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Expansivity and shadowing in fuzzy metric space

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Abstract

We define expansive and topological stability to homeomorphisms on $\mathfrak{F}\,\mathfrak{M}$ - spaces. We offer some basic features of $\mathfrak{F}$ - expansive homeomorphism. Further, we prove Walter's theorem in the context from a $\mathfrak{F}\,\mathfrak{M}$ - space, i.e., F- expansive with the F- shadowing feature is $\mathfrak{F}$ - topologically stable.

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1. Introduction

Dynamical systems theory is sometimes said to be the study of the behaviors of map orbits. Among the many methods of studying dynamical systems, we select the path of general topology [7]. Among the concepts are topological stability and the pseudo-orbit tracking feature, also known as shadowing, which is essential in topological dynamics. Utz presented The concept of expansion ([2]), which is essential in dynamic systems theory on metric spaces. Another interesting setup for studying the dynamics is fuzzy clusters and fuzzy metric space. Fuzzy set theory was introduced by Zadeh in 1965, and later, Kramosel and Michalek in 1975 ([3]) defined fuzzy metric spaces, which different authors have defined. In this paper, we used the definition of expansive given in [4] and obtained valid results because the parameter t is constant while the definition of expansive in [1]. The purpose of

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using the expansive definition in [4] is that the proof method is clear and can be used to study other chaotic properties and obtain valid results without the need to take inf for all t values as in the definition in [1,8,9]. In section 2, we define fuzzy metric space,$t - \mathfrak{F}$ - expansive and $\mathfrak{F}$ $-$shadowing feature. In section 3, we construe the initial features of $\mathfrak{F}$ $-$expansiveness and shadowing features. In section 4, we show the essential conclusion of the article Theorem4.5, the $\mathfrak{F}-$version of Walter's Theorem. It stipulates each $\mathfrak{F}-$expansive homeomorphism satisfactory to the $\mathfrak{F}-$shadowing feature is $\mathfrak{F}-$topological stable.

2. Preliminaries

Let's remember the definition of fuzzy metric space [3].A binary process $* : [0,1] \times [0,1] \to [0,1]$called is a continuous triangular norm $(t - norm)$ if $*$ it meets the next terms:

1) $*$ is associative and commutative;
2) $*$ is continuous;
3) $\mathfrak{a} * 1 = \mathfrak{a}$ for everyone $\mathfrak{a} \in [0,1]$;
4) $\mathfrak{a} * \mathfrak{b} \leq \mathfrak{c} * \mathfrak{d}$ whenever $\mathfrak{a} \leq \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{d}, (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in [0,1])$.

Definition 2.1 (Kramosil and Michalek [3]): A fuzzy metric space (briefly $\mathfrak{F}\mathfrak{M}-$ space) is a ternary n $(X, \mathcal{M},*)$ where X is not empty set $*$ is a continuous $t-$ norm, and $\mathcal{M} : X \times X \times (0,\infty) \longrightarrow [0,1]$ is a mapping which has the next features :

to each $x, y, z \in X$ and $t, s > 0$:

a) $\mathcal{M}(x, y, t) > 0$;
b) $\mathcal{M}(x, y, t) = 1$ if and only if $x = y$;
c) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
d) $\mathcal{M}(x, z, t + s) \geq \mathcal{M}(x, y, t) * \mathcal{M}(y, z, s)$;
e) $\mathcal{M}(x, y,\bullet) : (0,\infty) \longrightarrow [0,1]$ is a continuous map.

We are considered a topology on a $\mathfrak{F}\,\mathfrak{M}-$ space $(X, \mathcal{M},*)$ as below: Given a sequence. (x_n) in X, that we say (x_k) fuzzy converges to x, where $x \in X$, if to each $0 < \varepsilon < 1$ there are $N \in \mathbb{N}$ s.t $\mathcal{M}(x, x_k, t) > 1 - \varepsilon$ to $each$ $k \geq N$. Following this, the converging sequence converges to a singular point. In such a situation, we say the sequence is convergent. Point out $x_k \overset{\mathcal{M}}{\to} x$ whenever x_k is fuzzy convergent to x. We say that a subset $O \subset X$ is $open$ if to each sequence (x_k) converging for some point in O, there is $N \in \mathbb{N}$ s.t $x_k \in O$ For everyone $k \geq N$.That we say $(X, \mathcal{M},*)$ is $\mathfrak{F} - compact$ if each sequence has a $convergent$ $sub-sequence$ in X.

