathrm{T})}{\partial\mathrm{T}^{\mathrm{n}}}, & \alpha=\mathrm{n}\in N,\end{cases}\tag{1}$$

where $\alpha\in(\mathrm{n}-1,\mathrm{n}],\mathrm{n}\in N$ and the Riemann-Liouville fractional partial derivative of order α, for $\mathrm{y}(\mathrm{x},\mathrm{T})$, is defined as:

$$D_{\mathrm{T}}^{\alpha}\mathrm{y}(\mathrm{x},\mathrm{T})=\frac{\partial^{\mathrm{n}}}{\partial\mathrm{T}^{\mathrm{n}}}\int_{0}^{\mathrm{T}}\frac{(\mathrm{T}-\tau)^{\mathrm{n}-\alpha-1}}{\Gamma(\mathrm{n}-\alpha)}\mathrm{y}(\mathrm{x},\tau)d\tau,\mathrm{T}>0.\tag{2}$$

Some properties of the D_{T}^{α} the operator is included in the following lemma:

Lemma 1 [**10**]: *For* $\mathrm{T}\geq 0,\mathrm{y}>-1,\alpha\in(\mathrm{n}-1,\mathrm{n}],\mathrm{n}\in N$ *and* $c\in\mathbb{R}$. *Then:*

- $D_{\mathrm{T}}^{\alpha}\mathrm{T}^{\mathrm{y}}=\frac{\Gamma(\mathrm{y}+1)}{\Gamma(\mathrm{y}+1-\alpha)}\mathrm{T}^{\mathrm{y}-\alpha}$.
- $D_{\mathrm{T}}^{\alpha}c=0$.
- $D_{\mathrm{T}}^{\alpha}\mathrm{j}_{\mathrm{T}}^{\alpha}\mathrm{y}(\mathrm{x},\mathrm{T})=\mathrm{y}(\mathrm{x},\mathrm{T})$.

- $j_{\tau}^{\alpha} D_{\tau}^{\alpha} w(x,\tau) = w(x,\tau) - \sum_{j=0}^{n-1} w_{\tau}^{(j)}(x,0^{+}) \frac{\tau^{j}}{j!}$.

Definition 2: The General Quasi-Linear Fractional Partial Differential Equations (FPDEs):

The following form describes the class of quasi-linear fractional partial differential equations:

$$D_{\tau(n-\ times\ )}^{n\alpha} w(x,\tau) = f(x,\tau, w(x,\tau), w_{x}(x,\tau), w_{xx}(x,\tau), .., w_{x(n-times)}(x,\tau)), \quad a \le x \le b, 0 < \tau \le T. \tag{3}$$

With initial conditions (ICs):

$$w_{x(n-\ times\ )}(x,0) = f_{I}(x),\ I = 0,1,2,\dots,n-1, \tag{4}$$

and the boundary conditions (BCs):

$$w(a,\tau) = \varphi_{1}(\tau),\ w(b,\tau) = \varphi_{2}(\tau),$$

Where $0 < \alpha \le 1$.

2.1. *The Finite Difference Method [14]*

One of the simplest and oldest techniques to solve differential equations is via Finite Difference Method (FDM), already recognized in one dimension by L. Euler (1707-1783) in 1768, it was likely expanded to two dimensions by C. Runge (1856-1927) around 1908. FDMs originally appeared in numerical applications in the early years of the 1950s, and the emergence of computers fueled the development of these techniques, it offered a practical framework for tackling tough scientific and industrial problems. Over the past 50 years, theoretical conclusions have been reached on the precision, convergence, and stability of the finite difference technique for partial differential equations, also developed FDM for solving fractional equations, the fractional finite difference system as bellows.

$$\frac{\partial^{n\alpha} w(x,\tau)}{\partial \tau^{n\alpha}}\Big|_{(x,\tau)=(x_I,\tau_j)} = f\left(x,\tau,u(x,\tau), \frac{\partial w(x,\tau)}{\partial x}, \frac{\partial^{2\alpha} w(x,\tau)}{\partial x^{2\alpha}}, \dots, \frac{\partial^{n\alpha} w(x,\tau)}{\partial x^{n\alpha}}\right)\Big|_{(x,\tau)=(x_I,\tau_j)}. \tag{5}$$

Here we give the fundamental formulae for finite differences of order one, two, three, etc.

