

Differential subordinations of meromorphic multivalent harmonic functions

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Abstract

This paper's primary goal is to investigate the harmonic function of the multivalent analytic functions' differential subordination result. For this, we get some results in the complex plane's open unit disk.

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1. Introduction

Let $S_p(\beta)$ denotes the class of functions $f = \theta + \bar{\eta}$ that are harmonic meromorphic multivalent and sense-preserving in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. Where $f = \theta + \bar{\eta} \in S_p(\beta)$, the analytic function θ and η express by

$$\theta(z) = z^{-p} + \sum_{n=0}^{\infty} \sigma_n z^{n-\beta}, \quad \eta(z) = z^{-p} + \sum_{n=0}^{\infty} \tau_n z^{n-\beta} \\ (p, \beta \in \mathbb{N} = \{1, 2, \dots\}; z \in U), \quad (1)$$

Denotes $H_p(\beta)$ subclass of $S_p(\beta)$ consisting of function $= \theta + \bar{\eta}$, where $\theta(z), \eta(z)$ given by $\theta(z) = z^{-p} - \sum_{n=0}^{\infty} \sigma_n z^{n-\beta}$, $\eta(z) = z^{-p} - \sum_{n=0}^{\infty} \tau_n z^{n-\beta}$.

We introduce a class $W_{t,\rho}^\delta(n, p)$ of harmonic function of (1.1) that satisfying the follow

$$Re\left\{(1-\delta)\left(-\frac{(z^p f'(z))'}{\vartheta(p,1)g'(z)}\right)^{ts} + \delta\left(-\frac{(z^p f'(z))'}{\vartheta(p,1)g'(z)}\right)^{t\alpha}\right. \\ \left. \left[p + t\left(\frac{z(z^p f'(z))''}{(z^p f'(z))''} - \frac{zg''(z)}{g'(z)}\right)\right] \geq \frac{\rho}{\vartheta(p,2)}\right\} \quad (2)$$

Where,

$$f(z) = \vartheta(p, 1)z^{-p} + \sum_{n=p+1}^{\infty} \vartheta(p, 1)a_n z^{n-\beta} \\ \vartheta(k_1, k_2) = \frac{k_1!}{(k_1 - k_2)!} = \begin{cases} 1 & k_1 = 0 \\ k_1(k_1 - 1) \dots (k_1 - k_2 + 1) & k_1 \neq 0 \end{cases}, \\ 0 \leq \rho < p, p \in \mathbb{N}, \delta \geq 0, s, \alpha \in \mathbb{C}, t \in \mathbb{C}/0.$$

Such type of study was carried out by several authors [1, 3-7, 9-10]

A function $f(z)$ is subordination to $g(z)$ in open disk U if there exists an analytic function $q(z)$ with $q(0) = 0$ and $|q(z)| < 1$ such that $f(z) = g(q(z))$ for $|z| < 1$. Denotes by $f(z) < g(z)$, let $H_s^*[J, K]$ subclasses of $H_p(\beta)$ defined as follows

$$H_s^*[J, K] = \{f \in H_p(\beta), \frac{z^{-p}(f'(z))'}{f'(z)} < \frac{1+Jz}{1+Kz}, \quad -1 \leq K \leq J \leq 1\}$$

We denote by $G_{t,p}^s(n, p)$ the subclass of $W_{t,\rho}^\delta(n, p)$ that satisfies the relation $G_{t,p}^s(n, p) = H_p(\beta) \cap W_{t,\rho}^\delta(n, p)$. See [2, 8]

2. Main Result

In the class $W_{t,\rho}^\delta(n, p)$, we derive a sufficient condition for f .

Theorem 2.1: Let $f = \theta + \bar{\eta}$. If

$$\sum_{n=p+1}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] |\sigma_n| + \sum_{n=p}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] |\tau_n| \leq \vartheta(P, 2) - \rho, \quad (3)$$

where $0 \leq \rho < p, p \in \mathbb{N}, \delta \geq 0$, then f harmonic multivalent in U .

Proof: Suppose

$$m(z) = (1 - \delta) \left(-\frac{(z^p f'(z))'}{\vartheta(p, 1)g'(z)} \right)^{ts} + \delta \left(-\frac{(z^p f'(z))'}{\vartheta(p, 1)g'(z)} \right)^{t\alpha} \left[p + t \left(\frac{z(z^p f'(z))''}{(z^p f'(z))''} - \frac{zg''(z)}{g'(z)} \right) \right]$$

to prove that $Re\{m(z)\} \geq \frac{\rho}{\vartheta(p, 2)}$, it is sufficient to show inequality that

