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# Fuzzy cofibration and fuzzy Serre cofibration

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**Abstract**

In this paper, we investigate and present the idea of fuzzy cofibration and fuzzy Serre cofibration and we establish some characteristics and theorems of these concepts. In addition, to studying the fuzzy pullback of a fuzzy cofibration.

***Subject Classification:*** *03E72, 06D72.*

***Keywords:*** *Fuzzy homotopy, Fuzzy lowering homotopy property, Fuzzy cofibration, Fuzzy Serre cofibration.*

## 1. Introduction and Preliminaries

Zadeh presented the essential concepts of fuzzy sets in his classical paper [7]. Fuzzy sets have applications in Several branches of engineering, social science, economics, etc, in mathematics, the theory of fuzzy topology, was developed by many researchers the first one was chang [4]. In fact, many classical topology concepts have been expanded to encompass fuzzy topological

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space (see [2] and [3]). Topological algebra at first studied by Hatcher [1], and he developed some of its concepts. The fuzzy homotopy theory was studied and submitted by Cuvalcioglu and Citil [5] and in 2019 Malathi and Uma [6] in India presented the concept of fuzzy covering space and the definition of fuzzy lifting map which opened the way for us to study the fuzzy fibration and fuzzy cofibration [8,9], In this paper we study the concepts of fuzzy lowering homotopy property, fuzzy cofibration and fuzzy Serre cofibration.

**Definition 1.1: [7]** A fuzzy set $\lambda$ is a function $\lambda: X \to I$ such th $X \neq \emptyset$ and $I = [0,1]$, the expression for the collection of all fuzzy sets is $I^X$.

**Definition 1.2: [4]** Let $X$ be any set and $\tau$ to be a collection of fuzzy subsets of X, then $\tau$ is said to be fuzzy topology on if its fulfills the conditions.

- $0_X$ and $1_X \in \tau$.
- If $\gamma, \mu \in \tau$, then $\gamma \wedge \mu \in \tau$.
- If $\gamma_i \in \tau$ for each $i \in I$, then $\bigvee \gamma_i \in \tau$.

The pair $(X, \tau)$ is said to be fuzzy topological space (F.T.S). Furthermore the components of $\tau$ are called fuzzy open sets and their complements are called fuzzy closed sets.

**Definition 1.3: [5]** Let $(X, \tau_1)$ and $(Y, \tau_2)$ be fuzzy topological spaces, and $f, g: X \to Y$ are fuzzy continuous functions then, $f$ is a fuzzy homotopy to $g$ if there exist a family of fuzzy continuous functions $F_t: X \to Y$ such that for all $t \in I$

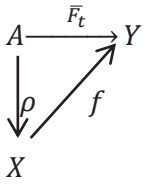
- $F_t(x) = f_t(x)$
- $F_0(x) = f(x)$
- $F_1(x) = g(x)$

If $f$ *and* $g$ are fuzzy homotopic functions then, we write "$f \sim g$"

For example every fuzzy continuous function is a fuzzy homotopic to itself.

## 2. Fuzzy cofibration and fuzzy Serre cofibration

**Definition 2.1**: A map $\rho: A \to X$ is said to have a fuzzy lowering homotopy property (FLHP) w. r. t Y if and only if given a fuzzy map $f: X \to Y$ and a fuzzy homotopy $\bar{F}_t: A \to Y$ satisfying $f\rho = \bar{F}_0$. Then there exist another fuzzy homotopy $F_t: X \to Y$, with $F_0 = f$ and $F_t\rho = \bar{F}_t$ for all $t \in I$.




- Let Q be a class of fuzzy topological spaces a fuzzy map $\rho: A \to X$ is a fuzzy cofibration with respect to Q if and only if $\rho$ has the (FLHP) with respect to each $X \in Q$.
- If Q is the class of all fuzzy CW-complex spaces, then $\rho: A \to X$ is called fuzzy Serre cofibration.
- If $A \subset X$, the inclusion map is a fuzzy cofibration.
- The pair (X, A) is called fuzzy cofibred pair.
- An essential condition to make (X, A) a cofibred pair is the existence of a retraction $\gamma: X \times I \to (X \times 0) \cup (A \times I)$. If A is closed, this condition is also sufficient.

