Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 4, pp. 849–855
DOI : 10.47974/JIM-1882

A new parallel Heronion Runge Kutta mean method by using forward backward technique

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Abstract

Runge Kutta methods regards one of the best methods to solve Initial Value Problems. Our new method used the Heronioun mean instead of the traditional arithmetic mean, and used the forward – backward technique with a delay of backward part to get a full advantage of parallelism, to make it suitable for this generation of computers.

Subject Classification: *65L06, 65L03.*

Keywords: *Initial value problems, Heronion mean method, Forward – backward technique, Stability region.*

1. Introduction

Since the papered of the Runge Kutta method by the German researcher Carl Runge in 1895 [1], and the development of it still until now. Montijano [2] introduced explicit RKN methods for numerical solution of second order linear inhomogeneous IVPs. Ajiya [3] presented numerical solution for optimal control problems by using predictor corrector methods. Jasim [4] found a new Runge Kutta method by using the Heronion mean instead of classical arithmetical mean and the forward backward technique. So, the need of new methods are necessary in this time [9,10]. Here, in our article we found a new connect between predictor and corrector parts to present our new method we called it (PPCHeM2) by parallelism employing Heronian mean, and we found the interval of absolute stability of it.

2. Formula of the Heronian mean :

Evans [5] presented the first Runge Kutta of the Heronion mean method which take the form,

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$$y_{i+1} - y_i = \frac{l}{i}\sum_{j=1}^{i}\left\{(m_j + m_{j+1} + \left(m_j m_{j+1}\right)^{1/2})/3\right\} \qquad (1)$$

where, $m_j = f\left(t_j + s_j l\,, y_j + l\sum_{j=1}^{i-1} r_{ji}\, m_j\right),\ s_j = \sum_{j=1}^{i-1} r_{ji}$, j=1,2,3,,, n (2)

Our PPCHeM2 forward part calculate y_{i+1}^{p} depending on both y_i^p and y_{i-1}^c, more than to calculate backward part y_i^c we need to use both y_{i-1}^c and y_i^p. This can allow to get the full advantage to calculate y_i^p and y_i^c simultaneously.

So, we can write our PPCHeM2 as :

$$y_{i+1}^{p} - y_i^p = \frac{l}{3}(m_1 + m_2 + (m_1 m_2)^{1/2}) \qquad (3)$$

Where,

$$m_1 = f\left(t_{j-1}, y_{i-1}^c\right),\ m_2 = f\left(t_{j-1} + l\,, y_{i-1}^c + l m_1\right) \qquad (4)$$

And,

$$y_{i+1}^{c} - y_i^c = \frac{l}{3}(n_1 + n_2 + (n_1 n_2)^{1/2}) \qquad (5)$$

Where,

$$n_1 = f\left(t_j, y_i^p\right),\ n_2 = f\left(t_j - l\,, y_i^p - l n_1\right) \qquad (6)$$

By used f as a function of y only. So, (4) and (6) becomes:

$$m_1 = f(y_{i-1}^c),\ m_2 = f(y_{i-1}^c + l m_1) \qquad (7)$$

$$n_1 = f\left(y_i^p\right),\ n_2 = f\left(y_i^p - l n_1\right) \qquad (8)$$

We can illustrate the difference between our new PPCHeM2 connection (shown in Figure 1 bellow) and the old method of Jasim [4] which (shown in Figure 2)

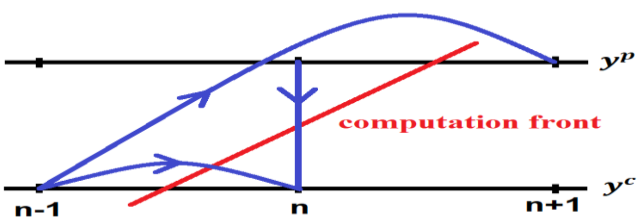

Figure 1
Represent the old method of Jasim [4].

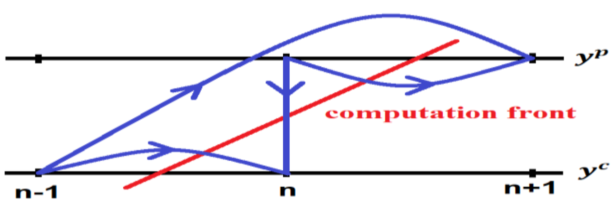

Figure 2
Represent our new PPCHeM2.

