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α–connected fibrewise topological spaces

Y. Y. Yousif *
Department of Mathematics
College of Education for Pure Science (Ibn Al–Haitham)
University of Baghdad
Baghdad
Iraq

Noor Allawyi Hussein †
Ministry of Education
Directorate of Education (Al-Kark-2)
Baghdad
Iraq

Abstract

The theory of Topological Space Fiber is a new and essential branch of mathematics, less than three decades old, which is created in forced topologies. It was a very useful tool and played a central role in the theory of symmetry. Furthermore, interdependence is one of the main things considered in topology fiber theory. In this regard, we present the concept of topological spaces α associated with them and study the most important results.

Subject Classification: *55R70, 54C08, 54C10.*

Keywords: *Fiberwise topology, α-open set, α-connected fiberwise topological space, α-connectedness of fiberwise subspaces.*

1. Introduction

The brilliant definition of the compact topology of fiberwise combined with fiber function space by Booth was introduced by Brown [12, 13] in 1978. In 1982, Neifeld [16] unified the fiberwise space and base space. Moreover, the Niefield condition weakened to natural topological space introduced via I. M. James [7] in 1989. A modern space in the optical fiberwise group began to

* *E-mail:* **yousif.y.y@ihcoedu.uobaghdad.edu.iq (Corresponding Author)**
† *E-mail:* **nooralawy26@gmail.com**

develop thanks to the subject of James [8] in 1989, in your publication titled "Fiberwise Topology", so I summarized the key findings of the latter.

Furthermore, the published "Fiber Homogeneity Theory" was written by M. C. Crabb, I. M. James [9] in 1998, promoting the fibewiser topological space. In literature [8, 7]. Most of the results in James (1984) and James (1989). Our goal of this work is to product the fiberwise topological space concept. We are studying several properties and got some new results for this structure. Also we investigate important theorems and fiberwise subspace properties in fiberwise topological spaces [1-6]. We define and study new concepts for the fiber topological space in this paper via the fiber topological space connected α on open α. We also introduce the concepts of open topological space α. We will base our research on some sources, including [13-22].

2. Preliminaries

In this part, we will show some information's and install the nomenclature, that have been saving in James [7], for use in all of our current studies .

Definition 2.1: [14] A subset F for a topological space (Z, τ) be called α-open if $\subseteq Int\ (cl\ (Int(F)))$.

Example 2.1: Consider Z = {1, 2, 3} with topology τ = {Z, ∅, {2} on Z so α- τ = {Z, ∅, {2},{1,2},{2,3}}.

Take 𝔅 as a base sets then fiberwise set over 𝔅 be a set Z balance and projected map ρ : Z→ 𝔅. Of any b ∈ 𝔅 , the subset $Z_b = \rho^{-1}(b)$ for Z it's called the fibre round b. The subspace for Z be fiberwise space round 𝔅, where the map ρ holds.

Definition 2.2: [12] A function ψ between two fiberwise sets Z, H, ρ a projection round 𝔅, q a projection round 𝔅 is known as fiberwise if $\rho o \psi =$ q.

By definition (2.2) we note the set $\psi(Z_b)$ is sub set for H_b , for all b ∈ 𝔅.

Definition 2.3: Suppose Z is a fiberwise set round 𝔅 be topology spaces. The topological on Z be called fiberwise topology if the projected function ρ be continuous.

Definition 2.4: [15] A fiberwise topological space Z round 𝔅 denoted by connected fiberwise topology spaces if of all two points z, n in Z_b, when b ∈ 𝔅, Z cannot be represent as the union for two nonempty opened and discrete subset for Z, said N and M, when n ∈ N and m ∈ M otherwise. It is said that Z is disconnected space.

Definition 2.5: [7] A fiberwise function $\psi: Z \to H$, where Z and H be fiberwise topological space round 𝔅 be called α-continuous if of all point $z \in Z_b$ where $b \in$ 𝔅 inveres image for all open set of $\psi(z)$ be α-open set for z.

Proposition 2.6: [8] Let ω : Z→ L is an open fiberwise function, when Z and L be spaces over 𝔅. If L is fiberwise α- opened, then Z is fiberwise opened.

Proposition 2.7: [8] Let ω : Z→ L be an α-opened fiberwise function, Where Z and L are spaces round 𝔅. If L as a fiberwise opened, then Z is a fiberwise α-opened.

