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Approximate analytical solutions of fractional order differential equations by conformable Sumudu decomposition method

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Abstract

This work offers a fast, approximate analytical technique for solving time fractional differential equations. CSTADM is a gentle combination of two powerful approaches to solve fractional PDE's. The hybrid approach suggested here produces more accurate findings and convergent solution series faster. Also, the proposed approach is used to get the analytical solution of nonlinear fractional PDE's. The fractional derivative's type is assumed to be conformable. Compared to standard fractional derivative definitions, this approach is effective and straightforward in applications involving fractional differential equations with complicated solutions. Furthermore, the results show that the acquired results increase the precision and efficacy of the proposed technique.

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1. Introduction

Many fields of study, including fluid mechanics, biology, physics, and control theory of dynamic systems, heavily rely on fractional ordinary /partial differential equations. Numerous academics have investigated the fractional differential equation using various numerical techniques. When Prof. Khalil and colleagues introduced a more straightforward and effective formulation of fractional derivatives, we became interested in solving fractional differential equations. The revised definition considers a conformable fractional derivative, a natural extension of the normal derivative. The conformable fractional derivative has many fascinating advantages, including the ability to generalize all of the ideas of ordinary calculus and to solve various fractional differential equations in all circumstances [1-10]. It may be used to represent a wide variety of applications and phenomena.

2. Test Example

Two illustrative examples will be solved to illustrate the proposed approach, and the obtained numerical results are sketched for different values of a [11,12].

Example 1: Consider a linear time - fractional Newell - Whitehead- Segel equation:

$$D_t^a f(\xi, t) = D_\xi^2 f(\xi, t) - 2f(\xi, t)., \ t \geq 0,.,0 < a \leq 1.$$

and

$$f(\xi, 0) = exp(x)$$

Applying the conformable Sumudu technique to both sides of the above equation, get:

$$\frac{1}{u} S_a[f(\xi, t)] - \frac{f(\xi,0)}{u} = S_a\left[D_\xi^2 f(\xi, t)\right] - 2S_a[f(\xi, t)]$$

Then, upon carrying out some simplifications, we have

$$\left(\frac{1}{u} + 2\right) S_a[f(\xi, t)] = S_a\left[D_\xi^2 f(\xi, t)\right] + \frac{e^\xi}{u}$$

$$\Rightarrow S_a[f(\xi, t)] = \frac{u}{1+2u} S_a\left[D_{\xi,}^2 f(\xi, t)\right] + \frac{e^{\xi,}}{1+2u}$$

The inverse Sumudu transform. S_a^{-1}, the above Eq can be explicitly given as:

$$f(\xi, t) = S_a^{-1}\left[\frac{u}{1+2u} S_a\left[D_{\xi,}^2 f(\xi, t)\right]\right] + e^{\xi,} S_a^{-1}\left[\frac{1}{1+2u}\right]$$

Then

$$\sum_{m=0}^{\infty} f_m(\xi, t) = e^x \, e^{-2\frac{t^a}{a}} + S_a^{-1}\left[\frac{u}{1+2u} S_a\left[D_{\xi,}^2 \sum_{m=0}^{\infty} f_m(\xi, t)\right]\right]$$

We obtain the following iterations: $f_0 = e^{\xi - 2\frac{t^a}{a}}$, $f_1 = \frac{t^a}{a} e^{\xi - 2\frac{t^a}{a}}$, $f_2 = \frac{t^{2a}}{a^2 2!} e^{\xi - 2\frac{t^a}{a}}$

Finally, we get the approximate solution as shown in the following Figure 1.

$$f(\xi,\mathrm{t}) = e^{\xi-2\frac{t^{a}}{a}} + \frac{t^{a}}{a}e^{\xi-2\frac{t^{a}}{a}} + \frac{t^{2a}}{a^{2}2!}e^{\xi-2\frac{t^{a}}{a}} + \frac{t^{3a}}{a^{3}3!}e^{\xi-2\frac{t^{a}}{a}} \ldots\ldots..$$

$$f(\xi,\mathrm{t}) = e^{\xi-2\frac{t^{a}}{a}}\left(1+\frac{t^{a}}{a}+\frac{t^{2a}}{a^{2}2!}+\frac{t^{3a}}{a^{3}3!}+\ \ldots..\right)$$

$$\Longrightarrow f(\xi,\mathrm{t}) = e^{\xi-\frac{t^{a}}{a}}$$

If we put $a = 1$ we can conclude the true solution: $f(\xi,\mathrm{t}) = e^{\xi-t}$

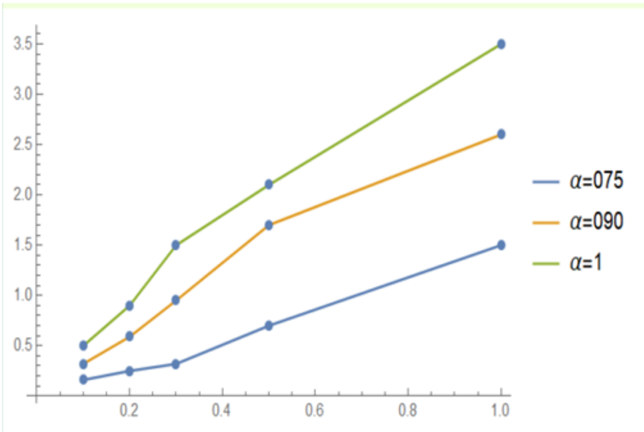

Figure 1

Let the Conformable Time fractional Linear Telegraph Equation (CTFLTE)

$$\frac{\partial^{2\mu}\omega(\xi,\mathrm{t})}{\partial t^{2\mu}} + 2\frac{\partial^{\mu}\omega(\xi,\mathrm{t})}{\partial t^{\mu}} + \omega(\xi,\mathrm{t}) = \frac{\partial^{2}\omega(\xi,\mathrm{t})}{\partial \xi^{2}}\ ,\ 0 < \mu \leq 1\ ,\ t \geq 0\ ,$$

with the initial condition $\omega(\xi,0) = e^{\xi},\ \omega_t(\xi,\mathrm{t}) = 2e^{\xi}$

