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Approximation by half Bernstein polynomials

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Abstract

This paper defines and studies the half Bernstein polynomials to approximate functions in $C[0,1]$. The pointwise convergence in ordinary approximation is given for these polynomials. Also, the order of approximation, the Voronovskaya-type asymptotic theorem and the error estimate are introduced. Finally, to show the ability convergence of these polynomials we provide some examples to approximate these polynomials to test functions and compare numerical results with the results corresponding to classical Bernstein polynomials.

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1. Introduction

In 1912 Bernstein introduced a sequence in proof of Weierstrass theorem [1]. This sequence is known as the classical Bernstein polynomials.

Some studies have introduced modifications or generalizations for the classical Bernstein polynomials to accelerate the order of this sequence, we refer to [2, 3, 5, 6,7,8]. For $\wp \in [0+\sigma, 1-\sigma] \subset [0,1]$, where $\sigma > 0$ is very small real value and $\mathfrak{g} \in C[0+\sigma, 1-\sigma]$ the half terms of the classical Bernstein polynomials define as

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$$\mathcal{H}_{\mathfrak{n},2\&}(\mathfrak{g}(\mathrm{t});\wp) = 2\sum_{\&=0}^{\left[\frac{\mathfrak{n}}{2}\right]} \mathscr{b}_{\mathfrak{n},2\&}(\wp)\mathfrak{g}\left(\frac{2\&}{\mathfrak{n}}\right) \tag{1}$$

where $\mathscr{b}_{\mathfrak{n},2\&}(\wp) = \binom{\mathfrak{n}}{2\&}(1-\wp)^{\mathfrak{n}-2\&}\wp^{2\&}, \wp \in [0+\sigma, 1-\sigma]$.

In this paper, the properties and the recurrence relation for the $\mathfrak{Y}$-th order moment for $\mathcal{H}_{\mathfrak{n},2\&}(\mathfrak{g}(\mathrm{t});\wp)$ are given. The pointwise convergence in ordinary approximation and the order of approximation by the Voronovskaya theorem are introduced. Also, the error estimate for $\mathfrak{g} \in C[0+\sigma, 1-\sigma], \mathfrak{g} \in C^1[0+\sigma, 1-\sigma]$ are shown for this polynomial via the modulus of continuity [9-11]. Finally, to show the ability convergence of $\mathcal{H}_{\mathfrak{n},2\&}(.;\wp)$ we provide numerical example to approximate these polynomials to the test function $g_1 = \sin(12\wp)$ and comparison the numerical results with the results corresponding classical Bernstein polynomials for some values of $\mathfrak{n} = 30, 60, 100$. Throughout this paper, we assume that M is a positive constant, although it may not always be the same and $[\gamma]$ denotes the value of the integer part of γ.

2. Preliminary Results

Lemma 1: For $\wp \in [0+\sigma, 1-\sigma]$ the condition hold

1) $\mathcal{H}_{\mathfrak{n},2\&}(1;\wp) = 1 + (1-2\wp)^{\mathfrak{n}} \to 1$ as $\mathfrak{n} \to \infty$;
2) $\mathcal{H}_{\mathfrak{n},2\&}(\mathrm{t};\wp) = \wp\,[1-(1-2\wp)^{\mathfrak{n}-1}] \to \wp$ *as* $\mathfrak{n} \to \infty$;
3) $\mathcal{H}_{\mathfrak{n},2\&}(\mathrm{t}^2;\wp) = \wp^2\,[1-(1-2\wp)^{\mathfrak{n}-2}] - \frac{\wp^2}{\mathfrak{n}}[1-(1-2\wp)^{\mathfrak{n}-2}] + \frac{\wp}{\mathfrak{n}}[1-(1-2\wp)^{\mathfrak{n}-2}] \to \wp^2$ as $\mathfrak{n} \to \infty$.

Proof: By using the fact that $\lim_{\mathfrak{n}\to\infty}(1-2\wp)^{\mathfrak{n}} \to 0\ \forall \wp \in [0+\sigma, 1-\sigma]$ and from Korovkin theorem [4] this lemma proved.