$$\frac{\partial^{\alpha} w(x,\tau)}{\partial \tau^{\alpha}}\Big|_{(x,\tau)=(x_I,\tau_j)} = \frac{w_{I,j+1} - w_{I,j-1}}{k}, \tag{6}$$

$$\frac{\partial^{2\alpha} w(x,\tau)}{\partial \tau^{2\alpha}}\Big|_{(x,\tau)=(x_I,\tau_j)} = \frac{w_{I,j+1} - 2w_{I,j} + w_{I,j-1}}{k^{2}}, \tag{7}$$

$$\frac{\partial^{3\alpha} w(x,\tau)}{\partial \tau^{3\alpha}}\Big|_{(x,\tau)=(x_I,\tau_j)} = \frac{w_{I,j+2} - 2w_{I,j+1} + 2w_{I,j-1} + w_{I,j-2}}{2\,k^{3}}, \tag{8}$$

$$\frac{\partial^{4\alpha} w(x,\tau)}{\partial \tau^{4\alpha}}\Big|_{(x,\tau)=(x_I,\tau_j)} = \frac{w_{I,j+2} - 4w_{I,j+1} + 6w_{I,j} - 4w_{I,j-1} + w_{I,j-2}}{k^{4}}. \tag{9}$$

3. Proposed RKM-Methods for Solving Fourth-Order FPDEs

Fourth-order ODEs with special properties, conditions where the function does not depend directly on $\left(w^{(n)}(x)\right)$, n = 1,2,3. Equations can be stated as follows:

$$w^{(4)}(x) = f\left(t, w(x)\right),\ x \geq x_0. \tag{10}$$

with initial conditions (I.C.)

$$w(x_0) = w_0,\ w'(x_0) = w'_0,\ w''(x_0) = w''_0,\ w^{(3)}(x_0) = w^{(3)}_0, \tag{11}$$

where $f: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Historically, scientists and engineers have solved ODEs of the fourth order by first converting them into a system of ODEs of the first order and then numerically solving this system. (see [11,12,15]). However, this method squandered a great deal of resources, both digital and human. Since this is the case, the straight numerical techniques would be the most effective approach to solving 4th ODEs. (see [13]).

3.1. *Problem statement*

The following form describes the class of fourth order quasi-linear fractional partial differential equations:

$$D_t^{4\alpha} w(x,t) = f\left(x, t, w(x,t), w_x(x,t), w_{xx}(x,t), w_{xxx}(x,t), w_{xxxx}(x,t)\right) \quad a \leq x \leq b, 0 < t \leq T \tag{12}$$

With initial conditions (ICs):

$$w(x,0) = f_1(x),\ w_x(x,0) = f_2(x),\ w_{xx}(x,0) = f_3(x),\ w_{xxx}(x,0) = f_4(x). \tag{13}$$

And the boundary conditions (BCs):

$$w(a,t) = \varphi_1(t),\ w(b,t) = \varphi_2(t), \tag{14}$$

Where $0 < \alpha \leq 1$.

3.2. *The Direct Numerical RKM method [13]*

To solve 4th ODEs $D^{(4)}w(t) = f\left(t, w(t)\right)$, the general form of the RKM method with an s-stage is as follows:

$$w_{n+1} = w_n + hw'_n + \frac{h^2}{2}w''_n + \frac{h^3}{6}w^{(3)}_n + h^4 \sum_{I=1}^{s} b_I^{(0)} K_I, \tag{15}$$

$$w'_{n+1} = w'_n + hw''_n + \frac{h^2}{2}w^{(3)}_n + h^3 \sum_{I=1}^{s} b_I^{(1)} K_I, \tag{16}$$

$$w''_{n+1} = w''_n + hw^{(3)}_n + h^2 \sum_{I=1}^{s} b_I^{(2)} K_I, \tag{17}$$

$$w^{(3)}_{n+1} = w^{(3)}_n + h \sum_{I=1}^{s} b_I^{(3)} K_I, \tag{18}$$

Where

$$K_1 = f\left(x_n, w_n\right), \tag{19}$$

$$K_I = f\left(x_n + c_I h, w_n + hc_I w'_n + \frac{h^2}{2}c_I^2 w''_n + \frac{h^3}{6}c_I^3 w^{(3)}_n + h^4 \sum_{j=1}^{I-1} a_{Ij} K_j\right). \tag{20}$$

for $I = 2,3, \dots, ş$. The RKM method assumes the following real values for the parameters $ç_I, a_{IJ}, ḅ_I^{(0)}, ḅ_I^{(1)}, ḅ_I^{(2)}, ḅ_I^{(3)}$ for $I, j̇ = 1,2, \dots, ş$. It is an explicit method if and only If $a_{Ij̇} = 0$ for $I \leq j̇$, and an implicit method otherwise. The following table of coefficients can be used to describe the RKM method in Butcher notation: (see Table 1)

Table 1
The Butcher Tableau for RKM technique

C	A
	$ḅ^T$
	$ḅ'^T$
	$ḅ''^T$
	$ḅ^{(3)T}$

RKM method order requirements for 4th ODEs have been obtained. We used the Taylor series extension method to determine the necessary ordering conditions. In this study, we apply the same method to the solution of 4th ODEs and derive the order conditions for ODEs up to order seven. (see Tables 2 and 3)

