*Journal of Interdisciplinary Mathematics*
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 4, pp. 761–766
DOI : 10.47974/JIM-1868

# Analytical solutions via coupled Elzaki adomian decomposition method for some applications

Asmaa Mustafa Mohammed
Huda Omran Altaie *
*Department of Mathematics*
*College of Education for Pure Sciences (Ibn Al-Haitham)*
*University of Baghdad*
*Baghdad*
*Iraq*

**Abstract**

An efficient combination of Adomian Decomposition iterative technique coupled Elzaki transformation (ETADM) for solving Telegraph equation and Riccati non-linear differential equation (RNDE) is introduced in a novel way to get an accurate analytical solution. An elegant combination of the Elzaki transform, the series expansion method, and the Adomian polynomial. The suggested method will convert differential equations into iterative algebraic equations, thus reducing processing and analytical work. The technique solves the problem of calculating the Adomian polynomials. The method's efficiency was investigated using some numerical instances, and the findings demonstrate that it is easier to use than many other numerical procedures. It has also been established that (ETADM) is a trustworthy technique for solving differential equations. Using the Mathematica 13.3 programme, the graphs of the approximate solutions are presented.

***Subject Classification:*** *49M27, 05C51.*

**Keywords:** *Elzaki transform, Decomposition technique, Ordinary differential equations, Iterative methods, Analytical solution.*

## 1. Introduction

Nonlinear equations of the Riccati differential equation (RDE) type are particularly important in optimal control, selection, and estimation issues. These equations are commonly found in science, engineering, and applied mathematics situations. While some authors solved the RNDE analytically or

* *E-mail:* dr.huda_hm2019@yahoo.com

computationally, many others derived their solutions by a variety of ways. Differential equations can be used as models for a wide range of engineering and physical problems. and as a result, sometimes it could be challenging to find closed form solutions for them, particularly for nonlinear ones.The fundamental theories of Riccati equation with applications to engineering science with a newer application to economics and finance. Various researchers had made attempts to the derivation of solutions to the problems by using the classical approach. Consequently, numerous numerical techniques for solving RDEs have been created [1-13]. The major goal of this study is to compute the analytical solutions by a novel mixture Elzaki transform and decomposition method in order to find the effective solutions of the RDEs.

## 2. Convergence Analysis

**Theorem 2.1**: *Let's assume* $\mathcal{J}$ *is a Banach space. Then, the expansion result of* $\psi(\xi, \mathrm{t})$ *converges,* $0 < \mathcal{W} < 1$, *so that* $\|\psi_i(\xi, \mathrm{t})\| \leq \mathcal{W}\|\psi_{i-1}(\xi, \mathrm{t})\|$, *for all* $i \in \mathbb{N}$.

***Proof:*** Consider $\mathcal{H}_i(\xi, \mathrm{t}) = \psi_0(\xi, \mathrm{t}) + \psi_1(x, t) + \psi_2(x, t) + \cdots + \psi_i(x, t)$.

It is vital to verify that successions of $i$-th partial sums $\mathcal{H}_i(\xi, \mathrm{t})$ are a Cauchy series in Banach space. In this regard, we consider the following:

$$\|\mathcal{H}_{i+1}(\xi, \mathrm{t}) - \mathcal{H}_i(\xi, \mathrm{t})\| \leq \|\mathcal{H}_{i+1}(\xi, \mathrm{t})\| \leq \rho\|\mathcal{H}_i(\xi, \mathrm{t})\| \leq \rho^2\|\mathcal{H}_{i-1}(\xi, \mathrm{t})\| \leq \cdots \leq \|\mathcal{H}_0(\xi, \mathrm{t})\|$$

For every $i, j \in \mathbb{N}, i \leq j$, it is obtained as

$$\left\|\mathcal{H}_i(\xi, \mathrm{t}) - \mathcal{H}_j(\xi, \mathrm{t})\right\| \leq \left\|\mathcal{H}_{j+1}(\xi, \mathrm{t}) - \mathcal{H}_j(\xi, \mathrm{t})\right\| + \left\|\mathcal{H}_{j+2}(\xi, \mathrm{t}) - \mathcal{H}_{j+1}(\xi, \mathrm{t})\right\| + \cdots + \|\mathcal{H}_i(\xi, \mathrm{t}) - \mathcal{H}_{i+1}(\xi, \mathrm{t})\|$$

Using the triangle inequality as follows :

$$\left\|\mathcal{H}_i(\xi, \mathrm{t}) - \mathcal{H}_j(\xi, \mathrm{t})\right\| \leq \mathcal{W}^{j+1}(1 + \mathcal{W} + \mathcal{W}^2 + \cdots + \mathcal{W}^{i-j-1})\|\mathcal{H}_0(\xi, \mathrm{t})\|,$$

$$\left\|\mathcal{H}_i(\xi, \mathrm{t}) - \mathcal{H}_j(\xi, \mathrm{t})\right\| \leq \mathcal{W}^{j+1}\left(\frac{1-\mathcal{W}^{i-j}}{1-\mathcal{W}}\right)\|\mathcal{H}_0(\xi, \mathrm{t})\|$$

Hence $0 < \mathcal{W} < 1$, and $1 - \mathcal{W}^{i-j} \leq 1$.

