Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 4, pp. 721–727
DOI : 10.47974/JIM-1757

Fekete-Szego inequalities for higher-order derivatives of multivalent analytic function with application to stealth combat aircraft

Sarah A. Al-Ameedee *
Department of Mathematics
College of Basic Education
Babylon University
Babylon
Iraq

Mohammed Baqer Hashim Al-Hakeem †
Ministry of Higher Education and Scientific Research
Baghdad
Iraq

Ali Kadhim Hussein Alghafil §
General Directorate of Education of Holy Karbala
Iraqi Ministry of Education
Karbala
Iraq

Abstract

In this paper, we find bounds for the Fekete-Szegö coefficient functional for the P-valent Function with Higher Order Derivatives of a new subclass $\mu_q^p(h)$ in the open unit disc, U. One of the most crucial strategic requirements for defense and space applications is reducing an object's visibility to the electromagnetic spectrum used for interrogation. The current method relies on Radar cross-section (RCS) for detection by adopting optimized physical designs that consider reflections and scattering from various edges and corners.

Subject Classification: *(2022) 30C45.*

Keywords: *Analytic function, P-valent, Subordinate, Coefficient, Stealth.*

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

* *E-mail:* **sarahabd54@gmail.com (Corresponding Author)**
† *E-mail:* **mbhmsc2015110@gmail.com**
§ *E-mail:* **alialshahen80@gmail.com**

© Taru Publications

1. Introduction

Let $A(p)$ denote the class of the function $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic in the open disc $\mathfrak{U}$.

Let $\mu_q^p(h)$ be the subclass of $A(p)$ and satisfying the analytic criterion

$$(1-\xi)\frac{(p-q+1)!}{p!}\frac{(f(z))^{q-1}}{z^{p-q+1}} + \xi\frac{(p-q)!}{p!}\frac{(f(z))^q}{z^{p-q}} \prec h(z), \tag{2}$$

Let $h(z)$ be the analytic function with $|h(z)| \le 1$ $(z \in \mathfrak{U})$ where

$$h(z) = d_0 + d_1 z + d_2 z^2 + \cdots, \quad (z \in \mathfrak{U}). \tag{3}$$

For two functions f and J in A, if there exist functions Ø(z) and ω(z) analytic in U, with ω(0)=0 such that |Ø(z)|<1, |ω(z)|<1, for all z∈U, then f(z)=J(ω(z)) f is said to be subordinate to J so that f(z)≺J(z) also if ω(z)=z, then f(z)=Ø(z)J(z) and it is said that f is majorized by J and written f(z) "J(z). Hence, it is obvious that quasi-subordination is the generalization of subordination and majorization, and many researchers introduced, for example, [1,2,4,5-9]and investigated subclasses of But the coefficient estimate problem of Fekete- Szego is still an open problem.

The following Lemma is required to demonstrate our findings.

Lemma 1. (Keogh and Merks [3]): *Let w be the analytic function in U with $w(0) = 0, |p(z)| < 1$ and*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots. \tag{4}$$

Then $|w_2 - t w_1^2| \le \max\{1, |t|\}$, where $t \in \mathbb{C}$. The result is sharp for functions $w(z) = z^2$ or $w(z) = z$.

Lemma 2. [3]**:** *Let $\vartheta(z)$ be an analytic function in U with $|\vartheta(z)| < 1$ given by (1). Then $|d_0| \le 1$ and $|d_n| \le 1 - |d_0|^2 \le 1$ for $n > 0$.*

2. Main Result

Theorem 1: Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 \neq 0)$. If $f(z) \in \mu_q^p(t, h)$, then

$$|a_{p+1}| \le \frac{B_1}{2(p-q+2)! + (p-q+1)!} \tag{5}$$

$$|a_{p+2} - t a_{p+1}^2| \le \frac{|\xi||B_1|p_1}{4((p-q+3)! + (p-q+2)!)} \max\left\{1, \left|\frac{B_2}{B_1} + 2k_1\right|\right\} \tag{6}$$

where

$$k_1 = \frac{B_1 p!}{4}\left(\frac{1}{(p-q+3)! + (p-q+2)!}\right) - \frac{t\xi B_1 p!}{((p-q+2)! + (p-q+1)!)^2}$$