3. Deriving of PPCHeM2 formula :

PPCHeM2 forward part has form:

$$y_{i+1}^{p} - y_{i}^{p} = \tfrac{l}{3}(am_1 + bm_2 + (abm_1m_2)^{1/2}) \tag{9}$$

$$m_1 = f(y_{i-1}^{c}),\ m_2 = f(y_{i-1}^{c} + clm_1) \tag{10}$$

For deriving forward part of PPCHeM2, we need to expansion of m_1 and m_2. so,

$$m_1 = f(y_{i-1}^{c}) = f,\ m_2 = f(y_{i-1}^{c} + clm_1) = f + clff_{y_{i-1}^{c}} + o(l^2)$$

Now,

$$am_1 + bm_2 = af(y_{i-1}^{c}) + bf(y_{i-1}^{c} + clm_1) = af + b(f + clff_{y_{i-1}^{c}}) + o(l^2) \tag{11}$$

$$(abm_1m_2)^{1/2} = (ab)^{1/2} f\left(1 + \tfrac{cl}{2} f_{y_{i-1}^{c}}\right) + o(l^2) \tag{12}$$

Gathering (11) and (12), we get:

$$am_1 + bm_2 + (abm_1m_2)^{1/2} = (a + b + (ab)^{1/2})f + (b + \tfrac{(ab)^{1/2}}{2})clff_{y_{i-1}^{c}} + o(l^2) \tag{13}$$

Substituting (13) in (9),

$$y_{i+1}^{p} - y_{i}^{p} = \tfrac{l}{3}\{(a + b + (ab)^{1/2})f + (b + \tfrac{(ab)^{1/2}}{2})clff_{y_{i-1}^{c}} + o(l^2)\} \tag{14}$$

By comparing the equivalent terms of Equation (13) with Taylor series of the form, $y_{i+1}^{p} - y_{i}^{p} = lf + \frac{l^2}{2} ff_{y_{i-1}^{c}} + o(l^3)$

Getting

$(a + b + (ab)^{1/2}) = 3$, $c\left(b + \frac{(ab)^{1/2}}{2}\right) = \frac{3}{2}$, Choosing $c = 1$ then $a = b = 1$

So (9) and (10) becomes,

$$y_{i+1}^{p} - y_{i}^{p} = \tfrac{l}{3}(m_1 + m_2 + (m_1m_2)^{1/2}) \tag{15}$$

$$m_1 = f(y_{i-1}^{c}),\ m_2 = f(y_{i-1}^{c} + lm_1) \tag{16}$$

Now, to deriving backward part of PPCHeM2 let

$$y_{i}^{c} - y_{i-1}^{c} = \tfrac{l}{3}(dn_1 + en_2 + (den_1n_2)^{1/2}) \tag{17}$$

$$n_1 = f\left(y_{i}^{p}\right),\ n_2 = f\left(y_{i}^{p} - cln_1\right) \tag{18}$$

Expansion of n_1 and n_2. so,

$$n_1 = f\left(t_j, y_{i}^{p}\right) = f,\ n_2 = f - clff_{y_{i}^{p}} - o(l^2)$$

$$dn_1 + en_2 == (d + e)f - eclff_{y_{i}^{p}} - o(l^2)) \tag{19}$$

$$den_1n_2 = def^2(1 - clf_{y_{i}^{p}}) - o(l^2)$$

$$(den_1n_2)^{1/2} = (de)^{1/2} f\left(1 - \tfrac{cl}{2} f_{y_{i}^{p}}\right) - o(l^2) \tag{20}$$

Gathering (19) and (20), we get:

$$dn_1 + en_2 + (den_1n_2)^{1/2} = \left(d + e + (de)^{1/2}\right)f - (e + \tfrac{(de)^{1/2}}{2})clff_{y_i^p} - o(l^2) \quad (21)$$

Substitute (21) in (17),

$$y_{i-1}^c - y_i^c = -\tfrac{l}{3}\{\left(d + e + (de)^{1/2}\right)f - \left(e + \tfrac{(de)^{1/2}}{2}\right)clff_{y_i^p} - o(l^2)\} \quad (22)$$

Comparing the equivalent terms of (22) with Taylor series of the form,

$y_{i-1} - y_i = -lf + \frac{l^2}{2}ff_{y_i} - o(l^3)$

We get,

$d + e + (de)^{1/2} = 3$, $\left(e + \frac{(de)^{1/2}}{2}\right) = \frac{3}{2}$, By choosing $d = 1$, so, $e = c = 1$.

Thus (22) becomes,

$$y_i^c - y_{i-1}^c = \tfrac{l}{3}(n_1 + n_2 + (n_1n_2)^{1/2}) \quad (23)$$

$$n_1 = f(y_i^p), n_2 = f(y_i^p - ln_1) \quad (24)$$

Finally equations (15), (16), (23) and (24) represent our new PPCHeM2.