Now by performing conformable Sumudu transform on above equation then we get

$$\frac{1}{v^{2}}S_{\mu}\{\omega(\xi,\mathrm{t})\} - \omega(\xi,\mathrm{t}) - v\omega(\xi,\mathrm{t}) + S_{\mu}\left[2\frac{\partial^{u}\omega(\xi,\mathrm{t})}{\partial t^{\mu}} + \omega(\xi,\mathrm{t}) - \frac{\partial^{2}w\omega(\xi,\mathrm{t}))}{\partial x^{2}}\right] = 0.$$

If we simplify the above, then we have

$$S_{\mu}\{\omega(\xi,\mathrm{t})\} = v^{2}\omega(\xi,\mathrm{t}) + v^{3}\omega_{t}(\xi,\mathrm{t}) - v^{2}S_{\mu}\left[2\frac{\partial^{\mu}\omega(\xi,\mathrm{t})}{\partial t^{\mu}} + \omega(\xi,\mathrm{t}) - \frac{\partial^{2}\omega(\xi,\mathrm{t})}{\partial \xi^{2}}\right]$$

Applying the inverse conformable Sumudu transform ,

$$\omega(\xi,\mathrm{t}) = S_{\mu}^{-1}\left[v^{2}\omega(\xi,0) + v^{3}\omega_{t}(\xi,\mathrm{t}) - v^{2}S_{\mu}\left[2\frac{\partial^{\mu}\omega(\xi,\mathrm{t})}{\partial t^{\mu}} + \omega(\xi,\mathrm{t}) - \frac{\partial^{2}\omega(\xi,\mathrm{t})}{\partial \xi^{2}}\right]\right].$$

Using the ADM procedure

$$\omega_0(\xi,t) = S_\mu^{-1}[v^2\omega(\xi,0) + v^3\omega_t(\xi,t)] = S_\mu^{-1}[v^2e^{\xi} - 2e^{\xi}v^3] = e^{\xi} - 2e^{\xi}\frac{t^\mu}{\mu}$$

$$\omega_{s+1}(\xi,t) = -S_\mu^{-1}\left[v^2 S_\mu\left[2\frac{\partial^\mu\omega_s(\xi,t)}{\partial t^\mu} + \omega_s(\xi,t) - \frac{\partial^2\omega_s(\xi,t)}{\partial\xi^2}\right]\right]$$

For s = 0, we obtain

$$\omega_1(\xi,t) = -S_\mu^{-1}\left[4e^x\mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)v^{2\frac{\mu-1}{\mu}+2}\right] = \frac{4e^x\mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}t^{3\mu-1}\right)}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)}$$

We get the subsequent terms recursively

$$\omega_2(\xi,t) = \frac{-8e^x\mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)t^{5\mu-2}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)}$$

And so on

When $\mu=1$ $\quad \omega\ (\xi,t) = e^{\xi}[1 - 2t + \left(\frac{(2t)^2}{21} - \frac{(2t)^2}{31} + \frac{(2t)^2}{41} - \cdots\right]$

This result is evaluated to the exact solution in a closed form

$$\omega(\xi,t) = e^{\xi-2t}$$

we observe that CFEDM solution is close to the true solution. The numerical solutions for several μ values are evaluated in Figure 1 and 2. From Figure 2, it has been observed that the solutions get closer to the true solution, when μ gets closer to 1.

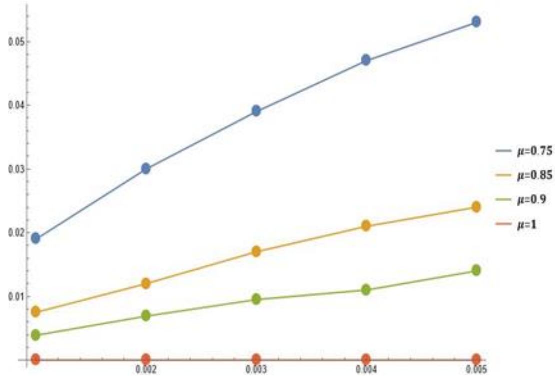

Figure 2
The true solution and approximate solution

3. Results and Discussion

For Example (1), figure (1) presents the true solution and approximate solutions. Here, we use several terms to get the true solution; we obtain the proposed method CSTADM that has a high convergence order and accuracy.

For Example (2), Figure (2) presents the true solution and approximate solution. Here, use three terms to get true solution, the proposed method CSTADM has a high convergence order and accuracy we get.

4. Conclusion

We profitably applied a new hybrid method, namely CFSTDM to solve the conformable time-fractional linear telegraph equations and linear time-fractional Newell- Whitehehead-Segel equation. During the investigation, the obtained solutions are illustrated in terms of plots with diverse values of space and time variables. We have observed that CFSTDM is powerful, fast and efficient method to solve the conformable differential equations.

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