3. α-connected Fibrewise Topological Space:

In this section, we studied the connect of α- connected fiberwise and we gave some new other definitions to reach the best relations between α-connected fiberwise and connectedness fiberwise.

Definition 3.1: A fiberwise topology spaces Z on W can be said to be a α-connectness fiberwise topology spaces if for all pointed n and m in $Z_{\_w}$, when w ∈W, Z cannot be represented by the union of two discrete non-empty points α-open subset for Z, said H and K , when $n \in H$ and $m \in K$ or else , Z is denoted by α-disconnected fiberwise topological space .

Example 3.2: When Z = {a, b, c} and τ_z = {Z, ∅ , {b}}. Then α-τ_z = {Z,∅,{b}, {a,b}, {b,c}}, with 𝔅 = {1, 2, 3} and $\tau_{𝔅}$ = {𝔅, ∅,{1}}. Define a projected function ρ : Z⟶ 𝔅 by ρ(a) = 2, ρ(b) = 1 and ρ(c) = 3. It's clear that α-τ_z is α-connected fiberwise topological space. Because ρ is an α-continuous, the inverse image of all open set in τ on 𝔅 is α-open in α-τ_z on Z . On other said Z is α-connected.

Example 3.3: Suppose Z = {a, b, c,} and τ_z = {Z, ∅, {b}, {a,c}} = α-τ_z, with 𝔅 = {a, b, c} and $\tau_{𝔅}$ ={𝔅, ∅,{a}}. Define a projection function ρ : Z⟶ 𝔅 by ρ(a) = b, ρ(c) = a and ρ(b) = c, Z is an α-disconnected fiberwise topological space.

In the following theorems and examples, we discuss the relation between connected fiberwise and α- connected fiberwise.

Theorem 3.4: *Every connected fiberwise topological space be α-connected fiberwise*

Proof: Suppose that Z is connected fiberwise. To show Z is α- connected fiberwise. Now, since Z is connected fiberwise, this means there Z over W such that for all two points n, m in Z_w, when $w \in W$. Notwo are separate open, nonempty subsets for Z said H, K when $n \in H, m \in K$ and $Z = H \cup K$. So by proposition (2.7), (2.6). The meaning of that H and K are α-open. Hence, Z is an α- connected fiberwise con. Fib.topological space.

The converse for theorem 3.4 dos't true in general we explain that by the following example.

Example 3.5: Let $Z = W = \{a, b, c\}$, $\tau_Z = \{z, ø, \{a\}\}$ and $\alpha - \tau_Z =$ {Z,ø, {a}, {a, b}, {a, c}}. $\alpha - \tau_Z$ is an α- connected fiberwise, topology space, but τ_Z be not connected fiberwise.topological space .

The following theorem explain to us the fiberwise α- connectedness be property we maintain it by fiberwise α- continuous function.

Theorem 3.6: *Let β be onto fiberwise α- continuous functions of an α- connectness fiberwise topology spaces Z to fiberwise topology spaces L. So L is α- connectness fiberwise topology spaces.*

Proof: Suppose z_1, z_2 are two points Z_w where $w \in W$ because β be an α- fiberwise functions, so found two points l_1, l_2 at l_W. So $l_1 = (z_1)$, $l_2 = (z_2)$. And this is paradoxical, we impose l is an α- disconnected fiberwise topological space. Thus, may is a performance by union for two actual disjoint and α-open subsets in l, called H , K in which $l_1 \in H$ and $l_2 \in K$. Wherefore, $z_1 \in \beta^{-1}(H)$ and $z_2 \in \beta^{-1}(K)$. Since for β be a fiberwise α- continuous function. Add to that $\beta^{-1}(H)$, $\beta^{-1}(K)$ the two α-open be disjoint, nonempty subsets from Z. The results of that $z_1 \in \beta^{-1}(H) \cup z_2 \in \beta^{-1}(K)$ be α- disconnectness fiberwise topology spaces, this be contradiction. So l , be α- connected fiberwise topology spaces .

The following theorems show the most important characteristics of the α- connectedness of fiberwise topology spacess.

Theorem 3.7: *Let Z be any α- connectness fiberwise topology spaces . So that the following statements be equivalent:*

(1) Z be α- connected fiberwise topology spaces.

