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Deriving a new centrodal Runge Kutta method with predictor corrector style

Mahmood D. Jasim
Department of Mathematics
College of Basic Education
University of Mosul
Mosul
Iraq

Abstract

In this research, we derived a new Runge - Kutta (RK) method with a centrodal mean, based on the principle of guesswork and correction, to obtain a new connection for this method, where we can use its two parts at the same time (in parallel). So, it's makes this method more suitable for multi-processor computers that exist at this time than sequential methods previous.

Subject Classification: *32E30, 32E05.*

Keywords: *Initial value problems, Predictor corrector, Runge kutta methods, Centrodal mean, Stability.*

1. Introduction

Since the advent and spread of computers with parallel processors, it has become necessary try to change the traditional numerical solution methods to others suitable for these computers. Miranker [1] introduced the idea of extending the premise of arithmetic and it states that "The computation front is an imaginary boundary between the former value and the counted value". Also, Evans and Yaakub used the central mean for the first time to calculate the traditional RK method (instead of the arithmetic mean)[2]. In 2022 Jasim [3] presented a new parallel method which depends on the previous corrector (C) to calculate new predictor (P) in Figure 2, (see[4-7]). Our new one (in this paper) depend on the previous predictor to calculate a new one, where we used the PC-style and we make one step delay between C and P to get our new method. we

E-mail: mahoodalani@uomosul.edu.iq, mahoodalani@yahoo.com

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called it newCeM2. This newCeM2 is very accurate comparing with the old methods and it is more sutable to used in the modern computers, since they used multi prosseors. Where our newCeM2 can calculate their parts in parallel, which reduce the coast of more time in calculations.

2. Deriving of newCeM2:

The general form of centrodal mean RK method is

$$Y_{n+1} - Y_n = \frac{\mathrm{h}}{n}\left(\sum_{k=1}^{n} \frac{2(i_k^2 + i_k i_{k+1} + i_{k+1}^2)}{3(i_k + i_{k+1})}\right) \tag{1}$$

Where, $i_k = \mathrm{f}\left(x_n + \xi_k \mathrm{h}, Y_n + \mathrm{h}\sum_{l=1}^{k-1} \delta_{lk}\, i_l\right), k = 1,2,3, \ldots., n$ (2)

$\xi_k = \sum_{l=1}^{k-1} \delta_{lk}$, and h is the step size.

For convenience we will assume that *f* depend on just Y_n. So, (2) becomes

$$i_k = \mathrm{f}\left(Y_n + \mathrm{h}\sum_{l=1}^{k-1} \delta_{lk}\, i_l\right), k = 1,2,3, \ldots., n \tag{3}$$

Now, we try to drive a new method which contain two parts, the predictor part and the corrector part. And we make a delay of one step in corrector part, to obtain the advantage of parallel while computing the parts of the corrector part. So, we assume that [9,10]:

$$Y_{n+1}^p - Y_n^p = \mathrm{h}\left(\frac{ui_1^2 + wi_1 i_2 + vi_2^2}{ti_1 + yi_2}\right) \tag{4}$$

$$i_1 = \mathrm{f}(Y_{n-1}^c),\ i_2 = \mathrm{f}(Y_{n-1}^c + \delta \mathrm{h} i_1) \tag{5}$$

Let $i_1 = f(Y_{n-1}^c) = f$

$$i_2 = \mathrm{f}(Y_{n-1}^c + \delta \mathrm{h} i_1) = \mathrm{f} + \delta \mathrm{h} i_1 \mathrm{f}_{Y_{n-1}^c} + o(\mathrm{h}^2) = \mathrm{f} + \delta \mathrm{h}\mathrm{f}\mathrm{f}_{Y_{n-1}^c} + o(\mathrm{h}^2)$$

$$ui_1^2 + vi_2^2 + wi_1 i_2 = (u + v + w)\mathrm{f}^2 + (2v + w)\delta \mathrm{h}\mathrm{f}^2 \mathrm{f}_{Y_{n-1}^c} + o(\mathrm{h}^2) \tag{6}$$

$$ti_1 + yi_2 = (t + y)\mathrm{f} + y\delta \mathrm{h}\mathrm{f}\mathrm{f}_{Y_{n-1}^c} + o(\mathrm{h}^2) \tag{7}$$

Sub. (6), (7) in (4) $\rightarrow$ $Y_{n+1}^p - Y_n^p = \left(\frac{(u+v+w)\mathrm{h}\mathrm{f}^2 + (2v+w)\delta \mathrm{h}^2 \mathrm{f}^2 \mathrm{f}_{Y_{n-1}^c} + o(\mathrm{h}^3)}{(t+y)\mathrm{f} + y\delta \mathrm{h}\mathrm{f}\mathrm{f}_{Y_{n-1}^c} + o(\mathrm{h}^2)}\right)$ (8)

