

Some inequalities of Hermite-Hadamard for product of two stochastic processes

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Abstract

In this work some inequalities of the Hermite-Hadamard are presented for product of two convex, s-convex, m-convex, h-convex and r-convex stochastic processes.

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1. Introduction

A stochastic processes is defined as function $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}$ where $\mathcal{I} \subset \mathbb{R}$ is an interval if for every $\tau \in \mathcal{I}$, with $\mathcal{S}_p(\tau, \cdot)$ being a random variable.

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If for all $\tau_0 \in \mathcal{I}$: $P\text{-}\lim_{\tau \rightarrow \tau_0} S_p(\tau, \cdot) = S_p(\tau_0, \cdot)$ then we say that $S_p(\tau, \cdot)$ is continuous in probability in interval $\mathcal{I}$. (see [1, 2, 3])

If for all $\tau_0 \in \mathcal{I}$: $\lim_{\tau \rightarrow \tau_0} E[(S_p(\tau, \cdot) - S_p(\tau_0, \cdot))^2] = 0$ then we say that $S_p(\tau, \cdot)$ is mean square continuous in the interval $\mathcal{I}$. with $E[S_p(\tau, \cdot)]$ representing the expectation value of the random variable $S_p(\tau, \cdot)$.

If a random variable exist $S'_p : \mathcal{E} \rightarrow \mathbb{R}$ such that for all $t_0 \in \mathcal{I}$ and $\lim_{\tau \rightarrow \tau_0} E\left[\frac{S_p(\tau, \cdot) - S_p(\tau_0, \cdot)}{t - t_0} - S'_p(\tau, \cdot)\right]^2 = 0$, then we say that $S_p(\tau, \cdot)$ is Mean-square differentiable at a point $\tau \in \mathcal{I}$.

For more informations the reader may refer to the work of D.S. Mitrovic, I.B. Lackovic, Hernandez JE and Gomez JF [7, 8, 6].

Definition 1.1: Let $S_p : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}$ be a stochastic process with $E[S_p(\tau, \cdot)^2] < \infty$ for all $\tau \in \mathcal{I}$. Let $[\alpha; \beta] \subset \mathcal{I}$; $\alpha = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \dots < \tau_n = \beta$ be a partition of $[\alpha; \beta]$ and $\lambda_k \in [\tau_{k-1}; \tau_k]$ for $k = 1, \dots, n$.

A random variable $Y : \mathcal{E} \rightarrow \mathbb{R}$ is called mean-square integral of the process S_{p1} on $[\alpha; \beta]$, if we have :

$$\lim_{n \rightarrow \infty} E[(\sum_{k=1}^n S_p(\lambda_k, \cdot)(\tau_k - \tau_{k-1}) - Y(\cdot))^2] = 0$$

For all normal sequence of partitions of the interval $[\alpha; \beta]$ and for all $\lambda_k \in [\tau_{k-1}; \tau_k]$ for all $k = 1, \dots, n$. Then, we write $Y(\cdot) = \int_{\alpha}^{\beta} S_p(\tau, \cdot) d\tau$.

Definition 1.2: Let $(\mathcal{E}, \mathcal{I}, P)$ be a probability space and $\mathcal{I} \subset \mathbb{R}$ and $h : [0, 1] \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$.

- The stochastic process $S_p : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}$ is convex, if for all $\lambda \in [0; 1]$ and $\alpha; \beta \in \mathcal{I}$ the inequality :

$$S_p(\lambda\alpha + (1-\lambda)\beta, \cdot) \leq \lambda S_p(\alpha, \cdot) + (1-\lambda) S_p(\beta, \cdot)$$

is satisfied.

- The stochastic process $S_p : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}$ is called s-convex in the second sense, if for all $\lambda \in [0, 1]$ and $\alpha; \beta \in \mathcal{I}$ and for some fixed $s \in [0; 1]$ the inequality :

$$S_p(\lambda\alpha + (1-\lambda)\beta, \cdot) \leq \lambda^s S_p(\alpha, \cdot) + (1-\lambda)^s S_p(\beta, \cdot)$$

is satisfied.

- The stochastic process $S_p : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}$ is called h-convex, if for all $\lambda \in [0, 1]$ and $\alpha; \beta \in \mathcal{I}$ the inequality:

$$\mathcal{S}_p(\lambda\alpha + (1-\lambda)\beta, \cdot) \leq h(\lambda)X(\alpha, \cdot) + h(1-\lambda)\mathcal{S}_p(\beta, \cdot)$$

is satisfied.