$$|(P - \rho) + \vartheta(P, 2)m(z)| \geq |(P + \rho) - \vartheta(P, 2)m(z)|.$$

Substituting for $m(z)$ and resorting to simple calculations, we have

$$\begin{aligned} |(p - \rho) + \vartheta(p, 2)m(z)| &= \left| (P - \rho) + \vartheta(p, 2)[(1 - \delta) \left(-\frac{(z^p(\vartheta(p, 2)(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n(z)^{(n-\beta-1)}))}{\vartheta(p, 2)(\sum_{n=0}^{\infty} a_n z^{(n-\beta-1)} + \sum_{n=0}^{\infty} b_n(z)^{(n-\beta-1)})} \right)^{ts} \right. \right. \\ &\quad \left. \left. + \delta \left(-\frac{Z^p(\vartheta(p, 2)(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-2} + \sum_{n=0}^{\infty} \tau_n(z)^{(n-\beta-2)}))}{\vartheta(p, 2)(\sum_{n=0}^{\infty} a_n z^{(n-\beta-2)} + \sum_{n=0}^{\infty} b_n(z)^{(n-\beta-2)})} \right)^{t\alpha} \right] \right| \\ &\geq \left[p + t \frac{z(Z^p((\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n(\bar{z})^{(n-\beta-1)}))^{n-\beta-2})}{(Z^p((\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n(\bar{z})^{(n-\beta-1)}))^{n-\beta-2})} \right. \\ &\quad \left. - \frac{z(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} (\tau_n(\bar{z})^{(n-\beta-1)})^{n-\beta-2})}{(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} (\tau_n(\bar{z})^{(n-\beta-1)})^{n-\beta-1})} \right] \\ &\geq (p + \vartheta(p, 2) - \rho) - \sum_{n=0}^{\infty} \vartheta(n, 1) [(p - 1) + \delta(n - p)] |\sigma_n| |z|^{n-p} - \sum_{n=0}^{\infty} \vartheta(n, 1) [(p - 1) + \delta(n - p)] |\tau_n| |z|^{n-p} \end{aligned} \quad (4)$$

and

$$\begin{aligned} |(p + \rho) - \vartheta(p, 2)m(z)| &= \left| (p + \rho) - \vartheta(p, 2)[(1 - \delta) \left(-\frac{(Z^p(\vartheta(p, 2)(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n(z)^{(n-\beta-1)}))}{\vartheta(p, 2)(\sum_{n=0}^{\infty} \sigma_n z^{(n-\beta-1)} + \sum_{n=0}^{\infty} \tau_n(z)^{(n-\beta-1)})} \right)^{ts} \right. \right. \\ &\quad \left. \left. + \delta \left(-\frac{Z^p(\vartheta(p, 2)(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-2} + \sum_{n=0}^{\infty} \tau_n(z)^{(n-\beta-2)}))}{\vartheta(p, 2)(\sum_{n=0}^{\infty} \sigma_n z^{(n-\beta-2)} + \sum_{n=0}^{\infty} \tau_n(z)^{(n-\beta-2)})} \right)^{t\alpha} \right] \right| \\ &\leq \left[p + t \frac{z(Z^p((\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n(\bar{z})^{(n-\beta-1)}))^{n-\beta-2})}{(Z^p((\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n(\bar{z})^{(n-\beta-1)}))^{n-\beta-2})} \right. \\ &\quad \left. - \frac{z(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} (\tau_n(\bar{z})^{(n-\beta-1)})^{n-\beta-2})}{(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} (\tau_n(\bar{z})^{(n-\beta-1)})^{n-\beta-1})} \right] \\ &\leq (p - \vartheta(p, 2) + \rho) + \sum_{n=0}^{\infty} \vartheta(n, 1) [(p - 1) + \delta(n - p)] |\sigma_n| |z|^{n-p} + \sum_{n=0}^{\infty} \vartheta(n, 1) [(p - 1) + \delta(n - p)] |\tau_n| |z|^{n-p} \end{aligned} \quad (5)$$

From (4) and (5), we have

$$|(p - \rho) + \vartheta(p, 2)m(z)| - |(p + \rho) - \vartheta(p, 2)m(z)| > 2(\vartheta(p, 2) - \rho) - \sum_{n=0}^{\infty} \vartheta(n, 1) [(p - 1) + \delta(n - p)] |\sigma_n| - \sum_{n=0}^{\infty} \vartheta(n, 1) [(p - 1) + \delta(n - p)] |\tau_n|.$$

The harmonic functions

$$f(z) = z^{-p} + \sum_{n=0}^{\infty} \frac{x_n}{\vartheta(n,1)[(p-1)+\delta(n-p)]} z^{n-\beta} + \sum_{n=0}^{\infty} \frac{\bar{y}_n}{\vartheta(n,1)[(p-1)+\delta(n-p)]} (\bar{z})^{n-\beta} \quad (6)$$