Let, $\mathrm{I} = (u + v + w)\mathrm{h}f^2 + (2v + w)\delta \mathrm{h}^2 f^2 f_{Y_{n-1}^c} + o(\mathrm{h}^3)$

$$\mathrm{II} = (t + y)\mathrm{f} + y\delta \mathrm{h}\mathrm{f}\mathrm{f}_{Y_{n-1}^c} + o(\mathrm{h}^2)$$

Then (8) becomes, $Y_{n+1}^p = Y_n^p + \frac{\mathrm{I}}{\mathrm{II}}$, from I and II we note that we have two different order of local truncation error $o(\mathrm{h}^2)$ *and* $o(\mathrm{h}^3)$. Therefore to solve this problem, we know that,

$$TAYLOR(Y_{n+1}^p) = Y_n^p + \mathrm{h}\mathrm{f} + \frac{\mathrm{h}^2}{2}\mathrm{f}\mathrm{f}_Y + o(\mathrm{h}^3) \tag{9}$$

$$Error = Taylor\ series - \frac{\mathrm{I}}{\mathrm{II}} \tag{10}$$

From (9) and (10), and comparing the order of $o(\mathrm{h})$ we have,

$$(t + y) - (u + v + w) = 0,\ (t + y) - (4v + 2w - 2y)\delta = 0$$

putting $u = v = w = 2, \delta = 1 \rightarrow t = y = 3$.

So, (4) becomes, $Y_{n+1}^{p} - Y_{n}^{p} = \mathrm{h}\left(\frac{2(i_1^2+i_1 i_2+i_2^2)}{3(i_1+i_2)}\right)$ (11)

$$i_1 = \mathrm{f}(x_{n-1}, Y_{n-1}^{c}) \, , \, i_2 = \mathrm{f}(x_{n-1} + \mathrm{h} \, , Y_{n-1}^{c} + \mathrm{h}i_1)$$

Equation (11) represent the first part of our newCeM2. Like the same way in (4), we derive the second part:

Let $Y_{n-1}^{c} - Y_{n}^{c} = -\mathrm{h}\left(\frac{uj_1^2+wj_1j_2+vj_2^2}{tj_1+yj_2}\right)$ (12)

$$j_1 = \mathrm{f}(\, Y_{n}^{p}\,) \, , \, j_2 = \mathrm{f}(\, Y_{n}^{p} - \delta \mathrm{h} j_1)$$

Now, $j_1 = f\left(\, Y_{n}^{p}\,\right) = f \, , \, j_2 = f\left(\, Y_{n}^{p} - \delta \mathrm{h} j_1\right) = f - \delta \mathrm{h} f f_{Y_n^p} + o(\mathrm{h}^2)$

$$uj_1^2 + vj_2^2 + wj_1j_2 = (u+v+w)\mathrm{f}^2 - (2v+w)\delta \mathrm{hf}^2 f_{Y_n^p} + o(\mathrm{h}^2) \quad (13)$$

$$tj_1 + yj_2 = (t+y)\mathrm{f} - y\delta \mathrm{hf} f_{Y_n^p} + o(\mathrm{h}^2) \quad (14)$$

Sub. (13)&(14) in (12) →

$$Y_{n-1}^{c} - Y_{n}^{c} = -\left(\frac{(u+v+w)\mathrm{hf}^2 - (2v+w)\delta \mathrm{h}^2 \mathrm{f}^2 f_{Y_n^p} + o(\mathrm{h}^3)}{(t+y)\mathrm{f} - y\delta \mathrm{hf} f_{Y_n^p} + o(\mathrm{h}^2)}\right) \quad (15)$$

Let $\mathrm{I} = -(u+v+w)\mathrm{h}f^2 + (2v+w)\delta \mathrm{h}^2 f^2 f_{Y_n^p} + o(\mathrm{h}^3)$

$$\mathrm{II} = (t+y)\mathrm{f} - y\delta \mathrm{hf} f_{Y_n^p} + o(\mathrm{h}^2)$$

Then (15) becomes,

$$Y_{n-1}^{c} = Y_{n}^{c} - \frac{\mathrm{I}}{\mathrm{II}} \quad (16)$$

We use the following relationships with equation (16)

$Taylor = Y_{n-1} - Y_n = -\mathrm{h}f + \frac{\mathrm{h}^2}{2} f f_{Y_n} + o(\mathrm{h}^3)$, $Error = Taylor\ series - \frac{\mathrm{I}}{\mathrm{II}}$