Proof: If $f \in \mu_q^p(t, h)$, then, there exist analytic functions h, w, with $|h(z)| \le 1$, w(0)=0 and $|w(z)| < 1$ such that

$$(1-\xi)\frac{(p-q+1)!}{p!}\frac{\left(f(z)\right)^{q-1}}{z^{p-q+1}}+\xi\frac{(p-q)!}{p!}\frac{(f(z))^{q}}{z^{p-q}}=\varphi(h(z)) \tag{7}$$

Since

$$\frac{((p-q+2)!+(p-q+1)!)}{p!}a_{p+1}=\frac{\xi B_1C_1}{2} \tag{8}$$

$$\frac{((p-q+3)!+(p-q+2)!)}{p!}a_{p+2}=\xi\left(\frac{B_1C_1}{2}-\frac{B_1C_1}{4}+\frac{B_1C_1^2}{4}\right) \tag{9}$$

By using this fact in Lemma (1.2) and $|w_1|\le 1$, we get the result (5). From some $t\in\mathbb{C}$, we obtain from (8) and (9)

$$a_{p+2}-ta_{p+1}^2=\frac{\xi 2B_1C_2-B_1C_1^2+B_2C_1^2}{4(p-q+3)!+(p-q+2)!}p!-t\frac{B_1^2C_1^2}{4}\left(\frac{\xi p!}{(p-q+2)!+(p-q+1)!}\right)^2. \tag{10}$$

Or equivalently,

$$a_{p+2}-ta_{p+1}^2=\frac{\xi p_1B_1}{4(p-q+3)!+(p-q+2)!}(C_2-vC_1^2)$$

$$v=\frac{1}{2}-\frac{B_2}{2B_1}-k_1 \tag{11}$$

$$k_1=\frac{B_1p!}{4}\left(\frac{1}{(p-q+3)!+(p-q+2)!}\right)-\frac{t\xi B_1p!}{\left((p-q+2)!+(p-q+1)!\right)^2}$$

Our outcome now derives from Lemma 1.1.

Theorem 2.2: *Let $f\in A(p)$ of the form (1), and if it satisfies the condition*

$$\frac{(1-\xi)(p-q+1)!}{p!}\frac{(f(z))^{q-1}}{z^{p-q+1}}+\frac{\xi(p-a)!}{p!}\frac{(f(z))^{q}}{z^{p-q}}\text{"}\varphi(z). \tag{12}$$

Then

$$\left|a_{p+1}\right|\le\frac{(B_1+c_1)}{\xi(p-q+3)!+(p-q+2)!}$$

$$\left|a_{p+2}-ta_{p+1}^2\right|\le$$

$$\left|\frac{B_2p!}{(p-q+3)!+(p-q+2)!}+p!\left(\frac{c_2}{(p-q+3)!+(p-q+2)!}-\frac{tB_1^2c_1^2}{(p-q+2)!+(p-q+1)!}\right)\right|$$

Proof: If $f\in\mu_q^p(t,h)$, then, there exist analytic functions $\varphi(z)$such that

$$\frac{(1-\xi)(p-q+1)!}{p!}\frac{(f(z))^{q-1}}{z^{p-q+1}}+\frac{\xi(p-a)!}{p!}\frac{(f(z))^{q}}{z^{p-q}}=\varphi(z)p(z)$$