**Example 2.2:** The fuzzy inclusion map $i: \mu_{S^{n-1}} \to \mu_{D^n}$ of the boundary sphere of solid disk is a fuzzy cofibration. In addition, if the space CW-complex then, the fuzzy inclusion map $i: \mu_{S^{n-1}} \to \mu_{D^n}$ is a fuzzy Serre cofibration.

**Definition 2.3:** The fuzzy homotopy extension property (FHEP) with respect to Y of a subset $A \subset Y$, if and only if any fuzzy partial homotopy $\bar{H}: X \times \{0\} \cup A \times I \to Y$ is possible to be extended to a fuzzy homotopy $H: X \times I \to Y$. If $A \subset X$ has the (FHEP) with respect to every Y, then $A \subset X$is said to have the absolute fuzzy homotopy extension property (AFHEP).

**Theorem:** *The cofibred pair (X, A) has (FHEP) with respect to Y, if and only if given a fuzzy homotopy $f_t: A \to Y$, and a fuzzy map $h_0: X \to Y$ such that $f_0 = h_0|A$, there exists a fuzzy homotopy $h_t: X \to Y$ that starts from the given fuzzy map $h_0$ and extends the fuzzy homotopy $f_t$, in the sense that $f_t = h_t|A$.*

***Proof:*** Suppose $\gamma: X \times I \to (X \times 0) \cup (A \times I)$ is a retraction. A fuzzy homotopy $f_t$ and a fuzzy map $h_0$ as above combine to give a fuzzy map $g: X \times \{0\} \cup A \times I \to Y$. The desired fuzzy homotopy is $h_t(x) = g\big(\gamma(x,t)\big)$. Conversely, take Y=$(X \times 0) \cup (A \times I)$, $f_t(a) = (a,t)$, and $h_0(x) = (x,0)$.if (X, A) has the FHEP, the resulting homotopy $h_t: X \to Y$ can be rewritten as a retraction $X \times I \to Y$.

**Example 2.4:** The same pair $(\mu_{D^n}, \mu_{S^{n-1}})$ has the FHEP. If given a fuzzy homotopy $f_t: \mu_{S^{n-1}} \to Y$ and a map $h_0: I^{D^n} \to Y$ such that $h_0|\mu_{S^{n-1}} = f_0$. We assemble these to form a fuzzy map $g: \mu_{D^n} \to Y$ from $\mu_{D^n} = \{\mu_x: D^2 \to I: \|x\| \leq 2\}$ by setting

$$g(\mu_x) = \begin{cases} h_0(\mu_x) & \text{for} \|x\| \le 1 \\ f_{\|\mu_x\| \le 1}\left(\frac{\mu_x}{\|\mu_x\|}\right) & \text{for } 1 \le \|x\| \le 2 \end{cases}.$$

The desired fuzzy homotopy $h_t: \mu_{D^n} \to Y$ is then just $h_t = g((1+t)x)$.

**Theorem 2.5:** *Let $\rho: A \to X$ be a fuzzy Serre cofibring of a fuzzy compact Hausdorff A into a fuzzy Hausdorff space X, then ρ must be an imbedding.*

***Proof:*** First let us prove that $\rho$ must be injective. This will be proved by contradiction. Assume that $\rho$ is not one-one, then there exist two distinct points u and v in A with $\rho(u) = \rho(v)$. Since A is fuzzy normal, and {u}, {v} are closed subsete of A, there exists a fuzzy continuous function $\gamma: A \to I$, with $\gamma(u) = 0$ and $\gamma(v) = 1$. Define a fuzzy homotopy $\bar{F}_t: A \to I$ by $\bar{F}_t(a) = t\gamma(a)$, $a \in A, t \in I = [0,1]$ and define $f: X \to I$, with $f(x) = 0$ for all $x \in X$. Then we have $f\rho = \bar{F}_0$. But $\rho: A \to X$ is a fuzzy Serre cofibration, hence there exists a fuzzy homotopy $F_t: X \to Y$, with $F_0 = f$ and $F_t\rho = \bar{F}_t$ for all $t \in I$.

Now $\gamma(u) = \bar{F}_1(u) = F_1(\rho(u))$ and $\gamma(v) = \bar{F}_1(v) = F_1(\rho(v))$

But $\rho(u) = \rho(v) \Longrightarrow \gamma(u) = \gamma(v)$

However, $\gamma(u) = 0 \neq 1 = \gamma(v) \Rightarrow \rho$ must be injective.