A higher idea can also be obtained through the opened covers. Given $f : X \to X$ and $x_0 \in X$, we say that f is $\mathfrak{F} -$ $continuous$ in if x_0 to each sequence $x_n$in X like that$x_k \overset{\mathcal{M}}{\to} x_0$ it holds that. $f(x_k) \overset{\mathcal{M}}{\to} f(x)$ .Equivalently, to every $0 < \varepsilon < 1$ there is $0 < \delta < 1$ with the feature that if $y \in X$ satisfies $\mathcal{M}(x_0, y, t) > 1 - \delta$ then $\mathcal{M}(f(x_0), f(y), t) > 1 - \varepsilon$. We say that a map is a $\mathfrak{F}-$ *homeomorphism* if it is $\mathfrak{F}-$ *continuous bijection* map with $\mathfrak{F}-$ *continuous inverse* [1].

Remark 2.1[4]: Suppose(X, d) is being metric space(briefly $\mathfrak{M}-$ space). Then $\mathcal{M}(x, y, t) = \mathcal{M}_d(x, y, t) = \frac{t}{t+d(x,y)}$ with the $t-$ norm $a * b = ab$ *is a* $\mathfrak{F}-$ metric defined on X (is called the criterion $\mathfrak{F}\mathfrak{M}-$ space [5]), the topology τ_d resulting from the metric d and the topology $\tau_{\mathcal{M}}$ are same.

Definition 2.2 [4]: Suppose$(X, \mathcal{M}, *)$ be a $\mathfrak{F}\mathfrak{M}-$ space. We say so $\mathfrak{F}-$ *homeomorphism* $f : X \to X$ has $\mathfrak{F}-$ *pseudo* $-$ *orbit tracing features on* X, if to every $0 < \varepsilon < 1$ *and* $t > 0$, there are $\delta \in (0,1)$ so that for a given sequence $\{x_k\}$ with $\mathcal{M}(f(x_k), x_{k+1}, t) > 1 - \delta$ for $k \in \mathbb{Z}$ (called δ-*pseudo orbit*) there are a point $p \in X$ s.t $\mathcal{M}(f^k(p), x_k, t) > 1 - \varepsilon$ for $k \in \mathbb{Z}$.

Definition 2.3 [1]: Let $(X, \mathcal{M}, *)$ be a $\mathfrak{F}\mathfrak{M}-$ spac.We say that $\mathfrak{F}-$ *homeomorphism* $f : X \to X$ is $t - \mathfrak{F}-$ expansive if there is $e \in (0,1)$ s.t to given $x \neq y$ and $t > 0$,there are $k \in \mathbb{Z}$ s.t $\mathcal{M}(f^k(x), f^k(y), t) \leq 1 - e$.

Theorem 2.4 [6]: *If* $(X, \mathcal{M}, *)$ *is a fuzzy metric space, then:*

i) *For given* $a, b \in [0,1], a * b = 1$ *implies* $a = b = 1$*;*
ii) *The inequality* $t \leq s$ *implies* $\mathcal{M}(x, y, t) \leq \mathcal{M}(x, y, s)$, *where* $x, y \in X$

Proof: One can deduce the first property immediately. To prove (ii) put$= t + r$ then

$\mathcal{M}(x, y, s) = \mathcal{M}(x, y, t + r) \geq \mathcal{M}(x, y, t) * \mathcal{M}(y, y, r) = \mathcal{M}(x, y, t) * 1 = \mathcal{M}(x, y, t)$. One can prove the following theorem by direct calculations.