where

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots$$

$$\frac{\xi((p-q+2)!+(p-q+1)!)}{p!}a_{p+1}=(B_1+c_1),$$

we get

$$\left|a_{p+1}\right|\le\frac{(B_1+c_1)}{\xi(p-q+3)!+(p-q+2)!} \tag{13}$$

$$\frac{\xi((p-q+3)!+(p-q+2)!)}{p!}a_{p+2}=c_2+B_1c_1+B_2$$

$$a_{p+2} = p!\frac{c_2+B_1c_1+B_2}{\xi((p-q+3)!+(p-q+2)!)}$$

$$a_{p+1}^2 = \left(\frac{(B_1+c_1)}{(p-q+3)!+(p-q+2)!}\right)^2$$

$$a_{p+2} - ta_{p+1}^2 = p!\frac{c_2+B_1c_1+B_2}{\xi((p-q+3)!+(p-q+2)!)} - t\frac{B_1^2+B_1c_1+c_1^2p^2!}{(p-q+3)!+(p-q+2)!},$$

which, together with (13) establishes the desired result of Theorem (2)

Corollary 1: *Let* $f(z) \in A$ *satisfy*

$$\alpha < Re\left\{1+\xi\frac{f(z)}{z}\right\} < \beta \tag{14}$$

Then

$$|a_3 - ta_2^2| \le \frac{\xi(\sigma-\tau)}{\sqrt{2\pi}}\sqrt{1-\frac{cos2\pi(1-\tau)}{(\sigma-\tau)}}max\left\{1,\left|\frac{\theta_2}{\theta_1}(1-t)\theta_1\right|\right\},$$

where

$$\theta_n = \frac{(\sigma-\tau)}{n\pi}i\left[1-e^{2n\pi i((1-\tau)/(\sigma-\tau))}\right].$$

Proof: Let

$$\varphi(z) = 1+\frac{(\sigma-\tau)}{\pi}i\log\left(\frac{1-e^{2n\pi i((1-\tau)/(\sigma-\tau))}z}{1-z}\right).$$

We can see that $\varphi(z)$maps $\mathfrak{U}$ onto the convex domain conformally and is of the form

$$k(z) = 1+\sum_{n=1}^{\infty}\theta_n z^n$$

where $\theta_n = \frac{(\sigma-\tau)}{n\pi}i\left[1-e^{2n\pi i((1-\tau)/(\sigma-\tau))}\right]$. The inequality (14)can be rewritten as the form $1+\xi\frac{f(z)}{z} \prec \varphi(z)$ by Theorem 1, we get the result.

Let $p=1$ and $q=0$ in Theorem 2.1, we get the result.

Corollary 2: *Let* $\varphi(z) = 1+B_1z+B_2z^2+B_3z^3+\cdots(B_1\neq 0)$. *If* $f(z) \in \mu_0^1(t,h)$, *then*

$$|a_3 - ta_2^2| \le \frac{|\xi||B_1|}{16!+3!}\max\left\{1,\left|\frac{B_2}{B_1}+2k_1\right|\right\}$$

where

$$k_1 = \frac{B_1}{4}\left(\frac{1}{30}\right) - \frac{t\xi B_1}{64}$$

Corollary 3: *Let* $f(z) \in A$ *satisfy*

$$\alpha < Re\left\{1+\xi\frac{f''(z)}{z^2}\right\} < \beta$$

Then

$$|a_3 - ta_2^2| \le \frac{\xi(\sigma-\tau)}{\sqrt{2\pi}}\sqrt{1-\frac{cos2\pi(1-\tau)}{(\sigma-\tau)}}max\left\{1,\left|\frac{\theta_2}{\theta_1}\left(1-\frac{3}{2}t\right)\theta_1\right|\right\},$$

where

$$\theta_n = \frac{(\sigma-\tau)}{n\pi} i\left[1 - e^{2n\pi i((1-\tau)/(\sigma-\tau))}\right].$$