Table 2
The RKM Method (Three-Stage, Fourth-Order)

0	0		
5/6	½	0	
1/3	½	½	0
	1/60	1/3240	1/81
	1/20	1/180	1/9
	1/10	1/15	1/3
	1/10	2/5	1/2

Table 3
The RKM Method (Four-Stage, Fifth-Order)

0	0			
1/2	$3/160 - \sqrt{15}/240$	0		
1/2 $-\sqrt{15}/10$ 1/2 $+\sqrt{15}/10$	1/2 $-51/100$ $-22/75\sqrt{15}$	$-1/2$ $1/100$ $+\sqrt{15}$ $/5$	0 1/2	0 0
	0	1/108	$7/432 + \sqrt{15}/240$	$7/432 - \sqrt{15}/240$
	0	1/18	$1/18 + \sqrt{15}/72$	$1/18 - \sqrt{15}/72$
	0	2/9	$5/36 + \sqrt{15}/36$	$5/36 - \sqrt{15}/36$
	0	4/9	5/18	5/18

3.3. *Solution Algorithm*

In this section, we introduce a computational strategy for solving equation (12) with I.Cs (13) and B.Cs (14) using a hybrid of the method of lines and the RKM method, as follows, Specifically, we presume that the x-axis interval [ą, ƀ] and the Ƭ-axis 0–T interval correspond to the numerical solution, with $Þ = \frac{ƀ-ą}{n}$ and k = T/m. Given that there are h points in the x̧-direction in [ą,ƀ], and m points in the Ƭ-direction in [0, T]. Combining method of lines (MOL) and the RKM method, we can solve problem (12) given initial and boundary conditions (13), (14) as shown below:

This algorithm is followed to solve problem (12) with conditions (13) and (14).

1. while $1 \le I \le n$, do the steps $1 - 6$.
2. FPDE (12) can be transformed into the following equation by fixing the value of $x̧ = x̧_I$ at the point $(x̧, Ƭ)$.

$$ẉ_I^{(4\alpha)}(Ƭ) = f̧\left(x̧, Ƭ, ẉ(x̧, Ƭ), \frac{\partial ẉ(x̧,Ƭ)}{\partial x̧}, \frac{\partial^2 ẉ(x̧,Ƭ)}{\partial x̧^2}, \frac{\partial^3 ẉ(x̧,Ƭ)}{\partial x̧^3}, \frac{\partial^4 ẉ(x̧,Ƭ)}{\partial x̧^4}\right), \tag{21}$$

where

$$ẉ_I^{(4\alpha)}(Ƭ) = \frac{d^4 ẉ(x̧,Ƭ)}{dx̧^4},\ I = 1,2,\dots,3. \tag{22}$$

3. The derivatives on the right-hand side of ODE (21), when substituted with finite difference formulas defined in equations (6)- (9), yield a system of FODEs at the fourth-order.

$$ẅ_I^{(4\alpha)}(Ƭ) = f̧(x̧_I, Ƭ, ẅ_{I-2}(Ƭ), ẅ_{I-1}(Ƭ), ẅ_I(Ƭ), ẅ_{I+1}(Ƭ), ẅ_{I+2}(Ƭ)), \tag{23}$$

for I = 1,2, … ,3.

4. starting conditions (with J =1)

$$ẅ_I(0) = f̧_1(x̧_I),\ ẅ_I^{\alpha}(0) = f̧_2(x̧_I),\ ẅ_I^{2\alpha}(0) = f̧_3(x̧_I),\ ẅ_I^{3\alpha}(0) = f̧_4(x̧_I). \tag{24}$$

if $2 \le J \le m$, so the (I.C.) is

$$\begin{aligned} w_I(\tau_{j-1}) &= w(x_I, \tau_{j-1}), \\ w_I^{\alpha}(\tau_{j-1}) &= \left.\frac{dw(x,\tau_{j-1})}{dx}\right|_{x=x_l}, \\ w_I^{2\alpha}(\tau_{j-1}) &= \left.\frac{d^2w(x,\tau_{j-1})}{dx^2}\right|_{x=x_l}, \\ w_I^{3\alpha}(\tau_{j-1}) &= \left.\frac{d^3w(x,\tau_{j-1})}{dx^3}\right|_{x=x_l}, \end{aligned} \tag{25}$$

5. put (B.C.):

$$\begin{aligned} w_{0,j} &= w(a, \tau_j) = g_1(\tau_j), \\ w_{n,j} &= w(b, \tau_j) = g_2(\tau_j), \end{aligned} \tag{26}$$

6. By using the RKM method, solve the system of fourth-order FODEs (23), given (I.C.) (25) and (B.C.) (26), at $\tau = \tau_j$.