Definition 1: For $\mathfrak{Y} \in N^0$ the $\mathfrak{Y}$ -th order moment $\Upsilon_{n,\mathfrak{Y}}(\wp)$ for $\mathcal{H}_{\mathfrak{n},2\&}(\mathfrak{g}(\mathrm{t});\wp)$ is defined as:

$$\Upsilon_{n,\mathfrak{Y}}(\wp) = \mathcal{H}_{\mathfrak{n},2\&}((\mathrm{t}-\wp)^{\mathfrak{Y}};\wp) \tag{2}$$

Lemma 2: *For the function* $\Upsilon_{\mathfrak{n},\mathfrak{Y}}(\wp)$ *one has:*

$\Upsilon_{\mathfrak{n},0}(\wp) = 1 + (1-2\wp)^{\mathfrak{n}};\ \Upsilon_{\mathfrak{n},1}(\wp) = 2\wp\,(\wp-1)(1-2\wp)^{\mathfrak{n}-1};$

Now, $\Upsilon_{\mathfrak{n},\mathfrak{Y}}(\wp)$ has the following recurrence relations:

$$\mathfrak{n}\Upsilon_{\mathfrak{n},\mathfrak{Y}+1}(\wp) = \wp\,(1-\wp)\,\mathfrak{Y}\,\Upsilon_{\mathfrak{n},\mathfrak{Y}-1}(\wp) + \wp\,(1-\wp)\Upsilon'_{\mathfrak{n},\mathfrak{Y}}(\wp), \mathfrak{Y} \geq 2. \tag{3}$$

Further, one gets $\Upsilon_{\mathfrak{n},\mathfrak{Y}}(\wp)$ is a polynomial in $\wp$ of degree $\mathfrak{Y}$ for sufficiently large $\mathfrak{n}$ and for every $\wp \in [0+\sigma, 1-\sigma]$, $\Upsilon_{\mathfrak{n},\mathfrak{Y}}(\wp) = O\left(\mathfrak{n}^{-\left[\frac{\mathfrak{Y}+1}{2}\right]}\right)$.

Proof: Using lemma 1 we can easily prove this lemma

Lemma 3: *For* $\mathfrak{Y} \geq 1$, *one has*

$\mathcal{H}_{\mathfrak{n},2\&}(\mathrm{t}^{\mathfrak{Y}};\wp) = \left\{\frac{\mathfrak{n}!}{\mathfrak{n}^{\mathfrak{Y}}(\mathfrak{n}-\mathfrak{Y})!}\wp^{\mathfrak{Y}} + \frac{\mathfrak{Y}\,(\mathfrak{Y}-1)\,\mathfrak{n}!}{2\,\mathfrak{n}^{\mathfrak{Y}}(\mathfrak{n}-\mathfrak{Y}+1)!}\wp^{\mathfrak{Y}-1}\right\}[1-(1-2\wp)^{\mathfrak{n}-\mathfrak{Y}}].$

Proof: By direct computation one gets $\mathcal{H}_{\mathfrak{n},2k}(\mathrm{t}^{\mathfrak{Y}};\wp) = 2\sum_{k=0}^{\left[\frac{\mathfrak{n}}{2}\right]} b_{\mathfrak{n},2k}(\wp)\left(\frac{2k}{\mathfrak{n}}\right)^{\mathfrak{Y}}$

$$= \frac{1}{\mathfrak{n}^{\mathfrak{Y}}}\{[\mathfrak{n}\,(\mathfrak{n}-1)\,(\mathfrak{n}-2)\ldots(\mathfrak{n}-\mathfrak{Y}\,+1)]\wp^{\mathfrak{Y}}$$
$$+\frac{\mathfrak{Y}\,(\mathfrak{Y}\,-1)}{2}[\mathfrak{n}\,(\mathfrak{n}-1)(\mathfrak{n}-2)\ldots(\mathfrak{n}-\mathfrak{Y}\,+2)]\wp^{\mathfrak{Y}\,-1}\}$$

Lemma 4: *Let* $\delta,\alpha\in R^{+}$ *and* $[a,b]\subset[0,1]$ *then*

$$\sup_{\wp\in[\,a,b]}\left|\sum_{\left|\frac{2k}{\mathfrak{n}}-\wp\right|<\delta} b_{\mathfrak{n},2k}(\wp)\,\mathrm{t}^{\alpha}\right| = O\left(\mathfrak{n}^{-\beta}\right)$$

Proof: This lemma has easily proven, from Taylor's theorem, Cauchy-Schwarz and lemma 2 (b), hence the details are omitted as shown in Figure 1.