3. Application to stealth combat aircraft

An object's RCS is the cross-section it projects to an electromagnetic wave that is probing it. Using camouflage patterns on the object, the RCS is modifiable. The RCS can also be smaller than the physical size. Stealth or Low Observable (LO) Technology is the term used to describe this technique of making objects less visible [10]. However, the ideal situation would be for the thing to be undetectable. According to Fermat's principle, perfect invisibility cannot be achieved in a material with a constant refractive index. In electromagnetic cloaking, an item is enclosed in a cloak to make it invisible. We know that an item only becomes visible when incident light reflects off of it, or when it makes its presence known by casting a shadow or functioning as a radiation source. (See, for example, [11,12]) indicates that a cloak made of met material, where the refractive index changes spatially, may be tailored to allow an incident electromagnetic wave to be guided through it, affording free space when viewed from the outside [15]. To ensures the cloak does not reflect or throw a shadow in the transmitted field. A viewing device does not pick up on the cloak. Furthermore, if the faults are exponentially tiny, the cloak reduces radiation dispersion from the object. (See Figure 1)

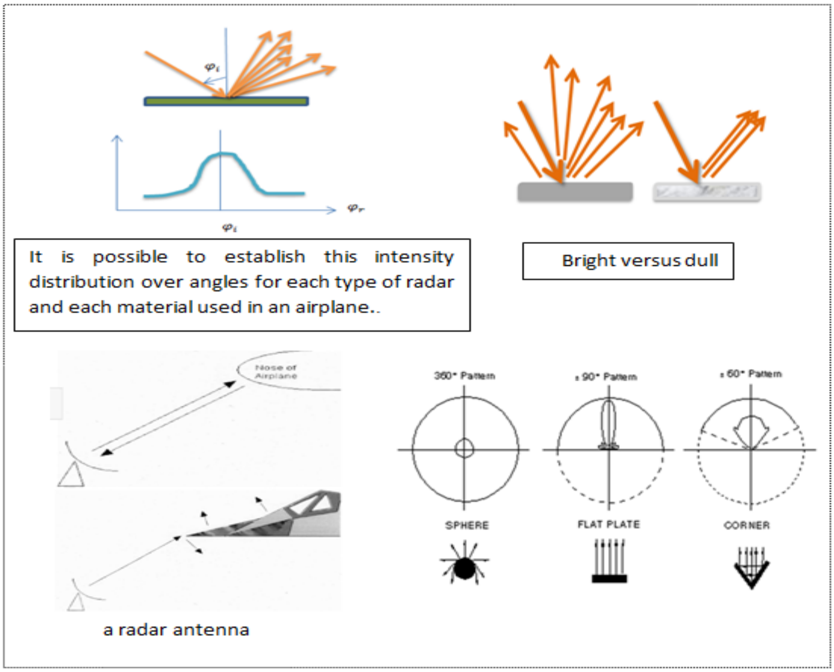

Figure 1
Edge reflections and scattering

As a result, the object's visibility decreases to the point where the detector's resolving power is insufficient to detect it. As a result, the item is no longer visible to the detector. There are claims in the published literature [11,12] that electromagnetic cloaking, previously considered technologically unfeasible, is now achievable when the cloak and the shrouded item exhibit circular symmetry in at least one plane, notably spheres and cylinders. It is a laminar or two-dimensional cloaking in the cross-section (or projection). See, for example, [13,14] for some notable contributions to this field of study. The two-dimensional cloak and the shrouded item are mathematically represented as simple linked areas in the complex plane, with the latter being a subset of the former. Both areas are identical to conformal maps on the unit disc U, according to the Riemann mapping theorem. It is necessary that $g(z) \prec Q(z)$ $(z \in U)$ suppose the cloaked item is represented by g(z) and the cloak by $Q(z)$.

