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Elzaki transform decomposition approach to solve Riccati matrix differential equations

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Abstract

Elzaki Transform Adomian decomposition technique (ETADM), which an elegant combine, has been employed in this work to solve non-linear Riccati matrix differential equations. Solutions are presented to demonstrate the relevance of the current approach. With the use of figures, the results of the proposed strategy are displayed and evaluated. It is demonstrated that the suggested approach is effective, dependable, and simple to apply to a range of related scientific and technical problems.

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Keywords: *Riccati matrix differential equations, Non-linear differential equations, Adomian decomposition method, Elzaki transform.*

1. Introduction

The Riccati matrix differential equation (RMDE) was numerically investigated by [1], who used a linear system and a quadratic feedback controller connected to it in reduced form to illustrate an effective computational approach for solving the problem with a finite terminal time. The problem becomes one of analyzing matrix differential equations when this approach is used. It is common for computational modeling of several border physical systems to produce nonlinear differential equations. Analytic solutions to these differential equations are quite difficult to find in most circumstances. The Elzaki decomposition technique is an approximate analytical method that can be used to a variety of equations. Numerous authors have conducted research on RDE and RMDEs. In both pure and applied mathematics, there is a long history

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of studying differential equations and/or algebraic matrices, going back to the early days of mathematical analysis, because these equations are the most common types of nonlinear equations, particularly in mathematical [1-9].

2. Definitions of Elzaki Transformation

Definition 2.1*:* If $\mathcal{J}(t)$ is a function defined for all$t \geq 0$, its Elzaki transform is the integral of $\mathcal{J}(t)$ is defined by $\mathbb{E}$ *[𝒥]* $= \mathbb{T}(s) = s\int_0^\infty \mathcal{J}(t)e^{-\frac{t}{s}}dt.$

Theorem 2.1: *The power series function's coefficients expand by the Elzaki transform as follows*:

$$\mathbb{E}[\mathcal{J}(t)] = \mathbb{T}(s) = \sum_{n=0}^{\infty} n!\, c_n s^{n+2}.$$

Theorem 2.2: *Elzaki transform of n^{th} deriveative $\mathcal{J}^{(n)}(t)$ of $\mathcal{J}(t)$, then for* $n \geq 1$.

$$\mathbb{T}_n(v) = \frac{\mathbb{T}(v)}{v^n} = \sum_{k=0}^{n-1} v^{2-n+k}\mathcal{J}^{(k)}(0).$$

And Elzaki transform of partial derivative as follows:

$$\mathbb{E}\left(\frac{\partial \mathcal{J}(x,\tau)}{\partial \tau}\right) = \frac{1}{v}\mathbb{T}(x,v) - vf(x,0),$$

$$E\left(\frac{\partial^2 \mathcal{J}(x,t)}{\partial t^2}\right) = \frac{1}{v^2}\mathbb{T}(x,v) - \mathcal{J}(x,0) - v\frac{\partial \mathcal{J}(x,0)}{\partial t}$$

3. Solutions of Riccati Matrix Differential Equation (RMDE)

Consider

$$\dot{\mathcal{X}} + \mathcal{X}A + \mathrm{A}^{\mathrm{T}}\mathcal{X} - \mathcal{X}B\mathcal{X} + C(t) = 0 \tag{1.1}$$

where A, B and C are constants $n \times n$ matrices such that $B = B^T$ and $C = C^T$ with initial conditions:

$$\mathcal{X}(0) = G \tag{1.2}$$

When we use ETADM with the initial condition to both sides of (1.1), we obtain:

$$\mathbb{E}\mathcal{X}(t) = G + \mathbb{E}\big(-\mathcal{X}A - \mathrm{A}^{\mathrm{T}}\mathcal{X} + \mathcal{X}B\mathcal{X} - C(t)\big) \tag{1.3}$$

According to the ADM, an infinite sequence of components could be used to decompose the solution (t) :

$$\mathcal{X}(t) = \sum_{n=0}^{\infty} \mathcal{X}_n(t).$$

The nonlinear term $\mathcal{X}B\mathcal{X}$ in equation (1.1) can be expressed as:

$$N(\Upsilon) = \mathcal{X}B\mathcal{X} = \sum_{n=0}^{\infty} \mathcal{A}_n \tag{1.4}$$