4. Analysis of stability for PPCHeM2:

By using the equation $\acute{y} = \lambda y$ where $\lambda = \partial f/\partial y$ constant [6-8],we try to determine the absolute stability interval of PPCHeM2 by applying it on the forward part, which has the form,

$$y_{i+1}^p - y_i^p = \tfrac{l}{3}(m_1 + m_2 + (m_1m_2)^{1/2}) \quad (25)$$

$$m_1 = f(y_{i-1}^c) , m_2 = f(y_{i-1}^c + lm_1)$$

Since $\acute{y} = \lambda y$ then,

$$m_1 = f(y_{i-1}^c) = \lambda y_{i-1}^c \quad (26)$$

$$m_2 = \lambda y_{i-1}^c + l\lambda^2 y_{i-1}^c \quad (27)$$

Sub. (26) and (27) in (25)

$$y_{i+1}^p - y_i^p = \lambda y_{i-1}^c \tfrac{l}{3}(2 + l\lambda + (1 + l\lambda)^{1/2})$$

$$s = \frac{y_{i+1}^p - y_i^p}{y_{i-1}^c} = \frac{l\lambda}{3}(2 + l\lambda + (1 + l\lambda)^{1/2}) \quad (28)$$

The last equation (28) is fulfilled the condition of absolute stability if $|s| < 1$,

and this achieving when $l\lambda \in (-2.732,0)$, which represent stability region of PPCHeM2 by consider h=0.05 as shown in Figure 3, and h=0.1 as shown in Figure 4.

5. Applying PPCHeM2 method in an example:

We used the IVPs $\acute{y} = y$, $y(0) = 1$, step size *h = 0.05, 0.1* . In Table 1, 2.

Table 1
Results of PPCHeM2 when step size h =0.05.

t	*y exact slution*	y^c**backward**	y^p**forward**	$\lvert y - y^c \rvert$	$\lvert y - y^p \rvert$
0	1	1	1	0	0
0.05	1.0512710963	1.0512764369	1.0100501670	5.3405723e-06	0.04122092929
0.10	1.1051709180	1.1051818730	1.0613266040	1.0954961e-05	0.04384431404
0.15	1.1618342427	1.1618510999	1.1152320401	1.6857206e-05	0.04660220260
0.20	1.2214027581	1.2214258202	1.1719012670	2.3062066e-05	0.04950149114
0.25	1.2840254166	1.2840550017	1.2314759873	2.9585056e-05	0.05254942937
0.30	1.3498588075	1.3498952500	1.2941051688	3.6442486e-05	0.05575363874
0.35	1.4190675485	1.4191112000	1.3599454171	4.3651505e-05	0.05912213144
0.40	1.4918246976	1.4918759277	1.4291613671	5.1230138e-05	0.06266333045
0.45	1.5683121854	1.5683713828	1.5019260948	5.9197336e-05	0.06638609062
0.50	1.6487212707	1.6487888437	1.5784215499	6.7573020e-05	0.07029972078
0.55	1.7332530178	1.7333293960	1.6588390108	7.6378135e-05	0.07441400706
0.60	1.8221188003	1.8222044350	1.7433795630	8.5634699e-05	0.07873923730
0.65	1.9155408290	1.9156361948	1.8322546021	9.5365856e-05	0.08328622684
0.70	2.0137527074	2.0138583034	1.9256863619	0.00010559594	0.08806634551
0.75	2.1170000166	2.1171163671	2.0239084704	0.00011635053	0.09309154611
0.80	2.2255409284	2.2256685850	2.1271665342	0.00012765652	0.09837439426
0.85	2.3396468519	2.3397863941	2.2357187521	0.00013954218	0.10392809982
0.90	2.4596031111	2.4597551483	2.3498365611	0.00015203723	0.10976654996
0.95	2.5857096593	2.5858748322	2.4698053154	0.00016517292	0.11590434383
1	2.7182818284	2.7184608105	2.5959249993	0.00017898209	0.12235682913

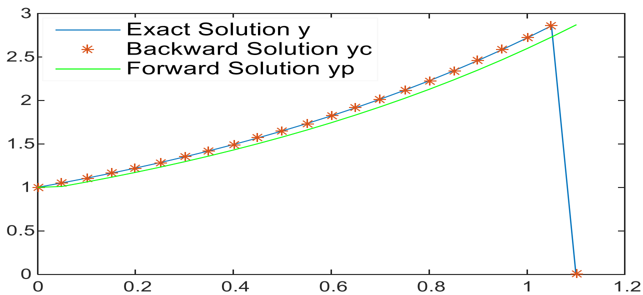

Figure 3
Results of PPCHeM2 when step size h =0.05.

