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Common fixed point theorem in fuzzy Fréchet space

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Abstract

By using the concepts of commuting and compatibility, we establish a common fixed point for mappings in fuzzy Fréchet space. By assuming two commuting self-mappings under some restrictions, we get the unique common fixed point in a fuzzy Fréchet space. Moreover, by adding the condition of compatibility for four self-mappings with some restrictions in a fuzzy Fréchet space(FFS), the unique common fixed point is satisfied.

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1. Introduction

L. A. Zadeh defined a fuzzy set in 1965 [1]. C. L. Chang [2], in 1968, first defined a "Fuzzy Topological Space". It was developed by "C. K. Wong [3], R. Lowen" [4], and another researcher. "A. K. Katsaras and D. B. Liu [5] in 1977, was constructed a fuzzy topology vector Space". In 1984, A. K. Katsaras defined the concept of fuzzy semi-norm [6]. In 2007, sadiqi and F. Kia developed the concept of fuzzy semi-norm [7]. In 2021, Ahmed Ghanawi and Al. "Nafie Z. D." defined a fuzzy space and conducted a study on continuity within this space [8]. In this paper, the commuting mappings in fuzzy Fréchet space are defined. We aim to demonstrate the existence of a shared fixed point for mappings in a fuzzy Fréchet space by utilizing commuting and compatibility principles.

2. Preliminaries

Definition 2.1 [8]: "A fuzzy topology generated by a countable set of fuzzy semi-norms that can be distinguished "fuzzy Fréchet space (FFS)".

Definition 2.2 [9-11]: If F and T are self-mappings on X. If $u = Tu = Fu$, A point u is said to be a common fixed point of F and T .

3. Main Results

Let X be an FFS with its fuzzy topology τ_Γ induced by $\Gamma = \{P_\omega\}_{\omega\in J}$ a "Countable Separation of Fuzzy Semi-norms ".

Definition 3.1: If F and T are self-mappings on X. The pair (T, F) is said to be commuting if $P_\omega(TFu - FTu, r) = 1, \forall P_\omega \in \Gamma, \forall r > 0, \forall u \in X$.

Theorem 3.2: *If F and T are self mappings on X, such that*

1. $F(\mathrm{X}) \subseteq T(\mathrm{X})$.
2. T is $FF -$ sequentially continuous
3. $\exists\, h \in (0,1)$ s.t $\forall u, v \in \mathrm{X}$

$$hP_\omega(Fu - Fv, r) \geq P_\omega((Fu - Fv, r), \forall r > 0, \forall P_\omega \in \Gamma$$

then, F and T have a unique common fixed point in X provided (T, F) commuting on X.

Proof: Let $u_0 \in X$. From (i) we can take $u_1 \in \mathrm{X}$ such that $Tu_1 = Fu_0$. We construct $\{u_n\}$ in X, with $Tu_n = Fu_{n-1}, n = 1,2,3, \ldots$

Now, $\forall P_\omega \in \Gamma$

$$\begin{aligned}
&h^n P_\omega(Tu_n - Tu_{n+1}, r) \\
&\quad = h^n P_\omega(Fu_{n-1} - Fu_n, r) = h^{n-1}(hP_\omega(Fu_{n-1} - Fu_n, r) \\
&\quad \geq h^{n-1}P_\omega(Tu_{n-1} - Tu_n, r) = h^{n-1}P_\omega(Fu_{n-2} - Fu_{n-1}, r) \\
&\quad = h^{n-2}(hP_\omega(Fu_{n-2} - Fu_{n-1}, r) \geq h^{n-2}P_\omega(Tu_{n-2} - Tu_{n-1}, r) \\
&\quad \geq \cdots \geq P_\omega(Tu_0 - Tu_1, r), \forall n
\end{aligned}$$

Therefore

$$
\begin{aligned}
&P_\omega(Tu_n - Tu_{n+p}, r) \\
&\geq \min\left\{P_\omega\left(Tu_n - Tu_{n+1}, \frac{r}{p}\right), P_\omega\left(Tu_{n+1} - Tu_{n+2}, \frac{r}{p}\right), \dots, P_\omega\left(Tu_{n+p-1} - Tu_{n+p}, \frac{r}{p}\right)\right\} \\
&= \min\left\{\begin{array}{c} \frac{1}{h^n}\, h^n P_\omega\left(Tu_n - Tu_{n+1}, \frac{r}{p}\right), \frac{1}{h^{n+1}}\, h^{n+1} P_\omega \\ \left(Tu_{n+1} - Tu_{n+2}, \frac{r}{p}\right), \dots, \frac{1}{h^{n+p-1}}\, h^{n+p-1} P_\omega\left(Tu_{n+p-1} - Tu_{n+p}, \frac{r}{p}\right) \end{array}\right\} \\
&\geq \min\left\{\frac{1}{h^n}\, P_\omega(Tu_0 - Tu_1, r_1), \frac{1}{h^{n+1}}\, P_\omega(Tu_0 - Tu_1, r_1), \dots, \frac{1}{h^{n+p-1}}\, P_\omega(Tu_0 - Tu_1, r_1)\right\} \\
&\qquad \geq \frac{1}{h^n}\, P_\omega(Tu_0 - Tu_1, r_1)
\end{aligned}
$$

Where $r_1 = \frac{r}{p}$ and $p \neq 0$. thus we have $P_\omega(Tu_n - Tu_{n+p}, r) \geq \frac{1}{h^n}\, P_\omega(Tu_0 - Tu_1, r_1)$. Since $h \in (0,1)$, then $\lim_{n\to\infty} P_\omega(Tu_n - Tu_{n+p}, r) \geq \lim_{n\to\infty} \frac{1}{h^n}\, P_\omega(Tu_0 - Tu_1, r_1) > 1$.