3. Public features

Lemma 3.1: *Suppose* $(X_1, \mathcal{M}_1, *_1)$ *and* $(X_2, \mathcal{M}, *_2)$ *be two fuzzy compact metric spaces (briefly* $\mathfrak{F}\mathfrak{C}\mathfrak{M}-$ *spac,* $\varphi : X_1 \to X_2$ *be* $\mathfrak{F}-$ *homeomorphism .To each* $0 < \varepsilon < 1$ *there are* $\delta \in (0,1)$ *with the property*

$$\mathcal{M}_1(x, y, t) > 1 - \delta, then\ \mathcal{M}_2(\varphi(x), \varphi(y), t) > 1 - \varepsilon.$$

Proof: Suppose that there is a pair of sequences. (x_k) , (y_k) in X_1 with property that $\mathcal{M}_1(x_k, y_k, t) > 1 - \frac{1}{k}$. By fuzzy compact, we can assume that $x_k \xrightarrow{\mathcal{M}_1} x$ and $y_k \xrightarrow{\mathcal{M}_1} y$ with $x, y \in X_1$. since given $t > 0$ and each $0 < \eta < 1$we have for $k \in N$ we have

$$\mathcal{M}_1(x, y, t) \geq \mathcal{M}_1\left(x, x_k, \frac{t}{3}\right) *_1 \mathcal{M}_1\left(x_k, y_k, \frac{t}{3}\right) *_1 \mathcal{M}_1\left(y_k, y, \frac{t}{3}\right) > (1 - \eta) *_1 \left(1 - \frac{1}{k}\right) *_1 (1 - \eta).$$

Then $x = y$

Suppose $m \in \mathbb{N}$ s.t

$(1 - \frac{1}{m}) *_2 (1 - \frac{1}{m}) > 1 - \varepsilon$. And then with continuation to φ in x,

$$\mathcal{M}_2(\varphi(x_k), \varphi(y_k), t) \geq \mathcal{M}_2\left(\varphi(x_k), \varphi(x), \frac{t}{2}\right) *_2 \mathcal{M}_2\left(\varphi(y), \varphi(y_k), \frac{t}{2}\right),$$
$$> \left(1 - \frac{1}{m}\right) *_2 \left(1 - \frac{1}{m}\right),$$
$$> 1 - \varepsilon.$$

Corollary 3.2: *Let* $(X_1, \mathcal{M}_1, *_1)$ *be* $\mathfrak{F}\,\mathfrak{C}\,\mathfrak{M}$ *−space and* $f: X_1 \to X_1$ *is being* $\mathfrak{F} - homeomorphism$. *For every* $0 < \varepsilon < 1\ and\ N \geq 1\ there\ exists\ 0 < \delta < 1$ *with the features that if* $\mathcal{M}_1(x, y, t) > 1 - \delta$, *thereafter* $\mathcal{M}_1(f^k(x), f^k(y), t) > 1 - \varepsilon\ for\ each\ |k| \leq N$.

We say that $\varphi: X_1 \to X_1$ is $a\ semi - conjugation$ (or a conjugation) between g and f if it is a $\mathfrak{F} - continuous$ map (or an $homeomorphism$) on $(X_1, \mathcal{M}_1, *_1)$ like that $\varphi \circ f = \mathrm{g} \circ \varphi$. In the latter case, g is named disorder of f.

Proposition 3.3: Suppose $(X_1, \mathcal{M}, *_1)$ and $(X_2, \mathcal{M}_2, *_2)$ be two $\mathfrak{F}\mathfrak{C}\mathfrak{M} -$ spaces. Suppose $f: X_1 \to X_1\ and\ \mathrm{g} : X_2 \to X_2$ be two $\mathfrak{F} - homeomorphisms$ and $\varphi : X_1 \to X_2$ be a $conjugation$ between f and g. Then $\mathrm{g}\ is\ t - \mathfrak{F} -$ expansive if and only if f is so.

Proof: We suppose that $0 < \delta < 1$ is expansive fixed from g .

$$\mathcal{M}_2(\mathrm{g}^k(\varphi(x), \mathrm{g}^k(\varphi(y), t) > 1 - \delta.$$

Let $0 < e < 1$ of Lemma 3.1 to the homeomorphism φ^{-1} and $\delta > 0$. Given $x, y \in X_1$. Since $f = \varphi^{-1} \circ \mathrm{g} \circ \varphi$ then for every $k \in \mathbb{Z}$ it follows that $\mathcal{M}_2(\mathrm{g}^k(\varphi(x), \mathrm{g}^k(\varphi(y), t) =$

$$\mathcal{M}_1(\varphi^{-1}(\mathrm{g}^k(\varphi(x))), \left(\varphi^{-1}\left(\mathrm{g}^k(h(y))\right), t\right) = \mathcal{M}_1(f^k(x), f^k(y), t) > 1 - e.$$

We conclude that f is $t - \mathfrak{F}$ −expansive.