So, we get $-(t+y) + (u+v+w) = 0$, $(t+y) - 2(2v+w-y)\delta = 0$ $putting\ u = v = w = 2, \delta = 1 \ \rightarrow t = y = 3.$

So (12) becomes,

$$Y_{n}^{c} - Y_{n-1}^{c} = \mathrm{h}\left(\frac{2(j_1^2+j_1j_2+j_2^2)}{3(j_1+j_2)}\right) \quad (17)$$

At last equations (11) and (17) together represent our new CeM2.Our newCeM2 compute Y_{n+1}^{p} based on values of Y_{n}^{p} , Y_{n-1}^{c}. And computing Y_{n}^{c} based on values of Y_{n-1}^{c} and Y_{n}^{p}. The following Figure 1, shows how newCeM2 works as in the fo:

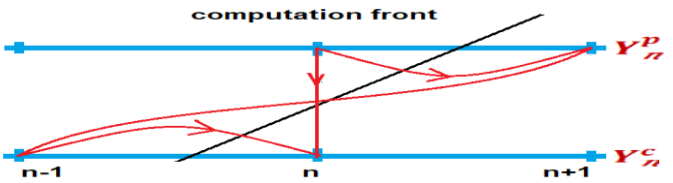


Figure 1
Represents our newCeM2

While the old connect of Jasim [3] represented in Figure 2 below:

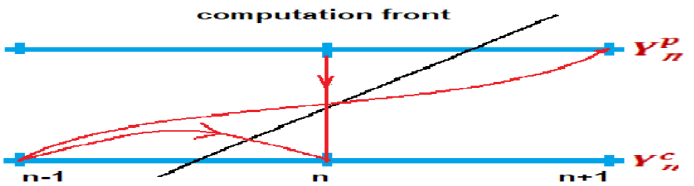


Figure 2
Represents the old method of Jasim [3].

3. Stability region of newCeM2

To discover absolute stability interval of newCeM2, utilizing test equation $\frac{dY}{dx} = \xi Y$ where $\xi = \frac{\partial f}{\partial Y}$>0 constant [8,11-13]. We test the predictor part of newCeM2 which represented by (11). Since,

$$i_1 = \mathrm{f}(x_{n-1}, Y^c_{n-1}) \, , \, i_2 = \mathrm{f}(x_{n-1} + \mathrm{h} \, , Y^c_{n-1} + \mathrm{h}i_1)$$

$$i_1 = \mathrm{f}(x_{n-1}, Y^c_{n-1}) = \xi Y^c_{n-1} \, , \, i_2 = \mathrm{f}(x_{n-1} + \mathrm{h} \, , Y^c_{n-1} + \mathrm{h}i_1) = \xi(Y^c_{n-1} + \mathrm{h}\xi Y^c_{n-1})$$

$$\text{So, } \; Y^p_{n+1} - Y^p_n = \tfrac{2}{3}\mathrm{h}\left(\frac{(\xi Y^c_{n-1})^2 + (\xi Y^c_{n-1})(\xi\, Y^c_{n-1} + \mathrm{h}\xi^2 Y^c_{n-1}) + (\xi\, Y^c_{n-1} + \mathrm{h}\xi^2 Y^c_{n-1})^2}{(\xi Y^c_{n-1}) + (\xi\, Y^c_{n-1} + \mathrm{h}\xi^2 Y^c_{n-1})}\right) \qquad (18)$$

$$Y^p_{n+1} - Y^p_n = \mathrm{h}\xi Y^c_{n-1}\left(\frac{1+\mathrm{h}\xi}{1+\frac{\mathrm{h}}{2}\xi}\right) \rightarrow \text{Let } \; s = \frac{Y^p_{n+1} - Y^p_n}{Y^c_{n-1}} = \mathrm{h}\xi\left(\frac{1+\mathrm{h}\xi}{1+\frac{\mathrm{h}}{2}\xi}\right) \qquad (19)$$

Then equation (19) if it meets the stability requirement $|S|$<1, and this hold when h$\xi\epsilon$(-1.280775 ,0).

4. Appling newCeM2 by an example

We applied our newCeM2 by equation $\acute{Y} = Y$ and the condition $Y(0) = 1$, using step-size h =0.05,0.02 respectively, where the exact solution is $Y = e^x$ in table 1,2 below, where it show that the results of the new method.