$f(re^{i\varphi}), z = re^{i\varphi}$, (see Figure 2)

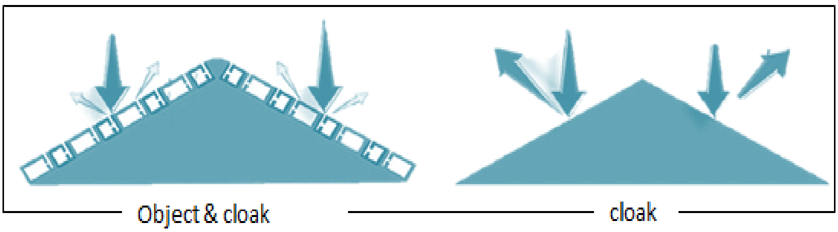

Figure 2
The cloaked object $g(z)$ and the cloak $Q(z)$.

References

[1] AL-Ameedee, S. A., Atshan, W. G., & AL-Maamori, F. A. Second, Hankel determines for certain subclasses of bi-univalent functions. In *Journal of Physics: Conference Series*, Vol. 1664, No. 1, p. 012044. IOP Publishing (2020, November).

[2] Al-Ameedee, S. A., Atshan, W. G., & Al-Maamori, F. A., Coefficients estimates of bi-univalent functions defined by new subclass function. In *Journal of Physics: Conference Series*, Vol. 1530, No. 1, p. 012105. IOP Publishing (2020, May).

[3] Keogh, F. R., & Merkes, E. P., A coefficient inequality for certain classes of analytic functions. *Proceedings of the American Mathematical Society*, 20(1), 8-12 (1969).

[4] Murugusundaramoorthy, G., & Janani, T., Sigmoid function in the space of univalent λ-pseudo starlike functions. *Int. J. Pure Appl. Math*, 101(1), 33-41 (2015).

[5] Fadipe-Joseph, O. A., Oladipo, A. T., & Ezeafulukwe, U. A., Modified sigmoid function in univalent function theory. *Int. J. Math. Sci. Eng. Appl*, 7(7), 313-317 (2013).

[6] Duren, P. L., Coefficients of univalent functions. *Bulletin of the American Mathematical Society*, 83(5), 891-911 (1977).

[7] Olatunji, S. O., Gbolagade, A. M., Anake, T., & Fadipe-Joseph, O. A., Sigmoid function in the space of univalent function of Bazilevic type. *Scientia Magna*, 9(3), 43 (2013).

[8] Olatunji, S. O., Dansu, E. J., & Abidemi, A., On a Sakaguchi type class of analytic functions associated with quasi-subordination in the space of modified sigmoid functions. *Electronic Journal of Mathematical Analysis and Applications*, 5(1), 97-105 (2017).

[9] Olatunji, S. O., Sigmoid function in the space of univalent λ-pseudo starlike function with Sakaguchi type functions. *J. Progr. Res. Math*, 7(4), 1164-1172 (2016).

[10] Rao, G. A., & Mahulikar, S. P., Integrated review of stealth technology and its role in airpower. *The Aeronautical Journal*, 106(1066), 629-642 (2002).

[11] Leonhardt, U., Optical conformal mapping. *Science*, 312(5781), 1777-1780 (2006).

[12] Schurig, D., Mock, J. J., Justice, 1. B., Cummer, S. A., Pendry, J. B., Starr, A. F., & Smith, D. R., Metamaterial electromagnetic cloak at microwave frequencies. *Science*, 314(5801), 977-980 (2006).

[13] Smith, D. R., Urzhumov, Y., Kundtz, N. B., & Landy, N. I., Enhancing imaging systems using transformation optics. *Optics Express*, 18(20), 21238-21251 (2010).

[14] Yang, R., Tang, W., & Hao, Y., Wideband beam-steerable flat reflectors via transformation optics. *IEEE Antennas and Wireless Propagation Letters*, 10, 1290-1294 (2011).

[15] Shelash, Hader Baqer, Ahmad, Hasanain Hamid & Obaid, Ahmed J. On characterizations some of subgroups of the dihedral group D2n, *Journal of Discrete Mathematical Sciences and Cryptography*, 1-6, DOI: 10.47974/JDMSC-1553.

Received October, 2023