But A is a fuzzy compact Hausdorff and X is a fuzzy Hausdorff space, hence ρ must be an imbedding.

**Lemma 2.6:** *A pair (X,A) is fuzzy closed cofibration iff there are a fuzzy mappings $D: X \times I \to X$ and $\varphi: X \to I$ such that:*

1) $D(x,0) = x$ for all $x \in X$
2) $D(a,t) = a$ for all $a \in A, t \in I$.
3) $A = \varphi^{-1}(1)$ and $D(\varphi^{-1}(0,1] \times 1) \subset A$.

***Proof:*** $A \subset X$ is a fuzzy cofibration if and only if there is a retraction $\gamma: X \times I \to (X \times 0) \cup (A \times I)$. If $A \subset X$ is a fuzzy cofibration, define $D: X \times I \to X$ by $D(x,t) = \pi_1\, \gamma(x,t)$ where $\pi_1$ is the projection on the first coordinate. Then, D satisfies 1 and 2 since $\gamma$ is a retraction. Define $\varphi_1: X \to I$ , by $\varphi_1(x) = \max_{t \in I} |\pi_2\, \gamma(x,t) - t|$. Then $\varphi_1(x) = 0$ iff $x \in A$ since "$\varphi_1(x) = 0$" $\Longrightarrow$ $\pi_2\, \gamma(x,t) = t$ for every $t \in I$, for $t \neq 0$ " $\gamma(x,t) \in A \times I$". However $\gamma(x,0) = (x,0)$ and thus $x \in A$. Define $\varphi_2: X \to I$, by $\varphi_2(x) = \pi_2\, \gamma(x,1)$. Then $\varphi_2(a)$=1 if $a \in A$ and if $\varphi_2(x) \neq 0$, $D(x,1) \in A$. Thus if we let $\varphi(x) = \big(1 - \varphi_1(x)\big) \cdot \varphi_2(x)$ then $\varphi$ satisfied condition "3". Conversely, given D and $\varphi$ are two fuzzy mappings, and define a retraction $\gamma: X \times I \to (X \times 0) \cup (A \times I)$. As follows:

1. $\gamma(x,t) = (x,0)$ if $\varphi(x) = 0$.
2. $\gamma(x,t) = (D(x, 2\varphi(x) \cdot t), 0)$ if $0 < \varphi(x) \le \frac{1}{2}$ .
3. $\gamma(x,t) = \left(D\left(x, \frac{t}{2(1-\varphi(x))}\right), 0\right)$ if $\frac{1}{2} \le \varphi(x) < 1$, $t \le 2\big(1 - \varphi(x)\big)$.

4. $\gamma(x,t) = \Big(D(x,1), t - 2\big(1-\varphi(x)\big)\Big)$, if $\frac{1}{2} \le \varphi(x) < 1$ and $2\big(1-\varphi(x)\big) \le t \le 1$.
5. $\gamma(x,t) = (x,t)$ if $\varphi(x) = 1$.

**Definition 2.7:** Let $\rho: A \to X$ be fuzzy cofibration and define Y to be any fuzzy topological space and $f: X \to Y$ be any fuzzy map and $Af \subset A \times Y$ be any fuzzy subspace such that $Af = \{(a,y) \in A \times Y;\ \rho(a) = f^{-1}(y)\}$ of the Cartesian product. Take $\rho^*: Af \to Y$ to be the projection $\rho^*(a,y) = y$ then $\rho^*$ is called fuzzy pullback of $\rho$ by $f$. Define a fuzzy map $\pi: A \to Af$ such that $\pi(a) = (a, \rho(a))$, it is immediate from the definitions that the diagram is commutative such that $\rho^* \circ \pi = f \circ \rho$

$$\begin{array}{ccc} Af & \xleftarrow{\ \pi\ } & A \\ \boxed{\rho^*}\downarrow & \circlearrowleft & \downarrow\boxed{\rho} \\ Y & \xleftarrow[\ f\ ]{} & X \end{array}$$

**Example 2.8:** Let we take the fuzzy cofibration $i: \mu_{S^{n-1}} \hookrightarrow \mu_{D^n}$ and let Y=$\mu_{S^n}$ , we have a fuzzy inclusion map $f: \mu_{D^n} \hookrightarrow \mu_{S^n}$ and $\mu_{S^{n-1}} f \subset \mu_{S^{n-1}} \times \mu_{S^n}$, the projection $\rho^*: \mu_{S^{n-1}} f \to \mu_{S^n}$ such that $\rho^*(\mu_x, \mu_y) = \mu_y$ is a fuzzy pullback of $i$ by $f$