Then $\left\|\mathcal{H}_i(\xi, \mathrm{t}) - \mathcal{H}_j(\xi, \mathrm{t})\right\| \leq \left(\frac{1-\mathcal{W}^{j+1}}{1-\mathcal{W}}\right)\|\mathcal{H}_0(\xi, \mathrm{t})\|$

Since$\mathcal{H}_0(\xi, \mathrm{t})$ is bounded, it is obtained as

$$\lim_{i,j\to\infty}\left\|\mathcal{H}_i(\xi, \mathrm{t}) - \mathcal{H}_j(\xi, \mathrm{t})\right\| = 0$$

## 3. Numerical experiments

In this section, we consider applying Elzaki Adomian Decomposition iterative approaches (ETADM) to solve the RNDEs.

**Example 3.1:** Consider

$$\Upsilon'(\mathrm{t}) = 1 - \Upsilon^2(\mathrm{t}) \ (1.1) \text{ and } \Upsilon(0) = 0 \ (1.2)$$

**Solution:** Applying the ETADM on the both side (1.1) with initial condition (1.2) will give: $E[\Upsilon(t)] - v\Upsilon(0) = v^2 - E[\Upsilon^2(t)]$ (1.3)

Taking inverse Elzaki transform on the both side of Eq.(1.5)

$$\Upsilon(t) = E^{-1}(v^3) - E^{-1}\left[vE[\Upsilon^2(t)]\right]$$

Here, the ADM represents solution $\Upsilon(t)$ of Eq.(1.1) as an infinite series of components in the form $\Upsilon(t) = \sum_{n=0}^{\infty} \Upsilon_n(t)$ and the non-linear term $\Upsilon^2(t)$ can be represented by infinite series $\Upsilon^2(t) = \sum_{n=0}^{\infty} A_n$ where $A_n$ are Adomian polynomials $\Upsilon_0, \Upsilon_1, \ldots \ldots$ calculated by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^n \Upsilon_n\right)\right]_{\lambda=0} .$$

Then, one gets

$$\Upsilon_o(t) = t \ , \Upsilon_{n+1} = -E^{-1}\left[vE\left[\sum_{n=0}^{\infty} A_n\right]\right],$$

So, the other components of the solution can be easily calculated

$$\Upsilon_1 = -E^{-1}\left[vE[A_0]\right] = -\frac{1}{3}t^3$$

$$\Upsilon_2 = -E^{-1}\left[vE[A_1]\right] = \frac{2}{15}t^5$$

$$\Upsilon_3 = -E^{-1}\left[vE[A_2]\right] = -\frac{17}{315}t^7$$

$$\Upsilon_4 = -E^{-1}\left[vE[A_3]\right] = \frac{62}{2834}t^9$$

$$\Upsilon_5 = -E^{-1}\left[vE[A_4]\right] = -\frac{652868}{73648575}t^{11}$$

Hence, $\Upsilon(t) = -\frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2834}t^9 + \cdots$ Which its closed form solution is $\Upsilon(t) = tanh\, t$ as shown in the following Figure 1.

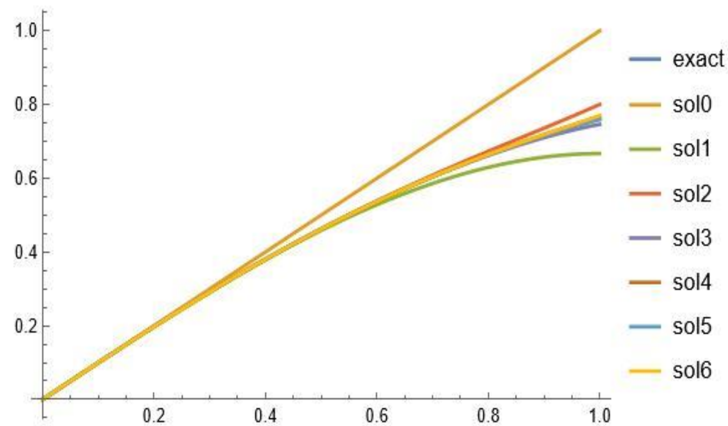


**Figure 1**
**Compasion solutions between the exact and several approximate solutions**

**Example 3.2:** Consider the Non-linear Telegraph Differential Equation as : telegraph $\frac{\partial^2 \mathcal{J}(\xi,\tau)}{\partial \tau^2} + 2\frac{\partial \mathcal{J}(\xi,\tau)}{\partial \tau} + \mathcal{J}(\xi,\tau) = \frac{\partial^2 \mathcal{J}(\xi,\tau)}{\partial \xi^2}$ ; $t \geq 0$