Table 1
Results of newCeM2 on IVPs. $\acute{Y} = Y$, $Y(0) = 1$with h =0.05.

x	Y	Y^c	Y^p	$\lvert Y - Y^c \rvert$	$\lvert Y - Y^p \rvert$
0	1	1	1	0	0
0.05	1.0512710963	1.0512924566	1.0100501670	2.13602870e-05	0.04122092929
0.10	1.1051709180	1.1052147338	1.0613426237	4.38157394e-05	0.04382829432
0.15	1.1618342427	1.1619016652	1.1152649008	6.74225074e-05	0.04656934182
0.20	1.2214027581	1.2214949977	1.1719518323	9.22396203e-05	0.04945092584
0.25	1.2840254166	1.2841437458	1.2315451648	0.000118329133	0.05248025182
…	….	…..	…..	….	…..
0.75	2.1170000166	2.1174653751	2.0242252187	0.000465358509	0.09277479788

0.85	2.3396468519	2.3402049683	2.2361016738	0.000558116430	0.10354517811
0.90	2.4596031111	2.4602112031	2.3502551354	0.000608091958	0.10934797571
0.95	2.5857096593	2.5863702891	2.4702613701	0.000660629787	0.11544828911
1	2.7182818284	2.7189976897	2.5964204561	0.000715861287	0.12186137227

Table 2

Represent results of newCeM2 on IVPs. $\acute{Y} = Y$, $Y(0) = 1$, h =0.02.

x	Y	Y^c	Y^p	$\lvert Y - Y^c \rvert$	$\lvert Y - Y^p \rvert$
0	1	1	1	0	0
0.02	1.0202013400	1.0202026867	1.0100501670	1.3467290e-06	0.010151172942
0.04	1.0408107741	1.0408134948	1.0302528538	2.7206638e-06	0.010557920352
0.06	1.0618365465	1.0618406688	1.0508636619	4.1223539e-06	0.010972884604
0.08	1.0832870676	1.0832926200	1.0718908359	5.5523601e-06	0.011396231691
0.10	1.1051709180	1.1051779293	1.0933427871	7.0112543e-06	0.011828130956
0.12	1.1274968515	1.1275053511	1.1152280964	8.4996201e-06	0.012268755165
0.14	1.1502737988	1.1502838169	1.1375555182	1.0018052e-05	0.012718280573
….	….	….	….	….	….
0.94	2.5599814183	2.5600854150	2.5194411743	0.00010399668	0.040540243961
0.96	2.6116964734	2.6118039177	2.5701355820	0.00010744428	0.041560891328
0.98	2.6644562419	2.6645672034	2.6218540847	0.00011096153	0.042602157139
1	2.7182818284	2.7183963782	2.6746173705	0.00011454983	0.043664457915

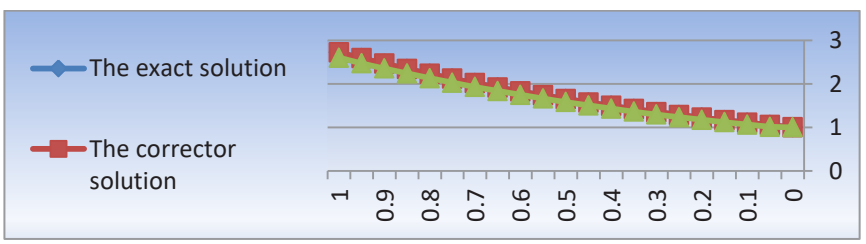


Figure 3

Represents our newCeM2 with step size h **=0.05.**

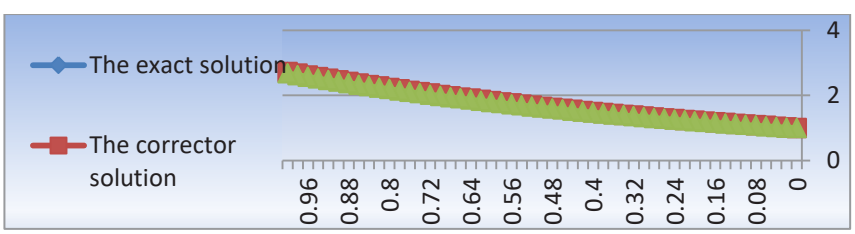


Figure 4

Represents our newCeM2 with step size h **=0.02.**

5. Discussion the results and Conclusion:

We see that the results of the corrector part of newCeM2(table1,2) in column 3 is very good as it observed in column 5 which represents the absolute difference between the exact solutions in column 2 and column 3, and much better than the predictor part in column 4, where the difference observed in column 6. More than we notice that the path of the corrective solution in red follows the exact solution very precisely (the exact solution in blue is located behind the corrective solution in the drawing, so it does not appear clearly in the drawing) (as is clear in Figures 3 and 4), while the predictor solution is slightly away from the exact solution.

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