- The stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \rightarrow [0, \infty)$ is said be r -convex, if for each $u, v \in \mathcal{I}$ and $\lambda \in [0, 1]$ we have :

$$\mathcal{S}_p(\lambda u + (1-\lambda)v, \cdot) \leq \begin{cases} (\lambda \mathcal{S}_p^r(u, \cdot) + (1-\lambda)\mathcal{S}_p^r(v, \cdot))^{1/r}, & r \neq 0 \\ (\mathcal{S}_p(u, \cdot))^\lambda (\mathcal{S}_p(v, \cdot))^{1-\lambda}, & r = 0 \end{cases}$$

Due to impottance in the Theory of Probability and Optimization the Hermite-Hadamard inequality became popular among researchers. (see [9, 10, 5]). For more information about the previous definitions and properties please refer to [11, 12, 13, 14, 15, 16, 17].

2. Main results

Theorem 2.1: Let $\mathcal{S}_{p_1}, \mathcal{S}_{p_2} : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}^+$ are convex stochastic processes where $\mathcal{I} \subset \mathbb{R}$.

Let $\alpha, \beta \in \mathcal{I}$ with $\alpha < \beta$, If $\mathcal{S}_{p_1}, \mathcal{S}_{p_2}$ are mean-square integrables stochastic process on $[\alpha, \beta]$, then we have

$$\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_{p_1}(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \leq \frac{\eta(\alpha; \beta)}{3} + \frac{\rho(\alpha; \beta)}{6}$$

and

$$2\mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2}; \cdot\right) \mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2}; \cdot\right) \leq \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_{p_1}(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau + \frac{\eta(\alpha; \beta)}{6} + \frac{\rho(\alpha; \beta)}{3}$$

Where : $\eta(\alpha; \beta) = \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)$ and $\rho(\alpha; \beta) = \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)$.

Proof: We suppose $\mathcal{S}_{p_1}, \mathcal{S}_{p_2}$ are convex on $[\alpha, \beta]$, then for all $t \in [0, 1]$ we have :

$$\begin{aligned} \mathcal{S}_{p_1} \mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta; \cdot) &\leq t^2 \mathcal{S}_{p_1} \mathcal{S}_{p_2}(\alpha; \cdot) + (1-\tau)^2 \mathcal{S}_{p_1} \mathcal{S}_{p_2}(\beta; \cdot) + \tau(1-\tau) \\ &\quad [\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)] \end{aligned}$$

Also

$$\begin{aligned} \mathcal{S}_{p_1}\mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta;.) &\leq (1-\tau)^2\mathcal{S}_{p_1}\mathcal{S}_{p_2}(\alpha;.) + t^2XY(b;.) \\ &\quad + \tau(1-\tau)[\mathcal{S}_{p_1}(\alpha;.)\mathcal{S}_{p_2}(\beta;.) + \mathcal{S}_{p_1}(\beta;.)\mathcal{S}_{p_2}(\alpha;.)] \end{aligned}$$

Then

$$\begin{aligned} \mathcal{S}_{p_1}\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta;.) + \mathcal{S}_{p_1}\mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta;.) &\leq (2t^2 - 2t + 1)[\mathcal{S}_{p_1}\mathcal{S}_{p_2}(\alpha;.) \\ &\quad + \mathcal{S}_{p_1}\mathcal{S}_{p_2}(\beta;.)] + 2\tau(1-\tau)[\mathcal{S}_{p_1}(\alpha;.)Y(b;.) + \mathcal{S}_{p_1}(\beta;.)\mathcal{S}_{p_2}(\alpha;.)] \end{aligned}$$

So,

$$\begin{aligned} &\int_0^1 [\mathcal{S}_{p_1}\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta;.) + \mathcal{S}_{p_1}\mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta;.)] d\tau \\ &\leq [\mathcal{S}_{p_1}\mathcal{S}_{p_2}(\alpha;.) + \mathcal{S}_{p_1}\mathcal{S}_{p_2}(\beta;.)] \int_0^1 (2t^2 - 2t + 1) d\tau + 2[\mathcal{S}_{p_1}(\alpha;.)\mathcal{S}_{p_2}(\beta;.) \\ &\quad + \mathcal{S}_{p_1}(\beta;.)\mathcal{S}_{p_2}(\alpha;.)] \int_0^1 \tau(1-\tau) d\tau \\ &= \frac{2}{3}[\mathcal{S}_{p_1}\mathcal{S}_{p_2}(\alpha;.) + \mathcal{S}_{p_1}\mathcal{S}_{p_2}(\beta;.)] + \frac{1}{3}[\mathcal{S}_{p_1}(\alpha;.)\mathcal{S}_{p_2}(\beta;.) + \mathcal{S}_{p_1}(\beta;.)\mathcal{S}_{p_2}(\alpha;.)] \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_p(\tau,.)\mathcal{S}_{p_2}(\tau,.) d\tau \leq \frac{1}{3}[\mathcal{S}_{p_1}(\alpha;.)\mathcal{S}_{p_2}(\alpha;.) + \mathcal{S}_{p_1}(\beta;.)\mathcal{S}_{p_2}(\beta;.)] \\ &\quad + \frac{1}{6}[\mathcal{S}_{p_1}(\alpha;.)\mathcal{S}_{p_2}(\beta;.) + \mathcal{S}_{p_1}(\beta;.)\mathcal{S}_{p_2}(\alpha;.)] \end{aligned}$$