Conversely, we suppose so $0 < e < 1$ is an expansive fixed to f .

$$\mathcal{M}_1(f^k(x), f^k(y), t) > 1 - e$$

Let $0 < \delta < 1$ be of Lemma 3.1 the homeomorphism φ and $e > 0$. Given $w, z \in X_2$. Since $\mathrm{g} = \varphi \circ f \circ \varphi^{-1}$then for every $k \in \mathbb{Z}$ it follows that

$$\mathcal{M}_1(f^k(x), f^k(y), t) = \mathcal{M}_2(\varphi(f^k(\varphi^{-1}(z))), \varphi(f^k(\varphi^{-1}(w))), t)$$
$$= \mathcal{M}_2(\mathrm{g}^k(z), \mathrm{g}^k(w), t) > 1 - \delta,$$

We conclude that g is $t - \mathfrak{F}$ −expansive.

Proposition 3.4: Suppose $(X_1, \mathcal{M}, *_1)$ and $(X_2, \mathcal{M}_2, *_2)$ be two $\mathfrak{F}\mathfrak{C}\mathfrak{M} -$ spaces. Suppose

$f : X_1 \to X_1\ and\ \mathrm{g} : X_2 \to X_2$ be two $\mathfrak{F} - homeomorphisms$ anφ : $X_1 \to X_2$be a $conjugation$ between f and g . Then g has $\mathfrak{F} - shadowing\ feature$ if and only if f is so.

Proof: We suppose $\mathfrak{g}$ has the $\mathfrak{F}$ −shadowing feature. Through Lemma 3.1, to each $0 < \varepsilon_1 < 1$ there is $0 < \varepsilon < 1$ to φ^{-1}. Of the $\mathfrak{F} - shadowing\ feature$ there is $\delta \in (0,1)$ with a feature that each $\delta - fuzzy\ oseudo\ orbit\ (y_k)_{k\in\mathbb{Z}}$ to $\mathfrak{g}$ maybe $\varepsilon - fuzzy\ tracking$ at some point in X_2. Suppose $\delta_1 > 0$ be a figure satisfactory Lemma $3.\mathbf{1}$ to δ and φ. Then, it is sufficient to see it every $\delta_1 - fuzzy\ pseudo\ orbit\ (x_k)_{k\in\mathbb{Z}}\ to\ f\ is\ \varepsilon_1$ −fuzzy tracking through some point.This situation $y_k = \varphi(x_k)$ to $k \in \mathbb{Z}$.Since $\mathcal{M}_1(f(x_k), x_{k+1}, t) \geq 1 - \delta_1$ for $k \in \mathbb{Z}$, we have

$$\mathcal{M}_2(\mathfrak{g}(y_k), y_{k+1}, t) = \mathcal{M}_2(\varphi(f(x_k)), \varphi(x_{k+1}), t) \geq 1 - \delta.$$

Thus $(\mathfrak{g}_k)$ is a $\delta -$ fuzzy pseudo orbit to g and so to $k \in \mathbb{Z}$. $\mathcal{M}_2(\mathfrak{g}^k(z), y_{k+1}, t) \geq 1 - \varepsilon$.For some $z \in X_2$. Subsequently, to $k \in \mathbb{Z}$, it follows that

$$\mathcal{M}_1(f^k(\varphi^{-1}(z)), x_i, t)$$
$$= \mathcal{M}_1(\varphi^{-1}(\mathfrak{g}^k(z)), \varphi^{-1}(x_{k+1}), t) \geq 1 - \varepsilon_1.$$

Hence, f has the $\mathfrak{F} - shadowing\ feature$.