Where

$$\mathcal{A}_0(t) = \mathcal{X}_0(t)B\mathcal{X}_0(t),\ \mathcal{A}_1(t) = \mathcal{X}_0(t)B\mathcal{X}_1(t) + \mathcal{X}_1(t)B\mathcal{X}_0(t)$$

$$\vdots$$

$$\mathcal{A}_n(t) = \frac{1}{n!}\frac{d^n}{d\xi^n}\Big[\Big(\sum_{i=0}^{\infty} \xi^i \mathcal{X}_i(t)\Big)B\Big(\sum_{i=0}^{\infty} \xi^i \mathcal{X}_i(t)\Big)\Big]_{\lambda=0}$$

Finally, after applying $\mathbb{E}^{-1}$ of both sides of Eq.(1.3) , we get

$\sum_{n=0}^{\infty} \mathcal{X}_n(t) = G + \mathbb{E}^{-1}[-\sum_{n=0}^{\infty} \mathcal{X}_n(t)\mathcal{A} - \mathcal{A}^T \sum_{n=0}^{\infty} \mathcal{X}_n(t)\mathcal{A} + \sum_{n=0}^{\infty} \mathcal{A}_n(t)]$

The following recursive formula determines the iterates from the previous equation: $\mathcal{X}_0(t) = G + \mathbb{E}^{-1}(-\mathcal{H})$

$$\mathcal{X}_{n+1}(t) = \mathbb{E}^{-1}(-\mathcal{X}_n A - A^T \mathcal{X}_n) + \mathbb{E}^{-1} A_n, \qquad n \geq 0$$

where $\mathcal{X}_0(t) = G + \mathbb{E}^{-1}(-\mathcal{H})$ express the zeroth component obtained by integrating the source term with the given initial condition. If $k \in \mathcal{R}$ then for the approximate solution we have:

$\psi_k = \sum_{n=0}^{k} \mathcal{X}_n(t) = \mathcal{X}_0(t) + \mathcal{X}_1(t) + \cdots + \mathcal{X}_k(t)$ with $\mathcal{X}(t) = \lim_{k\to\infty} \psi(t)$

4. Solution of Nonlinear Riccati Matrix Differential Equation

Two examples are resolved for RMDEs. These results are provided in tabular form that use ETADM.

Example 2.1: Consider RMDE as $\dot{\Upsilon} + \Upsilon\mathcal{A} + \mathcal{A}^T\Upsilon - \Upsilon\mathcal{B}\Upsilon + C(t) = 0$

$$\mathcal{A} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

With initial condition $\Upsilon(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Solution:

$$\begin{pmatrix} \dot{\Upsilon}_{11} & \dot{\Upsilon}_{12} \\ \dot{\Upsilon}_{21} & \dot{\Upsilon}_{22} \end{pmatrix} + \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{pmatrix} - \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{\Upsilon}_{11} & \dot{\Upsilon}_{12} \\ \dot{\Upsilon}_{21} & \dot{\Upsilon}_{22} \end{pmatrix} + \begin{pmatrix} -2\Upsilon_{12} & 2\Upsilon_{11} \\ -2\Upsilon_{22} & 2\Upsilon_{21} \end{pmatrix} + \begin{pmatrix} -2\Upsilon_{21} & -2\Upsilon_{22} \\ 2\Upsilon_{11} & 2\Upsilon_{12} \end{pmatrix} - \begin{pmatrix} 2\Upsilon_{11}\Upsilon_{12} + 2\Upsilon_{11}\Upsilon_{21} & 2\Upsilon^2{}_{12} + 2\Upsilon_{11}\Upsilon_{22} \\ 2\Upsilon_{11}\Upsilon_{22} + 2\Upsilon^2{}_{21} & 2\Upsilon_{12}\Upsilon_{22} + 2\Upsilon_{21}\Upsilon_{22} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\dot{\Upsilon}_{11} = 2\Upsilon_{12} + 2\Upsilon_{21} + 2\Upsilon_{11}\Upsilon_{12} + 2\Upsilon_{11}\Upsilon_{21} - 1$$

Applying the Elzaki transform on the both side

$$\frac{1}{v}E[\Upsilon_{11}(t)] - v\Upsilon_{11}(0) = E[2\Upsilon_{12} + 2\Upsilon_{21} + 2\Upsilon_{11}\Upsilon_{12} + 2\Upsilon_{11}\Upsilon_{21} - 1]$$