4. Numerical Examples

To test our method as shown in Figure 1, we used fourth-order FPDE in the following problems:

Problem 1: $D_{\tau}^{4\alpha} w(x,\tau) = (-1)^{4\alpha} w(x,\tau),\ 0 \le x \le 1,\ \tau > 0,$
with I. Cs,

$$\begin{aligned} w(x,0) &= x^4,\ w_x(x,0) = 4x^3,\ w_{xx}(x,0) = 12x^2, \\ w_{xxx}(x,0) &= 24x. \end{aligned}$$

and B.Cs,

$$w(a,\tau) = a^4 e^{-\tau},\ w(b,\tau) = b^4 e^{-\tau},$$

The exact solution is: $w(x,\tau) = x^4 e^{-\tau}$;

Problem 2: $D_{\tau}^{4\alpha} w(x,\tau) = (2)^{4\alpha-1}(w(x,\tau) + w_{xxxx}(x,\tau)),\ 0 \le x \le \pi,\ \tau > 0,$
with I.Cs,

$$\begin{aligned} w(x,0) &= \cos(x),\ w_x(x,0) = -\sin(x),\ w_{xx}(x,0) = -\cos(x), \\ w_{xxx}(x,0) &= \sin(x), \end{aligned}$$

and B. Cs,

$$w(a,\tau) = e^{2\tau} cos(a),\ w(b,\tau) = e^{2\tau} cos(b),$$

The exact solution is: $w(x,\tau) = e^{2\tau}\cos(x)$;

Problem 3: $D_{\tau}^{4\alpha} w(x,\tau) = (2)^{4\alpha-1}(w(x,\tau) + w_{xxxx}(x,\tau)),\ 0 \le x \le \pi,\ \tau > 0,$
with I. Cs,

$$\begin{aligned} w(x,0) &= \cos(x),\ w_x(x,0) = -\sin(x),\ w_{xx}(x,0) = -\cos(x), \\ w_{xxx}(x,0) &= \sin(x), \end{aligned}$$

and B. Cs,

$$w(a,\tau) = sin(a)\ e^{2\tau},\ w(b,\tau) = e^{2\tau}\ sin(b),$$

The exact solution is: $w(x,\tau) = e^{2\tau}\sin(x)$;

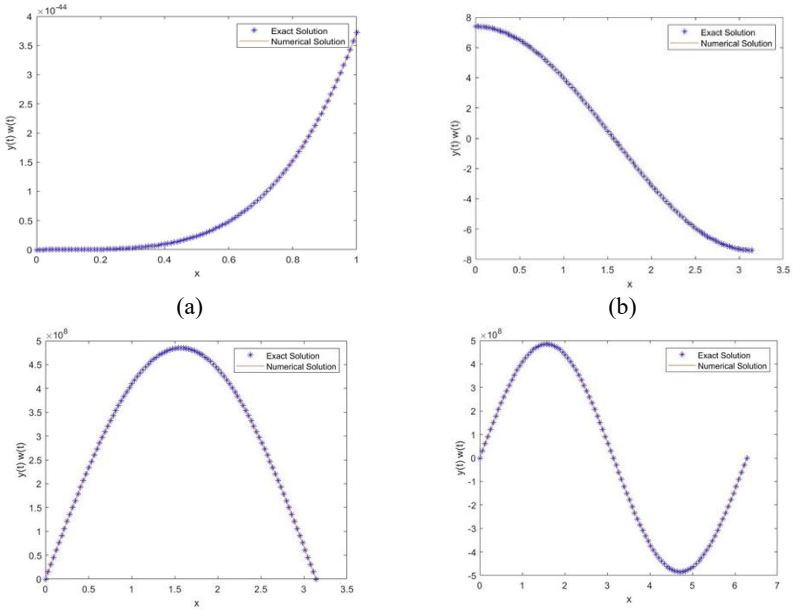


Figure 1

A Comparison between the Numerical Solutions evaluated by Generalized Numerical Methods versus the Analytical Solutions for Problems 1, 2, and 3 in (a), (b), (c), and (d) using the Generalized RKM Method

5. Discussion and Conclusion

The primary objective of this study is to present numerical approaches for solving FPDEs of various orders. By extending multiple types of RK-type numerical techniques that are useful for solving FPDEs numerically, the purpose of this study has been made clear.

We used the MATLAB software to apply the suggested algorithm to solve three Problem FPDEs. We evaluated the suggested approaches in this work by comparing the numerical solutions to the precise answers for four cases. The results showed the efficiency of the proposed method.

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