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A modified (CG) method for solving nonlinear systems of monotone equations

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Abstract

In this paper, a new modified of conjugate gradient method named the (HSH) algorithm is presented for solving nonlinear systems of monotone equations (NSME). The main goal is the (HSH) technique can decrease the number of iterations (N_i), the CPU-time (N_T), and the number of functions evaluation(N_f), by MATLAB R_2020a and a numerical results comparison with SCG method and PRP method. Under some standard conditions, the well- define and the convergence of the (HSH) algorithm are proved. The numerical experiment indicated the efficiency of the (HSH).

Subject Classification: *46N10, 47N10, 90C30.*

Keywords: *Nonlinear systems, Conjugate gradient method (CGM), Global convergence.*

1. Introduction

Consider the following (NSME) [1]

$$\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}\,, \forall\, \boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{i}} \tag{1}$$

where $\boldsymbol{F}: \boldsymbol{R}^{\boldsymbol{i}} \rightarrow \boldsymbol{R}^{\boldsymbol{i}}$; $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3} \ldots$ is continuous monotone function, that is $\boldsymbol{F}$ holds: $\left(\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})\right)^{\boldsymbol{T}}(\boldsymbol{x} - \boldsymbol{y}) \geq \mathbf{0}$; $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^{\boldsymbol{i}}$.

If the solution set of (1) is non empty then the solution set is convex. In addition the space R^i is vector space with Euclidean norm defined as in [2, 3].To solving (1), the (HSH) must choose a good initial point $x_0 \in R^i$ then the (HSH) generates a sequence $\{x_t\}$, when [4, 5]:

$$z_t = x_{t+1} \text{ s.t. } z_t = x_t + \alpha_t d_t. \tag{2}$$

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and the step length $\alpha_t = \max\{sr^{kt}; k = 0, 1, \dots\}$ satisfies the line search inequality [6, 7]:

$$-F(x_t + \alpha_t d_t)^T d_t \geq \rho\alpha_t \|F(x_t + \alpha_t d_t)\| \, \| d_t\|^2 \tag{3}$$

Where d_t is called the search direction defines as [8]:

$$d_t = \begin{cases} -F(x_t) \; if \; t = 0 \\ -\left(\frac{d_{t-1}^T y_{t-1}}{\| F_{t-1}\|^2} - \frac{d_{t-1}^T F_t F_t^T F_{t-1}}{\| F_t\|^2 \| F_{t-1}\|^2}\right) F(x_t) + \left(\frac{F_t^T y_{t-1}}{\| F_{t-1}\|^2}\right) d_{t-1} \; if \; t = 1,2, \dots \end{cases} \tag{4}$$

such that $y(x_t) = F(x_{t+1}) - F(x_t)$

1.1. The main steps of the (HSH) algorithm are:

Step 0. Take $x_0 \in R^i$ as an initial point and constants $\epsilon, s, r, \rho \in (0, 1)$.

Step 1. if $\|F(x_0)\| \leq \epsilon$ then stop, else go to step 2.

Step 2. Compute α_t form (3).

Step 3. Calculate x_{t+1} form (2), if $\|F(x_{t+1})\| \leq \epsilon$, then stop, else go to step 4.

Step 4. Use (4) to Calculate d_t.

Step 5. Set t: = t+1, go to step 2.

To proof the convergence of the (HSH) and the well- define of (3), we need the next lemma 1 and the following assumptions and accepting without proof.

Lemma 1: [9] *Let d_t given by (4) then:*

$$F_t^T d_t = -\| F_t\|^2, \forall\, t = 0,1, \dots \tag{5}$$

Consider the assumptions $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$, **and** $\mathbf{A_4}$ as in [10, 11]. From $\mathbf{A_4}$ we have:

a) If a $\{x_t\}$ is finite then $\lim_{t\to\infty} \| x_{t+1} - x_t\| = 0$.

b) If there exist a sequence $\{d_t\}$ is generated by (HSH) and there exist a constant $e > 0$, $\| F_t\| \geq e$ for non- negative integer t then a $\{d_t\}$ is bounded.