Conversely, we suppose f has a shadowing feature. By Lemma$3.\mathbf{1}$, to each $0 < \varepsilon < 1$ there is $0 < \varepsilon_1 < 1$ to φ .Of the $\mathfrak{F} - shadowing\ feature$ there is $0 < \delta_1 < 1$ with a feature that each $\delta_1 - fuzzy\ pseudo\ orbit (x_k)_{k\in\mathbb{Z}}$ to $f\ perhap\ \varepsilon_1 - fuzzy\ tracking$ at some point in X_1.Suppose $\delta > 0$ be a satisfactory figure Lemma 3.**1** to δ_1 and φ^{-1}. Then it's sufficient to see it every $\delta - fuzzy\ pseudo\ orbit\ (y_k)$ to $\mathfrak{g}$ is $\varepsilon - fuzzy\ tracking$ through some point. This situation $x_k = \varphi^{-1}(y_k)$ to $i \in \mathbb{Z}$.Since $\mathcal{M}_2(g(y_k)), y_{k+1}, t) \geq 1 - \delta$ to $k \in \mathbb{Z}$, we have

$$\mathcal{M}_1(f(x_k)), x_{k+1}, t)$$
$$= \mathcal{M}_1(\varphi^{-1}(\mathfrak{g}(y_k)), \varphi^{-1}(y_{k+1}), t) \geq 1 - \delta_1.$$

Thus (x_k) is a $\delta_1 - fuzzy\ pseudo\ orbit$ to f and to $k \in \mathbb{Z}$. $\mathcal{M}_1(f^k(y), x_{k+1}, t) \geq 1 - \varepsilon_1$. For some $y \in X_1$. Subsequently, to $i \in \mathbb{Z}$ it follows that

$$\mathcal{M}_2(\mathfrak{g}^k(\varphi(y)), y_i, t)$$
$$= \mathcal{M}_2(\varphi(f^k(y)), \varphi(x_{k+1}), t) \geq 1 - \varepsilon.$$

Therefore $\mathfrak{g}$ has the $\mathfrak{F} - shadowing\ feature$. □

Corollary 3.5: *The $\mathfrak{F}$ − expansiveness is independent of the select follower $\mathfrak{F}$ − metric on X_1 whenever the fuzzy metric induction himself compact $\mathfrak{F}$ − topology on X_1.*

4. Stability

Let $(X_1, \mathcal{M}_1, *_1)$ be a $\mathfrak{F}\mathfrak{M}$ −space and $\mathcal{F}, \mathcal{G} : X_1 \to X_1$ be $\mathfrak{F} - continuous$ maps, we define

$$\mathcal{W}(\mathcal{F}, \mathcal{G}) = \mathcal{M}_1(\mathcal{F}(x), \mathcal{G}(x), t).$$

Given $\mathfrak{F} - continuous$ map $\mathfrak{g} : X_1 \to X_1$ defined on the $\mathfrak{F}$ − metric space$(X_1, \mathcal{M}_1, *_1)$,$\delta \in (0,1)$ that we say $\mathfrak{g}$ is $\delta - fuzzy\ close$ to f whenever $\mathcal{W}(f, \mathfrak{g}) > 1 - \delta$.

With those symbols, we are considered the next definition of $\mathfrak{F}-topological\ stability$. Point out through I_{X_1} the identity map on X_1.

Definition 4.1 [1]**:** We say so $f\ is\ \mathfrak{F}-$ topological stable if per $\varepsilon \in (0,1)$ there are $\delta \in (0,1)$ like that to each $\mathfrak{F}-homeomorphism$ g $\delta-fuzzy\ close$ for f yonder a semi-conjugation

$\varphi : X_1 \rightarrow X_1\ between\ f$ and g, such that $\mathcal{M}(\varphi, I_{X_1}) \geq 1 - \varepsilon$.

Lemma 4.2: *Let* $(X_1, \mathcal{M}_1, *_1)$ *be* $\mathfrak{FCM}-$ *space and* $f : X_1 \rightarrow X_1$ *be* $\mathfrak{F}$ *−expansive homeomorphism with expansive fixed* $0 < e < 1$. *Given* $0 < \varepsilon < 1$ *there are* $N \geq 1$ *like that if* $\mathcal{M}_1(f^k x), f^k(y), t) \geq 1 - e, for\ every\ |k| \leq N, then$

$$\mathcal{M}_1(x, y, t) > 1 - \varepsilon.$$

Proof: Let $0 < \varepsilon < 1$ be given then to each $N \geq 1$ there exists two sequences $(x_N), (y_N)$ in X_1such that for all $|j| \leq N$