$(\sum_{n=0}^{\infty} |x_n| + \sum_{n=0}^{\infty} |y_n| = \vartheta(p, 2) - \rho)$ by the coefficient bound (3) and (6), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \vartheta(n, 1) [(p-1) + \delta(n-p)] |\sigma_n| + \sum_{n=0}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] |\tau_n| \\ &= \sum_{n=0}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \frac{|x_n|}{\vartheta(n,1)[(p-1)+\delta(n-p)]} + \\ & \quad \sum_{n=0}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \frac{|y_n|}{\vartheta(n,1)[(p-1)+\delta(n-p)]} \\ &= \sum_{n=0}^{\infty} |x_n| + \sum_{n=0}^{\infty} |y_n| = \vartheta(p, 2) - \rho. \end{aligned}$$

Now, we prove the condition (3) for function of (1) to be in the class $G_{t,p}^S(n, p)$.

Theorem 2.2: Let $f = \theta + \bar{\eta}$, then $f \in G_{t,p}^S(n, p)$ if and only if

$$\sum_{n=p+1}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \sigma_n + \sum_{n=p+1}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \tau_n \leq \vartheta(P, 1) - P$$

Proof: Since $G_{t,p}^S(n, P) \subset W_{t,p}^S(n, p)$,

So, we prove that “only if” part of this theorem

Let $f(z) \in G_{t,p}^S(n, P)$ by (2), we get

$$\begin{aligned} & Re\{\vartheta(p, 2) \left[(1-\delta) \left(\frac{(z^p \theta'(z) + \bar{\eta}'(\bar{z}))'}{\vartheta(p,1)(\theta'(z) - \bar{\eta}'(\bar{z}))'} \right)^{ts} + \delta \left(-\frac{(z^p \theta'(z) + \bar{\eta}'(\bar{z}))'}{\vartheta(p,2)(\theta'(z) - \bar{\eta}'(\bar{z}))'} \right)^{t\alpha} \right] \right. \\ & \quad \left. \left[p + t \left(\frac{z(z^p \theta'(z) + \bar{\eta}'(\bar{z}))''}{(z^p \theta'(z) + \bar{\eta}'(\bar{z}))''} \right) - \frac{z(\theta'(z) - \bar{\eta}'(\bar{z}))''}{(\theta'(z) - \bar{\eta}'(\bar{z}))'} \right] = \right. \\ & \quad \frac{Re\{\vartheta(p, 2)[(1-\delta) \left(-\frac{(z^p(\vartheta(p,2)(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-1)})) - \sum_{n=0}^{\infty} \vartheta(n,1) \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} \delta(n,1) \tau_n (\bar{z})^{(n-\beta-1)})^{n-\beta-1}}{\vartheta(p,2)(\sum_{n=0}^{\infty} \sigma_n z^{(n-\beta-1)} - \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-1)})^{n-\beta-1}} \right. \right. \\ & \quad \left. \left. + \vartheta(p, 2) \delta \left(-\frac{z^p(\vartheta(p,2)(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-2} - \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-2)}) - \sum_{n=0}^{\infty} \vartheta(n,1) \sigma_n z^{n-\beta-2} - \sum_{n=0}^{\infty} \delta(n,1) \tau_n (\bar{z})^{(n-\beta-2)})^{n-\beta-1}}{\vartheta(p,2)(\sum_{n=0}^{\infty} \sigma_n z^{(n-\beta-2)} - \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-2)})^{n-\beta-1}} \right) \right]} \\ & \quad \left. \left[p + t \frac{z(Z^p((\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-1)}))^{n-\beta-2}}{(Z^p((\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} + \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-1)}))^{n-\beta-2}} \right. \right. \\ & \quad \left. \left. - \frac{z(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-1)})^{n-\beta-2}}{(\sum_{n=0}^{\infty} \sigma_n z^{n-\beta-1} - \sum_{n=0}^{\infty} \tau_n (\bar{z})^{(n-\beta-1)})^{n-\beta-1}} \right] \right\} \\ &= Re\{\vartheta(p, 2) - \sum_{n=0}^{\infty} \vartheta(n, 1) [(p-1) + \delta(n-p)] \sigma_n z^{n-p} - \sum_{n=0}^{\infty} \vartheta(n, 1) \\ & \quad [(p-1) + \delta(n-p)] \tau_n (\bar{z})^{n-p}\} \geq \rho. \end{aligned}$$

If we choose z to be real and $z \rightarrow -1$, we get

$$\vartheta(p, 2) - \sum_{n=0}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \sigma_n - \sum_{n=0}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \tau_n \geq \rho.$$

So, we get

$$\sum_{n=0}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \sigma_n + \sum_{n=0}^{\infty} \vartheta(n, 1) [(P-1) + \delta(n-P)] \tau_n \leq \vartheta(P, 2) - \rho$$

This completes the proof of the Theorem 2.2.