Taking inverse Elzaki transform on the both side

$$\Upsilon_{11} = -t + E^{-1}\{ vE[-\Upsilon_{12} - \Upsilon_{21} - \Upsilon^2{}_{11} + \Upsilon_{12}\Upsilon_{21}]\}, \quad \Upsilon_{0_{11}}(t) = -t$$

$$\dot{\Upsilon}_{12} = -2\Upsilon_{11} + 2\Upsilon_{22} + \Upsilon^2{}_{12} + \Upsilon_{11}\Upsilon_{22}$$

Applying the Elzaki transform on the both side

$$\frac{1}{v}E[\Upsilon_{12}(t)] - v\Upsilon_{12}(0) = E[-2\Upsilon_{11} + 2\Upsilon_{22} + \Upsilon^2{}_{12} + \Upsilon_{11}\Upsilon_{22}]$$

$$E[\Upsilon_{12}(t)] = vE[-2\Upsilon_{11} + 2\Upsilon_{22} + \Upsilon^2{}_{12} + \Upsilon_{11}\Upsilon_{22}]$$

$$\Upsilon_{12}(t) = E^{-1}\{vE[-2\Upsilon_{11} + 2\Upsilon_{22} + \Upsilon^2{}_{12} + \Upsilon_{11}\Upsilon_{22}]\}, \quad \Upsilon_{0_{12}}(t) = 0$$

$$\dot{Y}_{21} = 2Y_{22} - 2Y_{11} + 2Y_{11}Y_{22} + 2Y^2{}_{21}$$

Applying the Elzaki transform on the both side

$$\frac{1}{v}E[Y_{21}(t)] - vY_{21}(0) = E[2Y_{22} - 2Y_{11} + 2Y_{11}Y_{22} + 2Y^2{}_{21}]$$

$$Y_{21}(t) = E^{-1}\{vE[2Y_{22} - 2Y_{11} + 2Y_{11}Y_{22} + 2Y^2{}_{21}]\}, \quad Y_{0\,21}(t) = 0$$

$$\dot{Y}_{22} = -2Y_{21} - 2Y_{12} + 2Y_{12}Y_{22} + 2Y_{21}Y_{22} - 1$$

Applying the Elzaki transform on the both side

$$\frac{1}{v}E[Y_{22}(t)] - vY_{22}(0) = E[-2Y_{21} - 2Y_{12} + 2Y_{12}Y_{22} + 2Y_{21}Y_{22} - 1]$$

$$Y_{22}(t) = -t + E^{-1}\{vE[-2Y_{21} - 2Y_{12} + 2Y_{12}Y_{22} + 2Y_{21}Y_{22}]\},$$

$$Y_{0\,22}(t) = -t$$

$$Y_0(t) = \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix}, \quad Y_1(t) = \begin{pmatrix} 0 & \frac{4}{3}t^3 \\ \frac{2}{3}t^3 & 0 \end{pmatrix},$$

$$Y_2(t) = \begin{pmatrix} t^4 - \frac{4}{5}t^5 & 0 \\ 0 & -t^4 - \frac{4}{5}t^5 \end{pmatrix}$$

$$Y_3(t) = \begin{pmatrix} 0 & -\frac{4}{5}t^5 + \frac{304}{315}t^7 \\ -\frac{4}{5}t^5 + \frac{184}{315}t^7 & 0 \end{pmatrix}$$

$$Y_4(t) = \begin{pmatrix} -\frac{16}{30}t^6 + \frac{2236}{2520}t^8 - \frac{1984}{2835}t^9 + \frac{16}{35}t^7 & 0 \\ 0 & \frac{16}{30}t^6 + \frac{2236}{2520}t^8 - \frac{1984}{2835}t^9 + \frac{16}{35}t^7 \end{pmatrix}$$

$$Y_5(t) = \begin{pmatrix} 0 & \frac{32}{105}t^7 - \frac{170}{189}t^9 - \frac{1118}{3150}t^{10} + \frac{130784}{155925}t^{11} \\ \frac{16}{105}t^7 - \frac{1319}{5670}t^9 - \frac{656}{51975}t^{11} - \frac{688}{22680}t^8 - \frac{559}{700}t^{10} & 0 \end{pmatrix}$$

and so, on we may compute which will be more complicated. Each component's first nine iterative solutions can be obtained with Mathematica software. For the component solutions of the analytical approximation solutions utilizing ETADM, Table (1) displays the Absolute Error of the 9-iterate.