3. Main Results

This section introduced the order of approximation by using Voronovskaya type asymptotic formula and an estimate error theorem in ordinary approximation for the sequence $\mathcal{H}_{\mathfrak{n},2k}(\mathfrak{g}(\mathrm{t}\,);\wp)$.

Theorem 1: *Suppose that* $\mathfrak{g}\in C[0+\sigma,1-\sigma]$ *and* $\mathfrak{g}''$ *exists at the point* $\wp\in[0+\sigma,1-\sigma]$ *then*

$$\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\left\{\mathcal{H}_{\mathfrak{n},2k}(\mathfrak{g}(\mathrm{t});\,\wp)-\mathfrak{g}(\,\wp)\right\} = \frac{\wp\,(1-\wp)}{2}\mathfrak{g}''(\,\wp). \tag{4}$$

Proof: By Taylor 's theorem of $\mathfrak{g}$, one gets

$\mathfrak{g}(\mathrm{t}) = \mathfrak{g}(\wp)+\mathfrak{g}'\,(\wp)\left(\frac{2k}{\mathfrak{n}}-\wp\right)+\frac{1}{2}\,\mathfrak{g}''(\wp)\left(\frac{2k}{\mathfrak{n}}-\wp\right)^{2}+\varepsilon\left(\frac{2k}{\mathfrak{n}},\wp\right)\left(\frac{2k}{\mathfrak{n}}-\wp\right)^{2},$

where $\varepsilon\left(\frac{2k}{\mathfrak{n}},\ \wp\right)\to 0\ as\ \frac{2k}{\mathfrak{n}}\to\wp$. Hence operating by $\mathcal{H}_{\mathfrak{n},2k}(\mathfrak{g}(\mathrm{t});\wp)$ one gets $\Sigma_1+\Sigma_2$. Using lemma 2, one gets $\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\left\{\mathcal{H}_{\mathfrak{n},2k}(\mathfrak{g}\,(\mathrm{t});\wp)-\mathfrak{g}(\wp)\right\} = \frac{\wp\,(1-\wp)}{2}\mathfrak{g}''\,(\wp)+\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\,\Sigma_2$. Now we prove that $\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\,\Sigma_2\to 0$ as $\mathfrak{n}\to\infty$, given $\varepsilon>0$ arbitrary there exist $\delta>0$ such that $\left|\frac{2k}{\mathfrak{n}}-\wp\right|<\delta$ implies that $\left|\varepsilon\left(\frac{2k}{\mathfrak{n}},\ \wp\right)\right|<\varepsilon$

$$|\Sigma_2|\le 2\ \sum_{\left|\frac{2k}{\mathfrak{n}}-\wp\right|<\delta} b_{\mathfrak{n},2k}(\wp)\left|\varepsilon\left(\frac{2k}{\mathfrak{n}},\wp\right)\right|\left(\frac{2k}{\mathfrak{n}}-\wp\right)^{2}$$
$$+2\ \sum_{\left|\frac{2k}{\mathfrak{n}}-\wp\right|\ge\delta} b_{\mathfrak{n},2k}(\wp)\left|\varepsilon\left(\frac{2k}{\mathfrak{n}},\wp\right)\right|\left(\frac{2k}{\mathfrak{n}}-\wp\right)^{2} = \Sigma_3+\Sigma_4.$$

$\Sigma_3 = \varepsilon\,\Upsilon_{\mathfrak{n},2}(\wp) = \varepsilon\,O(\mathfrak{n}^{-1}) = o(1)$. Using Lemma 4, one has $\Sigma_4 = \mathfrak{n}\,M\,\mathrm{O}\left(\mathfrak{n}^{-\beta}\right) = M\,\mathrm{O}\left(\mathfrak{n}^{1-\beta}\right)$ then $\Sigma_4 = o(1)$ whenever $\beta>1$. Hence, $\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\Sigma_2\to 0$ as $\mathfrak{n}\to\infty$.