**Theorem 2.9:** *Fuzzy pullback of a fuzzy cofibration is also fuzzy cofibration.*

***Proof:*** Let Z be fuzzy topological space we will show that $\rho^*$ has the (FLHP) with respect to Z.

$$\begin{array}{ccccc} & \xleftarrow{F_t} & Af & \xleftarrow{\ \pi\ } & A \\ \boxed{Z} & \circlearrowleft & \boxed{\rho^*}\downarrow & \circlearrowleft & \downarrow\boxed{\rho} \\ & \xleftarrow[\ h\ ]{} & Y & \xleftarrow[\ f\ ]{} & X \end{array}$$

Let $h: Y \to Z$ be a fuzzy map and $F_t: Af \to Z$ be a fuzzy homotopy such that $h\rho^* = F_0$. Consider $F_t \circ \pi : A \to Z$ and $h \circ f : X \to Z$ are a fuzzy homotopic

$$\begin{array}{ccc} & \xleftarrow{F_t \circ \pi} & A \\ \boxed{Z} & \circlearrowleft & \downarrow\boxed{\rho} \\ & \xleftarrow[h \circ f]{} & X \end{array}$$

$h \circ f \circ \rho = F_t \circ \pi$ since ( $h\rho^* = F_0$ and $\rho^*$ is a fuzzy pullback $\Rightarrow f \circ \rho = \rho^* \circ \pi$)

$$h \circ \rho^* \circ \pi = F_t \circ \pi \Rightarrow F_0 \circ \pi = F_t \circ \pi \Rightarrow F_0 = F_t \ \ \forall\, t \in I$$

$$h \circ f \circ \rho = h \circ \rho^* \circ \pi$$

But $\rho$ is a fuzzy cofibration then, there exist a fuzzy homotopy $H_t: X \to Z$ such that $H_0 = h \circ f$ and $H_t \circ \rho = F_t \circ \pi$

Define a fuzzy homotopy $\overline{F}_t: Y \to Z$ such that $\overline{F}_t(y) = h(y)$

$\Longrightarrow \overline{F_0}(y) = h(y)$ and $\overline{F_t} \circ \rho^*(y) = h \circ \rho^*(y) = F_t$
$\Longrightarrow (X, Af)$ is a cofibred pair

**3. Discussion and conclusion**

In this paper we have studied The definition of fuzzy cofibration and fuzzy Serre cofibration as important results of the current study with giving theorems and properties of the two concepts and the definition of fuzzy pullback with its properties we can use our result to study the mixing case of the fuzzy cofibration and fuzzy Serre cofibration.

**References**

[1] Hatcher. A , "Algebraic topology", Cambridge University Press, New York, NY, USA (2001) .

[2] Azad. K. K, " On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity", *J. Math. Anal. Appl.* 82, pp. 14-32 (1981).

[3] Balasubramanian.G,"Maximal fuzzy topologies", KYBERNETIKA. 31, pp. 459-464 (1995).

[4] Chang C.L., "Fuzzy topological spaces", *J. Math. Anal. Appl.* 24, pp. 182-190 (1968).

[5] Cuvalcioglu.G and Citil.M., "On fuzzy homotopy theory", Adv. stud. Contemp. Math. (2006).

[6] Malathi R. and Uma M. K., "Fuzzy Covering spaces", The *International journal of analytical and experimental modal analysis*, India (2019).

[7] zadeh L.A., "Fuzzy sets", *information and control*, 8, pp.338-365 (1965).

[8] Shubham Sharma & Ahmed J. Obaid. Mathematical modelling, analysis and design of fuzzy logic controller for the control of ventilation systems using MATLAB fuzzy logic toolbox, *Journal of Interdisciplinary Mathematics*, 23:4, 843-849 (2020), DOI: 10.1080/09720502.2020.1727611.

[9] Shelash, Hader Baqer , Ahmad, Hasanain Hamid & Obaid, Ahmed J. On characterizations some of subgroups of the dihedral group D2n, *Journal of Discrete Mathematical Sciences and Cryptography*, 1-6, DOI: 10.47974/JDMSC-1553.

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