$$\mathcal{M}_1(f^j(x_N), f^j(y_N), t) \geq 1 - e$$

and there is a sequence(t_N) in $(0, +\infty)$ with the property that $\mathcal{M}_1(x_N, y_N, t_N) \leq 1 - \varepsilon$. By fuzzy compactness, we can assume that $x_N \overset{\mathcal{M}_1}{\longrightarrow} x$ and $y_N \overset{\mathcal{M}_1}{\longrightarrow} y$ with $x, y \in X_1$. Let $n \in \mathbb{Z}\ and\ t > 0$. For every $N \in$ such that$|n| \leq N$ we have

$$\mathcal{M}_1(f^n(x), f^n(y), t)$$
$$\geq \mathcal{M}_1\left(f^n(x), f^n(x_N), \tfrac{t}{3}\right) *_1 \mathcal{M}_1\left(f^n(x_N), f^n(y_N), \tfrac{t}{3}\right) *_1$$
$$\mathcal{M}_1\left(f^n(y_N), f^n(y), \tfrac{t}{3}\right)$$
$$\geq \mathcal{M}_1\left(f^k(x), f^k(x_N), \tfrac{t}{3}\right) *_1 (1 - e) *_1 \mathcal{M}_1(f^k(y_N), f^k(y), \tfrac{t}{3}).$$

By continuity, we obtain that $\mathcal{M}_1(f^k(x), f^k(y), t) \geq 1 - e$, then $x = y$.

Lemma 4.3: *Let* $(X_1, \mathcal{M}_1, *_1)$ *be* $\mathfrak{F}$ *−metric space and* $f : X_1 \rightarrow X_1$ $\mathfrak{F}-$ *expansive homeomorphism with the* $\mathfrak{F}-shadowing\ feature$. *Exists* $0 < \varepsilon < 1$ *and* $0 < \delta < 1$*as follows feature each* $\delta-fuzzy\ pseudo\ orbit(x_k)_{k\in\mathbb{Z}}$ *maybe* $\varepsilon-fuzzy\ tracking\ in\ f$ *singular point in* X_1.

Proof: Suppose $0 < e < 1$ be $\mathfrak{F}-$ expansive fixed from f .pick $0 < \varepsilon < 1$ achieving $(1 - \varepsilon) *_1 (1 - \varepsilon) \geq (1 - e)$ and select $\delta \in (0,1)$ conforming to ε of the $\mathfrak{F}-shadowing$ feature. Suppose that there are a $\delta-fuzzy\ pseudo\ orbit\ (x_k)_{k\in\mathbb{Z}}$ maybe to f that $\varepsilon-fuzzy\ traced$ by x and y.Given $k \in \mathbb{Z}$,$t > 0$ we have

$$\mathcal{M}_1(f^k(x), f^k(y), t) > \mathcal{M}_1(f^k(x), x_k, \tfrac{t}{2}) *_1 \mathcal{M}_1(x_k, f^k(y), \tfrac{t}{2}) >$$
$$(1 - \varepsilon) *_1 (1 - \varepsilon) \geq 1 - e.$$

Since f be $t-\mathfrak{F}$ −expansive map, we've $x = y$.

Theorem 4.5: *(Walter $\mathfrak{F}-$ version) Every $\mathfrak{F}-expansive$ homeomorphism with the $\mathfrak{F}-shadowing\ feature$ on an $\mathfrak{FCM}$ -space is fuzzy topological stable.*

Proof: Suppose $0 < e < 1$ be an expansive fixed from f. Pick $0 < \varepsilon < 1$ and $0 < \delta < 1$ of Lemma 4.**3**. We can assume so $(1-\varepsilon) *_1 (1-\varepsilon) *_1 (1-\varepsilon) > (1-e)$.

Allow $\mathfrak{g} : X_1 \to X_1$be a homeomorphism like that $\mathcal{W}(f,\mathfrak{g}) > 1-\delta$ $and\ x \in X_1$. We claiming it the orbit $(\mathfrak{g}^k(x))_{k\in\mathbb{Z}}$ is a $\delta - fuzzy\ pesudo\ orbit$ to f. Actually, through the identification of $\mathcal{W}(f,\mathfrak{g})$, to each $k \in \mathbb{Z}$ and $t > 0$ .we have

$$\mathcal{M}_1\big(f\left(\mathfrak{g}^k(x)\right), \mathfrak{g}^{k+1}(x), t\big) = \mathcal{M}_1\big(f\left(\mathfrak{g}^k(x)\right), \mathfrak{g}\big(\mathfrak{g}^k(x)\big), t\big) \geq \mathcal{W}(f,\mathfrak{g}) > 1 - \delta.$$