c) If z_t satisfies $F(z_t)^T(x_t - z_t) > 0$ then:

$$x_{t+1} = x_t - \frac{F(z_t)^T (x_t - z_t)}{\| F(z_t)\|^2} F(z_t) \tag{6}$$

Now, to prove that the (3) is well- define, as the same way in [12, 13]. From the contradiction we impose that for some iterate index t, the (3) is defined as:

$$-F(x_t + \alpha_t d_t)^T d_t < \rho\alpha_t \|F(x_t + \alpha_t d_t)\| \, \| d_t\|^2$$

Now if we take $\lim_{k\to\infty}$ for two sides then we get: $\| F(x_t)\|^2 < 0$. This contradiction, thus (3) is well- defined.

2. Convergence property of algorithm (HSH) [14, 15]

With the above preparation, Start proving the (HSH) is convergence.

Theorem 1: *If the* $\{x_t\}$ *generated by the (HSH) then* $lim\ inf_{t\to\infty}\|F(x_t)\| = 0$.

Proof: Assume $lim\ inf_{t\to\infty}\|F(x_t)\| \neq 0$ for some t, by A4,

$$\| F(x_t)\| \geq e. \tag{7}$$

Form (2), (3) and (6) we obtain:

$$x_{t+1} - x_t \geq -\frac{\rho\,(\alpha_t)^2 \,\| d_t\|^2}{\| F(z_t)\|} F(z_t).$$

Taking the norm and the $\lim\limits_{t\to\infty}$ and by A_4 obtain:

$$\lim_{t\to\infty} (\alpha_t) = 0.$$

Let $\alpha_t^* = r^{-1}\alpha_t$, s.t.

$$-F(x_t + \alpha_t^* d_t)^T d_t < \rho\alpha_t^*\|F(x_t + \alpha_t^* d_t)\| \, \| d_t\|^2. \tag{8}$$

Without loss of generate, Taking

$$(\lim_{t\to\infty} x_t = x^*, \lim_{t\to\infty} d_t = d^* \ \forall t \in N). \tag{9}$$

If taking the $\lim\limits_{t\to\infty}$ of (8), and by (9) then obtain:

$$F(x^*)^T d^* > 0. \tag{10}$$

From (7) and (5) we get:

$$(F(x_t))^T d_t < 0.$$

Taking the $\lim\limits_{t\to\infty}$ of the two sides and by (9) we get:

$$F(x^*)^T d^* < 0.$$

This is a contradiction with (10), thus $lim\ inf_{t\to\infty}\|F(x_t)\| = 0$

3. Numerical Experiment

From the MATLAB, by the numerical results of (HSH) and comparison with SCG method in [16] and PRP method in [17]. And problems from [18-21].

Table 1
The numerical results of N_f, N_i and N_T

p_i	Dim.	x_t	(HSH)			SCG			PRP		
			N_f	N_i	N_T	N_f	N_i	N_T	N_f	N_i	N_T
p_1	10000	x_0	**13**	**4**	**0.1**	30004	10001	116.1	30004	10001	1
	10000	x_1	**13**	**4**	**0.3**	30004	10001	88.5	30004	10001	0.2
	10000	x_2	**16**	**5**	**0.1**	30004	10001	96.8	30004	10001	0.2
	10000	x_3	**13**	**4**	**0.5**	30004	10001	110.6	30004	10001	0.4
	10000	x_4	**16**	**5**	**0.1**	30004	10001	121.8	30004	10001	0.2
	10000	x_5	**16**	**5**	**0.1**	30004	10001	101.7	30004	10001	0.1
	10000	x_6	**16**	**5**	**0.1**	30004	10001	109	30004	10001	0.2
	10000	x_7	**16**	**5**	**0.1**	30004	10001	107.2	30004	10001	0.2
p_2	1000	x_0	**13**	**4**	**0.1**	3004	1001	83.8	2113	704	0.6
	1000	x_1	**13**	**4**	**0.2**	3004	1001	88.4	3004	1001	1.3
	1000	x_2	**13**	**4**	**0.3**	52	17	1.5	3004	1001	1.1