Theorem 2.3: Let $f = \theta + \bar{\eta} \in S_p(\beta)$. Then $f \in H_s^*[J, K]$ if $\theta(z) *$

$$\left(\frac{z^{-p} + \frac{(v-j)}{J-K} z^{-2p}}{(1-z^2)^{-p+1}} \right) + \xi K \overline{\eta(z)} \left(\frac{v(\bar{z})^{-p} - \frac{(-1-jv)}{J-K} \bar{z}^{-2p}}{(1-\bar{z}^2)^{-p+1}} \right) \neq 0 \quad |v| = 1, 0 < |z| < 1$$

Proof: Let $h(z) = \frac{\theta(z) - \eta(z)}{1-c}$, then $h \in H_s^*[J, K]$ if and only if

$$-\frac{(z^{-p} h'(z))'}{\vartheta(p, 1) h'(z)} < \frac{-1 + Jz}{1 + Kz},$$

or

$$\frac{(z^{-p} h'(z))'}{h'(z)} \neq \frac{1 + J e^{i\theta}}{1 + K e^{i\theta}} \quad 0 \leq \theta \leq 2\pi, \quad z \in U.$$

It follows that

$$[(z^{-p} h'(z))' - h'(z) \frac{1 + J e^{i\theta}}{1 + K e^{i\theta}}] \neq 0.$$

Since $(z^p h'(z))' = h'(z) * \frac{z^{-p}}{(1-z^2)^{2p}}$, the above inequality is equivalent to

$$\begin{aligned} 0 &\neq h'(z) * \left[\frac{z^{-p}}{(1-z^2)^{-p+1}} - \frac{1 + J e^{i\theta}}{1 + K e^{i\theta}} \frac{z^{-p}}{(1-z)^{-p}} \right] \\ &= \frac{1}{\gamma e^{it}} \left\{ h'(z) * \left[\frac{z^{-p} + \frac{(-e^{i\theta}-j)}{(J-K)} z^{-2p}}{(-e^{i\theta}-j)(1-z^2)^{-p+1}} \right] \right\}, \quad 1-c = \gamma e^{it} \\ &= \frac{1}{\gamma e^{it}} \left\{ \theta(z) * \frac{z^p + \frac{(-e^{i\theta}-j)}{(J-K)} z^{-2p}}{(-e^{i\theta}-j)(1-z^2)^{-p+1}} - \eta(z) * \left(\left\{ \frac{z^{-p} + \frac{(-e^{i\theta}-j)}{(J-K)} z^{-2p}}{(J-K)(e^{i\theta}/j)} \right\} / (1-z)^{-2p} (-K - e^{i\theta}) \right) \right\} \\ &= \frac{1}{\gamma} \left\{ \theta(z) * \left(\frac{z^{-p} + \frac{(-e^{i\theta}-j)}{(J-K)} z^{-2p}}{e^{it}(1-z^2)^{-p+1}} - \eta(z) * \left(\frac{K e^{i\theta} z^{-p} + \frac{(-e^{i\theta}-j)}{(J-K)} z^{-2p}}{e^{it}(-K - e^{i\theta})(1-z^2)^{-p+1}} \right) \right) \right\}. \end{aligned}$$

Now if $z_1 - z_2 \neq 0$ and $|z_1| \neq |z_2|$ then $z_1 - \xi \bar{z}_2 \neq 0, |\xi| = 1$, i.e.,

$$\begin{aligned} &\frac{1}{\gamma(-K - e^{i\theta})} \left[\theta(z) * \frac{z^{-p} + \frac{(-e^{i\theta}-j)}{(J-K)} z^{-2p}}{e^{it}(1-z^2)^{-p+1}} \right] - \xi \overline{\eta(z)} * \left(\frac{K e^{i\theta} z^{-p} + \frac{(-1-j e^{i\theta})K}{(J-K)} z^{-2p}}{e^{it}(-K - e^{i\theta})(1-z^2)^{-p+1}} \right) \\ &= \frac{1}{\gamma(-K - e^{i\theta})} \left[\theta(z) * \frac{z^{-p} + \frac{(-e^{i\theta}-j)}{(J-K)} z^{-2p}}{e^{it}(1-z^2)^{-p+1}} \right] - \xi \overline{\eta(z)} * \left(\frac{(-K)(-e^{i\theta}(\bar{z})^{-p} + \frac{(-1-j e^{i\theta})K}{(J-K)} \bar{z}^{-2p})}{e^{it}(1-\bar{z}^2)^{-p+1}} \right). \end{aligned}$$

Since $\arg(1-c) = t \neq \pi$, we have the result.

3. Conclusions

This article presents more findings on the harmonic and conformal functions, along with the subordination function's applications in a novel class of derivatives of multivalent analytic functions in the open unit disk of the complex plane that are connected to linear operators.

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