$\lim_{n\to\infty} P_\omega(Tu_n - Tu_{n+p}, r) = 1, \forall P_\omega \in \Gamma$. Then $\{Tu_n\}$ is a Cauchy sequence in X.

$\exists v \in X$ such that $Tu_n \to v$ as $n \to \infty$ in X. Since $Tu_{n+1} = Fu_n$, it follows that $Fu_n \to v$ as $n \to \infty$ in X. The continuity of T implies the continuity of F by (iii), Then

$$FTu_n \to Sv \text{ as } n \to \infty \tag{1}$$

Since T is $FF-$ sequentially continuous and $Fu_n \to v$ as $n \to \infty$, then

$$TFu_n \to Tv \text{ as } n \to \infty \tag{2}$$

However, since T and F are commuting on X then $TFu_n = FTu_n$. Then from (1), (2) and since the limits is unique, we get $Tv = Fv$ and $TTv = TFv$.

Now, $\forall P_\omega \in \Gamma$

$$
\begin{aligned}
&P_\omega(Fv - FFv, r) = \frac{1}{h} h P_\omega(Fv - FFv, r) \geq \frac{1}{h} P_\omega(Tv - TFv, r) \\
&= \frac{1}{h} P_\omega(Tv - FTv, r) = \frac{1}{h} P_\omega(Fv - FFv, r) \geq \cdots \geq \frac{1}{h^n} P_\omega(Fv - FFv, r)
\end{aligned}
$$

$P_\omega(Fv - FFv, r) \geq \lim_{n\to\infty} \frac{1}{h^n} P_\omega(Fv - FFv, r) > 1$.

Then $P_\omega(Fv - FFv, r) = 1, \forall P_\omega \in \Gamma$.

Hence $FFv = Fv$ and $Fv = FFv = FTv = TFv$.

Fv is a Common Fixed Point of F and T.

For uniqueness, if $w, x \in X$ are common fixed points of F and T. From (iii)

$$
\begin{aligned}
1 \geq\ P_\omega(x - w, r) =\ & P_\omega(Fx - Fw, r) = h\frac{1}{h}\, P_\omega(Fx - Fw, r) \\
&\geq \frac{1}{h}\, P_\omega(Tx - Tw, r) = \frac{1}{h}\, P_\omega(x - w, r) \\
&= \frac{1}{h^2} h\, P_\omega(Fx - Fw, r) \geq \frac{1}{h^2}\, P_\omega(Tx - Tw, r) = \cdots \\
&\geq \frac{1}{h^n}\, P_\omega(x - w, r), \forall P_\omega \in \Gamma.
\end{aligned}
$$

Since $h \in (0,1)$, then $1 \geq P_\omega(x-w,r) \geq \lim_{n\to\infty} \frac{1}{h^n} P_\omega(x-w,r) \geq 1$. Then $P_\omega(x-w,r)=1$, $\forall P_\omega \in \Gamma$ and $x=w$..■

Definition 3.3: Let D and E be a self mappings on X. The pair (D,E) is said to be compatible if $\lim_{n\to\infty} P_\omega(DEu_n - EDu_n, r) = 1, \forall r>0, \forall P_\omega \in \Gamma$, $\{u_n\}$ is a sequence in X with $\lim_{n\to\infty} Du_n = \lim_{n\to\infty} Eu_n = u, u \in$ X.

Theorem 3.4: *Let be a self-mappings D,E,F and T on X satisfying the following conditions:* $\forall P_\omega \in \Gamma$

1. $\forall u,v$ in X, $h \in (0,1), r>0$, $P_\omega(Du-Ev,hr) \geq \min\{P_\omega(Fu-Tv,r), P_\omega(Du-Fu,r), P_\omega(Ev-Tv,r), P_\omega(Du-Tv,r), P_\omega(Fu-Ev,2r)\}$.
2. $D(X) \subseteq T(X), E(X) \subseteq F(X)$.
3. One of D,E,For T is $FF-$ sequentially continuous.
4. The pair (D,F) and (E,T) are compatible.

   Then the unique common fixed point of $D,E,F$and T is satisfied.

Proof: Let $D(X) \subseteq$ T(X) and $B(X) \subseteq F(X)$ then there exists $u_1,u_2 \in X$ such that $Du_0 = Tu_1, Eu_1 = Fu_2$. Construct a sequence $\{v_n\}$ in X such that $v_{2n+1} = Tu_{2n+1} = Du_{2n}$ and $v_{2n} = Fu_{2n} = Eu_{2n-1}$, for $n = 1,2,3,\ldots$. It is easy to see that $\{v_n\}$ is a cauchy sequence in X and $v_n \to w$ as $n\to\infty$ in X. By using the condition of (iii) and (iv), we get w is a common fixed point of $D,E,F$and T. Let $z = Dz = Ez = Fz = Tz$. Putting $u=w$ and $v=z$ in (i), $\forall P_\omega \in \Gamma$ we have $P_\omega(Dw-Ez,hr) \geq \min\{P_\omega(Fw-Tz,r),\ P_\omega(Dw-Fw,r),\ P_\omega(Ez-Tz,r),\ P_\omega(Dw-Tz,r), P_\omega(Fw-Ez,2r)\} \forall P_\omega \in \Gamma$ we have $P_\omega(w-z,hr) \geq \min\{\ P_\omega(w-z,t),1\}$. Which implies $w=z$.

4. Conclusion

The notations of commuting and compatibility in a fuzzy Fréchet space (FFS) has been established. With some restrictions in (FFS), for two and four self-mappings, we got new common fixed point results

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