Table 2
Results of PPCHeM2 when step size *h =0.1*

t	*y exact slution*	y^c**backward**	y^p**forward**	$\lvert y - y^c \rvert$	$\lvert y - y^p \rvert$
0	1	1	1	0	0
0.1	1.10517091807	1.1052147338	1.0100501670	4.38157394e-05	0.0951207509
0.2	1.22140275816	1.2214949977	1.1152649008	9.22396203e-05	0.1061378572

0.3	1.34985880757	1.3500045638	1.2315451648	0.000145756285	0.1183136427
0.4	1.49182469764	1.4920295989	1.3600547309	0.000204901347	0.1317699666
0.5	1.64872127070	1.6489915374	1.5020797660	0.000270266749	0.1466415046
0.6	1.82211880039	1.8224613070	1.6590417045	0.000342506690	0.1630770958
0.7	2.01375270747	2.0141750516	1.8325114741	0.000422344173	0.1812412333
0.8	2.22554092849	2.2260515067	2.0242252187	0.000510578237	0.2013157097
0.9	2.45960311115	2.4602112031	2.2361016738	0.000608091958	0.2235014373
1	2.71828182845	2.7189976897	2.4702613701	0.000715861287	0.2480204582

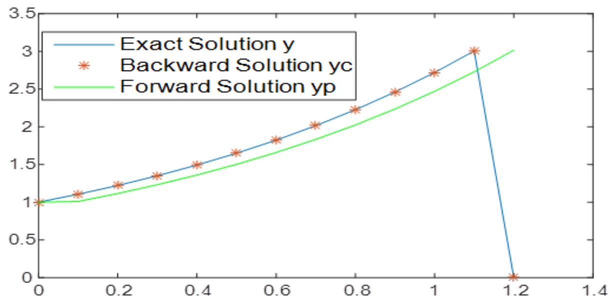

Figure 4
Results of PPCHeM2 when step size *h =0.1*.

6. Conclusions and discussion the numerical results of PPCHeM2

From observing the results of PPCHeM2 in the Figure 1 & 2. And the tables1 & 2, we notes that our new method is has a very good results comparing with the exact solution in column2, and we see that the difference between the results of backward part (in column5) with the exact (column2), are much better than the results of forward part (column6) with the exact (column2).

7. Acknowledgments

We are grateful to the University of Mosul, for their provided facilities.

References

[1] Butcher, J.C., " A history of Runge-Kutta methods", *Applied Numerical Mathematics*, vol., 20, pp 247-260, (1996).

[2] Montijano, J.I. etal, "Explicit Runge–Kutta-Nyström methods for the numerical solution of second order linear inhomogeneous IVPs", *Journal of Computational and Applied Mathematics*, Vol. 438, (2023), DOI: https://doi.org/10.1016/j.cam.2023.115533.

[3] Ajiya, Y. et al., "improve numerical method for solving optimal control problems using predictor corrector method", *Bima Journal of Science*

and Technology, Vol. 7 (2.1), (2023), DOI: https://doi.org/10.56892/bima.v7i02.%201.457.

[4] Jasim, M. D. "Implementation of the Heronion mean (HeM) in the predictor corrector mode," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 7, pp. 1917–1923, (2021), doi: 10.1080/09720502.2021.1964738.

[5] Evans, D. J. and N. Yaacob " A Forth Order Runge – Kutta Method Based on the Heronian Mean Formula", Parallel Algorithms Research Centre , University of Technology, Loughbrough, Leics, U.K., Report No. 879, (1994).

[6] Kouser, S., et al, "Beyond Traditional RK3: Exploring the Stability and Convergence of efficient Explicit Modified Runge- Kutta Techniques", *International Journal of Applied and Computational Mathematics*, (2023), DOI: 10.21203/rs.3.rs-3330773/v1.

[7] Shampine,L.F.," Stability of explicit Runge—Kutta methods", Volume 10, Issue 6, Pp. 419-432, (1984), Doi.org/10.1016/0898-1221(84)90072-5.

[8] Agbeboh,G. U. et al, "On the Stability Analysis of a Sixth-Stage Fifth-Order Runge-Kutta Method for Solving IVPs in ODEqs", *Nigerian Journal of Mathematics and Applications,* Volume 32, pp. 273 – 286, (2023).

[9] Abbood, Mohaimen M., Ebrahim, Hassan H., Al-Fayadh, Ali & Obaid, Ahmed J. () Measure defined on Γ – algebra and some of their generalizations, *Journal of Interdisciplinary Mathematics*, 1-7, DOI: 10.47974/JIM-1502.

[10] Shelash, Hader Baqer, Ahmad, Hasanain Hamid & Obaid, Ahmed J. On characterizations some of subgroups of the dihedral group D2n, *Journal of Discrete Mathematical Sciences and Cryptography*, 1-6, DOI: 10.47974/JDMSC-1553.

Received January, 2024