Through Lemma 4.3, there is a unique $\varphi(x) \in X_1$ like that $(\mathfrak{g}^k(x))_{k\in\mathbb{Z}}$ maybe $\varepsilon - fuzzy\ tracking$ through $\varphi(x)$ to f . This identifies $\varphi : X_1 \to X_1$ which satisfies

$$\mathcal{M}_1\big(f^k(\varphi(x)), \mathfrak{g}^k(x), t\big) > 1 - \varepsilon, \forall k \in \mathbb{Z}, \forall x \in X.$$

Following we display it φ is a$semi-conjugation\ between\ f$ and $\mathfrak{g}$.

$\mathcal{M}_1(f^k(\varphi(\mathfrak{g}(x))), \mathfrak{g}^{k+1}(x), t) > 1-\varepsilon, \forall k \in \mathbb{Z}$,and for every $k \in \mathbb{Z}$,

$$\mathcal{M}_1\left(f^k\left(f\left(\varphi(x)\right)\right), \mathfrak{g}^{k+1}(x), t\right)$$
$$= \mathcal{M}_1\big(f^{k+1}(\varphi(x)), \mathfrak{g}^{k+1}(x), t\big) > 1 - \varepsilon.$$

We have that $\varphi\big(\mathfrak{g}(x)\big)$ and $f\left(\varphi(x)\right)$ scoring in X_1, that $\varepsilon - fuzzy\ trace$ the δ −fuzzy pseudo orbit $\left(\mathfrak{g}^{k+1}(x)\right)_{k\in\mathbb{Z}}$. Through Lemma 4.3, it bears that $\varphi \circ \mathfrak{g} = f \circ \varphi$. At last, we display it $\varphi\ is\ \mathfrak{F}-continuous$. Suppose $0 < \sigma < 1$. Through Lemma 4.2**,** we can select N like that $\mathcal{M}_1(f^k(u), f^k(v), t) \geq 1 - e, \forall |k| \leq N$, which implies that$\mathcal{M}_1(u,v,t) > 1-\sigma$. Furthermore it, through Lemma 3.1 we select $\eta > 0$, like that $M_1(x,y,t) > 1 - \eta$ implies that

$$\mathcal{M}_1(\mathfrak{g}^k(x), \mathfrak{g}^k(y), t) \geq 1-\varepsilon, \forall |k| \leq N.$$

For $x, y \in X_1$ such that $\mathcal{M}_1(x,y,t) > 1 - \eta$ and $\xi > 0$ we have

$$\mathcal{M}_1\big(f^k\big(\varphi(x)\big), f^k\big(\varphi(y)\big), \xi\big)$$
$$= \mathcal{M}_1(\varphi(\mathfrak{g}^k(x)), \varphi(\mathfrak{g}^k(y)), \xi)$$
$$\geq \mathcal{M}_1\left(\varphi\big(\mathfrak{g}^k(x)\big), \mathfrak{g}^k(x), \tfrac{\xi}{3}\right) *_1 \mathcal{M}_1\left(\mathfrak{g}^k(x), \mathfrak{g}^k(y), \tfrac{\xi}{3}\right) *_1$$
$$\mathcal{M}_1\left(\mathfrak{g}^k(y), \varphi\big(\mathfrak{g}^k(y)\big), \tfrac{\xi}{3}\right)$$
$$\geq \mathcal{W}(\varphi, I_X) *_1 \mathcal{M}_1(\mathfrak{g}^k(x), \mathfrak{g}^k(y), \tfrac{\xi}{3}) *_1 \mathcal{W}(\varphi, I_X)$$
$$> (1-\varepsilon) *_1 (1-\varepsilon) *_1 (1-\varepsilon)$$
$$> 1 - e$$

For all $|k| < N$. Therefore $\mathcal{M}_1(x,y,t) > 1 - \eta \ implies \ that$
$\mathcal{M}_1(\varphi(x), \varphi(y), t) > 1 - \sigma$,

And so continuity φ is proved.

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