Table 1

The results of absolute error for the approximate solutions $Y_9(t)$

t	$ABSY_{11}$	$ABSY_{12}$	$ABSY_{21}$	$ABSY_{22}$
0.1	0	3.24556×10^{-15}	0	0
0.2	0	1.34584×10^{-12}	0	1.561×10^{-12}
0.3	0	7.063454×10^{-10}	0	8.09863×10^{-10}

0.4	0	4.7225×10^{-7}	0	5.02349×10^{-8}
0.5	0	2.13345×10^{-6}	0	1.7542×10^{-6}
0.6	0	1.6508×10^{-5}	0	3.5698×10^{-5}
0.7	0	3.1338×10^{-4}	0	4.0331×10^{-4}
0.8	0	5.4348×10^{-3}	0	2.78110×10^{-3}
0.9	0	2.4561×10^{-2}	0	1.7195×10^{-2}
1.0	0	1.34432×10^{-1}	0	5.84472×10^{-2}

Example 2.1: Use ETADM to solve the following Riccati matrix differential equation

$$\frac{dX}{dt} = -X^2(t) + \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 100\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

With initial condition $X(0) = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

Solution: Using ETADM to solve above equation, we get

$$X_1(t) = \begin{pmatrix}-t - \frac{103}{4}t^2 - \frac{10001}{6}t^3 & -2t - t^2 + \frac{3333}{2}t^3\\ \frac{99}{2}t^2 + \frac{3333}{2}t^3 & -t - \frac{103}{4}t^2 - \frac{10001}{6}t^3\end{pmatrix}$$

And

$$X_2(t) = \begin{pmatrix}t^2 + \frac{4}{3}t^3 + \frac{10003}{6}t^4 + \frac{1000001}{15}t^5 & 3t^2 + \frac{313}{6}t^3 - \frac{29371}{48}t^4 - \frac{333333}{5}t^5\\ -\frac{198}{3}t^3 - \frac{43527}{16}t^4 - \frac{333333}{5}t^5 & t^2 + \frac{4}{3}t^3 + \frac{10003}{6}t^4 + \frac{1000001}{15}t^5\end{pmatrix}$$

$$X_3(t) = \begin{pmatrix}-t^3 + \frac{277}{12}t^4 - \frac{49141}{120}t^5 - \frac{129457993}{1440}t^6 - \frac{1700000017}{630}t^7\\ \frac{165}{2}t^4 + \frac{14496}{40}t^5 + \frac{65919051}{480}t^6 + \frac{188888887}{70}t^7\\ -4t^3 - 104t^4 - \frac{255197}{120}t^5 + \frac{74242031}{1440}t^6 + \frac{188888887}{70}t^7\\ -t^3 + \frac{277}{12}t^4 - \frac{49141}{120}t^5 - \frac{129457993}{1440}t^6 - \frac{1700000017}{630}t^7\end{pmatrix}$$

and so, on we may compute which will be more complicated. Each component's first nine iterative solutions can be obtained with Mathematica software. For the component solutions of the analytical approximation solutions utilizing ETADM, Table (2) displays the Absolute Error of the 9-iterate.

Table 2
The approximate solutions of the fourth components of 9-iterative $\Upsilon_9(t)$

t	$ABSY_{11}$	$ABSY_{12}$	$ABSY_{21}$	$ABSY_{22}$
0.1	0.1	1.3198×10^{-10}	0.1	1.
0.2	0.2	5.26028×10^{-9}	0.2	0.9
0.3	0.345985	8.324127×10^{-8}	0.3000000001	0.9
0.4	0.423151	3.057381×10^{-7}	0.400000007	0.988899929
0.5	0.534529	7.766891×10^{-6}	0.5000000874	0.9888999818
0.6	0.6351207	8.535463×10^{-5}	0.6000002239	0.9999981723
0.7	0.6780456	0.0000435686	0.6999982404	1.000001390
0.8	0.7127870	0.00004523762	0.7999964286	0.9998894527
0.9	0.8156414	0.00015626308	0.9001202008	0.977894618

5. Conclusion

We employed ETADM successfully to obtain approximate analytical solutions for the Riccati Matrix differential equation. This strategy with the initial condition only requires a few iterations and fewer processing steps to achieve a good approximation. This is shown to be a useful, efficient, and trustworthy technique for solving differential equations. We used ETADM to assess the approximate solution of nonlinear RMDE. One example has been created to demonstrate how to use the ETADM to solve nonlinear RMDE. The absolute error have been obtained of the RMDE.

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