With initial condition, $\mathcal{J}(\xi,0) = e^{\xi}$ , $\mathcal{J}_\tau(\xi,0) = -2e^{\xi}$

**Solution:** Applying the Elzaki transform on both side of above equations

$$E_\tau \frac{\partial^2 \mathcal{J}(\xi,\tau)}{\partial \tau^2} + E_\tau 2\frac{\partial \mathcal{J}(\xi,\tau)}{\partial \tau} + E_\tau \mathcal{J}(\xi,\tau) = E_\tau \frac{\partial^2 \mathcal{J}(\xi,\tau)}{\partial \xi^2}$$

$$\frac{1}{v^2}E_\tau[\mathcal{J}(\xi,\tau)] - \mathcal{J}(\xi,\tau) - v\mathcal{J}_\tau(\xi,\tau) + 2\left[\frac{1}{v}E_\tau[\mathcal{J}(\xi,\tau)] - v\mathcal{J}(\xi,0)\right]$$

$$+E_\tau[\mathcal{J}(\xi,\tau)] = E_t[\frac{\partial^2 \mathcal{J}(\xi,\tau)}{\partial \xi^2}]$$

$$E_\tau[\mathcal{J}(\xi,\tau)] = \frac{v^2}{(1+v)^2}e^{\xi} + \frac{v^2}{(1+v)^2}E_\tau[\frac{\partial^2 \mathcal{J}(\xi,\tau)}{\partial \xi^2}]$$

Taking inverse Elzaki transform on both side, and the result will compare as shown in Figure 2.

$$\mathcal{J}(\xi,\tau) = e^{\xi-\tau} - \tau e^{\xi-\tau} + E_\tau^{-1}[\frac{v^2}{(1+v)^2}E_\tau\left[\frac{\partial^2 \mathcal{J}(\xi,\tau)}{\partial \xi^2}\right]]$$

$$\mathcal{J}_0(\xi,\tau) = e^{\xi-\tau} - \tau e^{\xi-\tau}$$

$$\mathcal{J}_1(\xi,\tau) = E_\tau^{-1}[\frac{v^2}{(1+v)^2}e^{\xi}\left(\frac{v^2}{1+v} + \frac{v^3}{(1+v)^2}\right)] = \frac{t^2}{2!}e^{\xi-\tau} - \frac{t^3}{3!}e^{\xi-\tau}$$

$$\mathcal{J}_2(\xi,\tau) = E_\tau^{-1}[\frac{v^2}{(1+v)^2}E_\tau\left[\frac{\partial^2 \mathcal{J}_1(\xi,\tau)}{\partial x^2}\right]] = \frac{t^4}{4!}e^{x-t} - \frac{t^5}{5!}e^{x-t}$$

$$\mathcal{J}_3(\xi,\tau) = \frac{\tau^6}{6!}e^{\xi-\tau} - \frac{\tau^7}{7!}e^{\xi-\tau}, \mathcal{J}_4(\xi,\tau) = \frac{\tau^8}{8!}e^{\xi-\tau} - \frac{\tau^9}{9!}e^{\xi-\tau}$$

$$\mathcal{J}_5(\xi,\tau) = \frac{\tau^{10}}{10!}e^{\xi-\tau} - \frac{\tau^{11}}{11!}e^{\xi-\tau},\ \mathcal{J}_6(\xi,\tau) = \frac{\tau^{12}}{12!}e^{\xi-\tau} - \frac{\tau^{13}}{13!}e^{\xi-\tau}$$

$$\mathcal{J}_7(\xi,\tau) = \frac{\tau^{14}}{14!}e^{\xi-\tau} - \frac{\tau^{15}}{15!}e^{\xi-\tau}.$$

Hence, $\mathcal{J}(\xi,\tau) = e^{\xi-\tau}(1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \frac{\tau^4}{4!} - \frac{\tau^5}{5!} + \cdots$

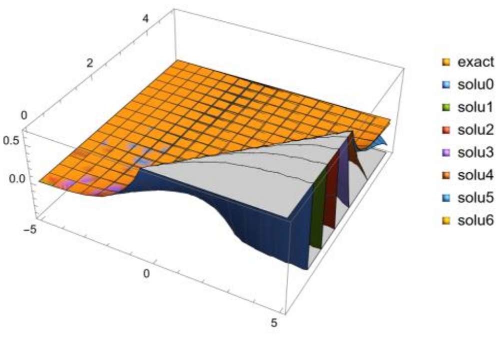


**Figure 2**
**Compasion solutions between the exact solution and several approximate solutions**

## 3. Conclusions

To get approximate analytical solutions for the Riccati differential equation and the Telegraph equation, we have successfully used ETADM in our work. In the aforementioned cases, we saw that, in contrast to ADM, the ETADM with the beginning conditions only takes a few iterations and less computational steps to provide a best approximation to the exact solution. This is demonstrated to be a practical, efficient, and trustworthy tool for solving differential equations.

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*Received January, 2024*