(2) Z can't put as the union for two sub-sets in Z be disjoint nonempty α- closed, said k_1 , k_2 . when $z_1 \in k_1$, $z_2 \in k_2$ for all z_1 , $z_2 \in Z_w$.

(3) No more α-clopen subset in Z , say l when $z \in l$, $z_1 \in l^c$. For all z , $z_1 \in Z_w$, $w \in W$.

(4) If L ⊂ Z such that $z \in L$ and $z_1 \in L^c$, $\forall z, z_1 \in Z_w$. So Bd (L) ≠ ϕ.

Proof:

(1) ⟹(2) Suppose $Z = k_1 \cup k_2$, when k_1 and k_2 be disjoint α-closed, nonempty subsets for Z and $z_1 \in k_1$, $z_2 \in k_2$, $\forall z_1, z_2 \in Z_w$ $w \in W$, so $k_1 = k_2^c$ and $k_2 = k_1^c$ them too a-open subsets in Z. Wherefore Z be not α- con. Fib., this is a contradiction.

(2) ⟹(3) Let $L \subset Z$ and α-clopen subset for Z, when $z \in L$, $z_1 \in L^c$, $\forall z, z_1 \in z_w$, $w \in W$, Furthermore, L is neither Z nor $\emptyset$. Leads to $L^c \subset Z$ and α- clopen. So $Z = L \cup L^c$, contradict in (2).

(3) ⟹(4) If $L \subset Z$ and $z \in L$ and $z_1 \in L^c$, $z, z_1 \in Z_w$ $w \in W, b(L) = \emptyset, cl(L) = \ln(L) \cup b(L)$, we have $cl(L) = \ln(L)$. Also, int$(L) \subset L$ and L⊂ cl (L), so that $L = int(L) = cl(L)$ thus, $L \subset Z$ and α-clopen, which be contradiction .

(4) $\Longrightarrow$(1) Let Z be appear as union of two nonempty disjointed α-opened subsets of Z , said H, K. $z_2 \in K$,$\forall z_1, z_2 \in z_b$, $b \in B$. Then H and K are also α-closed. So, $L = int(L) - cl(L)$.

Hence $b(L) = \emptyset$ be contradiction with (4).

Theorem 3.8: *The fiberwise topology spaces Z be α-connected $\Leftrightarrow$be not fiberwise α-continuous function from Z to L, when $L = \{0,1\}$ and has a discrete topology.*

Proof: **($\Longrightarrow$)** We say there exists fiberwise function is α-continuous from Z onto L. So , we have Z be α-connected fiberwise, L its hold be too (by Theorem 3.6). and that is not satisfy.

($\Longleftarrow$) Let Z be an α-disconnected fiberwise. Consequently Z we write it in the union for two disjoint open, nonempty subsets for Z, when $n \in H$, $m \in K$, for all $n, m \in Z_b$, $b \in \mathfrak{B}$.

Define $\beta: Z \Longrightarrow L$ by $\beta(m_*) = 0$, if $m_* \in H$ and $\beta(m_*) = 1$, if $m_* \in K$

So, $n \in B^{-1}(\{0\}) = H$ and $m \in B^{-1}(\{1\}) = K$

Then β be fiberwise α-continuous function, this be contradicted.

4. α-Connectedness of Fibrewise Subspaces

We define in the part α- connected fiberwise subspace also inspect standards of decision if subspace be α- connected fiberwise.

Definition 4.1: Let Z be a fiberwise topology spaces. A fiber subspace W in Z is called α- connectness fiberwise sub-space if for all pointed z_1 , z_2 in Z_b, when b ∈ B, W cannot be put as the union of two separate α-opened, non-empty subspaces of W, is denoted by H and K such that $z_1 \in H$ and $z_2 \in K$, W is called the α- connected fiberwise topology spaces.

Stain fibrewise subspace for α- connected fiberwise.spaces it doesn't have to be be α- connected fiberwise explained in the following example in addition to the α-connectedness for fiberwise doesn't hereditary property.