	1000	x_3	**13**	**4**	0.2	52	17	1.5	2080	693	0.1
	1000	x_4	**13**	**4**	0.2	52	17	1.6	2080	693	0.6
	1000	x_5	**16**	**5**	0.4	3004	1001	94.6	1981	660	0.1
	1000	x_6	**16**	**5**	0.4	3004	1001	87.2	1981	660	0.1
	1000	x_7	**16**	**5**	**0.1**	3004	1001	83	3004	1001	1
p_3	10000	x_0	**13**	**4**	**0.3**	30004	10001	725.1	1903	634	0.8
	10000	x_1	**13**	**4**	0.4	49	16	1.3	2197	732	0.1
	10000	x_2	**13**	**4**	**0.4**	30004	10001	672.4	30004	10001	10.6
	10000	x_3	**13**	**4**	**0.3**	745	248	18.4	30004	10001	10.1
	10000	x_4	**13**	**4**	**0.1**	745	248	16.7	30004	10001	11.5
	10000	x_5	**16**	**5**	**0.2**	30004	10001	776.9	2017	672	0.3
	10000	x_6	**16**	**5**	**0.3**	30004	10001	692	2017	672	0.4
	10000	x_7	**16**	**5**	**0.1**	30004	10001	677.6	2017	672	0.3
p_4	5000	x_0	**13**	**4**	**0.2**	15004	5001	125.9	15004	5001	0.3
	5000	x_1	**13**	**4**	0.1	15004	5001	120.9	15004	5001	0.1
	5000	x_2	**13**	**4**	**0.5**	15004	5001	121.5	15004	5001	0.4
	5000	x_3	**13**	**4**	**0.2**	15004	5001	83.5	15004	5001	0.5
	5000	x_4	**16**	**5**	**0.1**	15004	5001	77	15004	5001	0.2
	5000	x_5	**13**	**4**	**0.5**	15004	5001	80.2	15004	5001	0.4
	5000	x_6	**13**	**4**	**0.2**	15004	5001	85.8	15004	5001	0.3
	5000	x_7	**13**	**4**	**0.2**	15004	5001	74.8	15004	5001	0.4
p_5	50000	x_0	**16**	**5**	**0.3**	150004	50001	463.5	150004	50001	1.7
	50000	x_1	**16**	**5**	**0.6**	150004	50001	513.9	150004	50001	1.5
	50000	x_2	**16**	**5**	**0.6**	150004	50001	527.2	150004	50001	2.7
	50000	x_3	**16**	**5**	**0.1**	150004	50001	485.1	150004	50001	7.1
	50000	x_4	**16**	**5**	**0.5**	150004	50001	438.7	150004	50001	9.9
	50000	x_5	**16**	**5**	**0.5**	150004	50001	321	150004	50001	3.8
	50000	x_6	**16**	**5**	**0.6**	150004	50001	369.4	150004	50001	4.2
	50000	x_7	**16**	**5**	**0.6**	150004	50001	364	150004	50001	4.3
p_6	50000	x_0	**10**	**3**	**0.1**	34	11	24.2	2833	944	0.2
	50000	x_1	**10**	**3**	0.1	34	11	26.5	1237	412	0.1
	50000	x_2	**10**	**3**	0.1	34	11	22	1513	504	0.1
	50000	x_3	**10**	**3**	**0.1**	34	11	20.5	150004	50001	5.5
	50000	x_4	**10**	**3**	0.1	34	11	22.6	2095	698	0.1
	50000	x_5	**10**	**3**	0.6	34	11	21.8	1318	439	0.1
	50000	x_6	**10**	**3**	**0.4**	34	11	22.4	2518	839	0.5
	50000	x_7	**10**	**3**	0.3	34	11	22.1	1606	535	0.1

4. Discussion and Conclusion

From the table above, we conclude that the numerical results of (N_f), (N_i) and (N_T) indicators in dark tint of (HSH) are less than the SCG method and PRP method in most of problems, so the (HSH) is better in this side.

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