Theorem 2: *Suppose that* $\mathfrak{g}\in C[\,0+\sigma,1-\sigma]$ *and* $\wp\ \in[0+\sigma,1-\sigma]$ *then*

$$\left\|\mathcal{H}_{\mathfrak{n},2\mathscr{k}}(\mathfrak{g}(\mathrm{t});\wp)-\mathfrak{g}(\wp)\right\| \le \omega(\mathfrak{g};\delta)(2+(1-2\wp)^{\mathfrak{n}})+(1-2\wp)^{\mathfrak{n}}\,\|\mathfrak{g}\| \tag{5}$$

Proof: By lemma 2 and choosing $\delta = 2\wp\,(\wp-1)\,(1-2\wp)^{\mathfrak{n}-1}$ then (5) holds.

Theorem 3: *Suppose that* $\mathfrak{g} \in C^1[0+\sigma, 1-\sigma]$ *and* $\wp \in [0+\sigma, 1-\sigma]$, *then*

$$\left|\mathcal{H}_{\mathfrak{n},2\mathscr{k}}(\mathfrak{g}(\mathrm{t});\wp)-\mathfrak{g}(\wp)\right| \le \omega(\mathfrak{g}';\delta)\left\{\left(\frac{1}{\mathfrak{n}}\,c_{\mathfrak{n}}(\wp)\right)^{\frac{1}{2}}+1\right\}+(1-2\wp)^{\mathfrak{n}}\,|\mathfrak{g}(\wp)| \tag{6}$$

where

$$c_{\mathfrak{n}}(\wp) = \wp\,(1-\wp)\,[1+(1-2\wp)^{\mathfrak{n}-2}] + \wp^2\,(1-2\wp)^{\mathfrak{n}-2}[1+(1-2\wp)^2] + 2\,\wp\,(1-2\wp)^{\mathfrak{n}-1}.$$

Proof: By mean value theorem, From the linearity of $\mathcal{H}_{\mathfrak{n},2\mathscr{k}}$ and using Lemma 2, (6) holds easily.

4. Numerical Examples

This section uses numerical example to demonstrate how the test functions $g_1(\upsilon) = \sin(12\wp)$ on the interval $[0+\sigma, 1-\sigma]$ lead to convergence of the polynomials $\mathcal{H}_{\mathfrak{n},2\mathscr{k}}(.;\wp)$. Additionally, when $\mathfrak{n} = 30, 60, 100$, we compare the outcomes for this polynomial with the conventional Bernstein polynomials.

Example 4.1: We introduce the comparison of the convergence of $\mathcal{H}_{\mathfrak{n},2\mathscr{k}}(g_1;\wp)$ and $\mathbb{B}_{\mathfrak{n},\mathscr{k}}(g_1;\wp)$ to the test function $g_1(\wp) = \sin(12\wp)$ for $\mathfrak{n} = 30, 60$ and 100 respectively. These polynomials' error functions are also compared, and the results are shown in the accompanying figures.

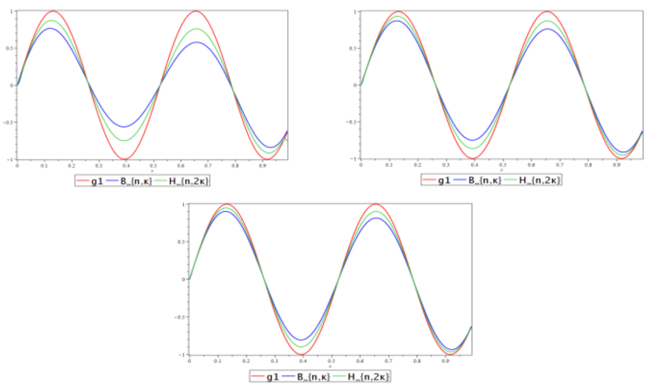

Figure 1
The Comparison between the convergent polynomials $\mathbb{B}_{\mathfrak{n},\mathscr{k}}(g_1;\wp)$ and $\mathcal{H}_{\mathfrak{n},2\mathscr{k}}(g_1;\wp)$ to the function $g_1(\wp)$

5. Conclusions

We can see from the example above that for $n = 30, 60$ and 100 the polynomial $\mathcal{H}_{n,2k}(.;\ \wp)$ performs better than the polynomial $\mathbb{B}_{n,k}(.;\ \wp)$. So, in the application, we advise using $\mathcal{H}_{n,2k}(.;\ \wp)$.

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