Example 4.2: Let $Z = \{1, 2, 3, \}$, τ_z = $\{Z, \emptyset, \{1\}$ with topology $\alpha - \tau_z = \{Z, \emptyset, \{1\}, \{1,2\}, \{1,3\}\}$ on Z and $W = \{1,2\}$ with topology $\tau_z = \alpha - \tau_W = \{W, \emptyset, \{1\}, \{2\}\}$, be an fiberwise topology spaces of Z. It is obvious that W is α-disconnected fiberwise subspace for Z because there are two points 1 and 2 in Z_1 when $1 \in B$ and two disjoint α-opened, non-empty subsets of A , $\{1\}$ *and* $\{2\}$, such that

$1 \in \{1\}$, $2 \in \{2\}$ and $W = \{1\} \cup \{2\}$. Hence Z is an α- connected fiberwise So, the α-connectedness for fiberwise isn't hereditary property .

Theorem 4.3: *Suppose Z is an fibewise. Topology spaces, it is based on the disjoint union of two α-open nonempty for Z , called H , K. So A be an α-connected fibewise topology spaces of , $n \in H \cap A$, $m \in K \cap A$, for all two n and m points in X_b , as $b \in B$, so $A \subset H$ or $A \subset K$*

Proof: If $H \cap A \neq \emptyset$, $K \cap A \neq \emptyset$ then $H \cap A$ and $K \cap A$ be disjoint subsets nonempty of A be α-open in A and so on A is not α-fiberwise connected. For that, $H \cap K = \emptyset$ and so $A \subset K$ or $K \subset A = \emptyset$ and as a result $A \subset H$.

Definition 4.4: Let Z is a fibewise. Topology spaces. The $\phi \neq H$, $K \subset Z$ be said fiberwise separated sets ⟺ for all two points $n \, and \, m$ in Z_b, when $b \in B$, $n \in H$, $m \in K$ and $H \cap cl(K) = cl(H) \cap K = \emptyset$.

The following theorems show the criterion on the basis of which the subspace is α-connected fiberwise or not.

Proposition 4.5: Let Z be fiberwise space. So the fiberwise sub-space A for Z be α-con. ⟺ A Cannot expressing as union for two fiberwise separating sets.

Proof: (⟹) If A be an α-disconnected fiberwise. So A we replace by union of two nonempty subsets for A be disjoint it's called H and K, where are α-clopen in A. Also , $n \in H$, $m \in K$. $cl(H) \cap K = \emptyset$, for each two points $n \, and \, m$ in Z_b, when $b \in \mathfrak{B}$. Let $n \subset H \cap cl(K)$ so because $H \subset A$, $n \in A \cap cl(K) = cl(K)$ in A. Thus $n \in H \cap K = \emptyset$ a contradiction. So $H \cap cl(K) = \emptyset$ and by the same way $cl(H) \cap K = \emptyset$.

(⟸) Let A= H ∪ K, when $\cap \, cl(K) = cl(H) \cap K = \emptyset$, and $n \in H, m \in K$, for each two points n and m in Z_b , $b \in \mathfrak{B}$. So $cl(H)$int A = A∩ $cl(K)$ = (H∩ $K) \cap cl(H) = H \cup \emptyset = S$.

Thus H is an α-closed in A . So , A is an α-disconnected fibreeise .

Proposition 4.6: Let A is an α-conn. fib. subspace for fiberwise space Z, A ⊂ ω ⊂ cl(A). Then ω be α- connected fiberwise subspace for Z.

Proof: Suppose ω be α-dis connected fiberwise. When ω=H⋃K , H and K are fiberwise-separated (proposition 4.5) . because A is an α- connected fiberwise, either A⊂H or A⊂ K. Suppose A⊂H Then cl (A) ⊂ cl(K) . So ω ⊂ cl(A) ⊂ cl(K). Therefore ω = H⋃K , then H ⊂ cl(K) . because H ⋂ cl(K) = ∅ , we come to a contradiction . Thus ω is α-connected fiberwise.

5. Discussion and Conclusion

The theory of fiberwise in topology spacess is one of the most important new branches in mathematics .The brilliant definition of compact topology of fiberwise combined with fiber function space be given by Booth with Broown [12,13] in. 1978 . In 1982 Niefield [16] specified the fiber space. in this work is to introduce the fiberwise topology spaces concept. We are studying several properties and got some new results for this structure. Also we investigate important theorems and fiberwise subspace properties in fiberwise topology spaces. Topological fiber space is currently one of the most important in algebraic topology and general topology. So it has many applications in various grops of mathematical such as differentially geometry and others. This paper define and study new concepts for the topology spaces of fibers via α- -open, α-connected, we mention and introduce several propositions related to these concepts.

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