Also we have,

$$\frac{\alpha+\beta}{2} = \frac{\tau\alpha+(1-\tau)\beta}{2} + \frac{(1-\tau)\alpha+t\beta}{2}$$

Then

$$\begin{aligned} \mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2};.\right)\mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2};.\right) &= \mathcal{S}_{p_1}\left(\frac{\tau\alpha+(1-\tau)\beta}{2} + \frac{(1-\tau)\alpha+t\beta}{2};.\right)\mathcal{S}_{p_2}\left(\frac{\tau\alpha+(1-\tau)\beta}{2} + \frac{(1-\tau)\alpha+t\beta}{2};.\right) \\ &\leq \frac{1}{4}[\mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta;.) + \mathcal{S}_{p_1}((1-\tau)\alpha + \tau\beta;.)][\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta;.) \\ &\quad + \mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta;.)] \leq \frac{1}{4}[(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau\alpha + (1-\tau)\beta;.) + (\mathcal{S}_{p_1}\mathcal{S}_{p_2})((1-\tau)\alpha + \tau\beta;.)] \\ &\quad + \frac{1}{4}[\tau\mathcal{S}_{p_1}(\alpha;.) + (1-\tau)\mathcal{S}_{p_1}(\beta;.)][(1-\tau)\mathcal{S}_{p_2}(\alpha;.) + \tau\mathcal{S}_{p_2}(\beta;.)] + \frac{1}{4}[(1-\tau)\mathcal{S}_{p_1}(\alpha;.) \\ &\quad + \tau\mathcal{S}_{p_1}(\beta;.)][t\mathcal{S}_{p_2}(\alpha;.) + (1-\tau)\mathcal{S}_{p_2}(\beta;.)] \leq \frac{1}{4}[(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau\alpha + (1-\tau)\beta;.) \\ &\quad + (\mathcal{S}_{p_1}\mathcal{S}_{p_2})((1-\tau)\alpha + \tau\beta;.)] + \frac{1}{4}\{2\tau(1-\tau)[(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\alpha;.) + (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\beta;.)] \\ &\quad + [t^2 + (1-\tau)^2][\mathcal{S}_{p_1}(\alpha;.)\mathcal{S}_{p_2}(\beta;.) + \mathcal{S}_{p_1}(\beta;.)\mathcal{S}_{p_2}(\alpha;.)]\} \end{aligned}$$

As consequence, we obtain :

$$\begin{aligned} & \mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2};\cdot\right)\mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2};\cdot\right)\leq\frac{1}{4}\int_0^1\{(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau\alpha+(1-\tau)\beta;\cdot) \\ & +(\mathcal{S}_{p_1}\mathcal{S}_{p_2})((1-\tau)\alpha+\tau\beta;\cdot)\}d\tau+\frac{1}{12}[\mathcal{S}_{p_1}(\alpha;\cdot)\mathcal{S}_{p_2}(\alpha;\cdot)+\mathcal{S}_{p_1}(\beta;\cdot)\mathcal{S}_{p_2}(\beta;\cdot)] \\ & +\frac{1}{6}[\mathcal{S}_{p_1}(\alpha;\cdot)\mathcal{S}_{p_2}(\beta;\cdot)+\mathcal{S}_{p_1}(\beta;\cdot)\mathcal{S}_{p_2}(\alpha;\cdot)] \end{aligned}$$

Finally,

$$\begin{aligned} & 2\mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2};\cdot\right)\mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2};\cdot\right)\leq\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau;\cdot)d\tau \\ & +\frac{1}{6}[(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\alpha;\cdot)+(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\beta;\cdot)]+\frac{1}{3}[\mathcal{S}_{p_1}(\alpha;\cdot)\mathcal{S}_{p_2}(\beta;\cdot)+\mathcal{S}_{p_1}(\beta;\cdot)\mathcal{S}_{p_2}(\alpha;\cdot)] \end{aligned}$$

Hermite-inequality Hadamard's has recently attracted the attention of numerous academics, and a number of improvements and generalizations have been made (See [4, 5])

Theorem 2.2: Let $\mathcal{S}_{p_1}, \mathcal{S}_{p_2} : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}$ are two stochastic processes where $\mathcal{I} \subset \mathbb{R}^+$. Let $\alpha, \beta \in \mathcal{I}$ with $a < b$. We suppose that $\mathcal{S}_{p_2}$ and $\mathcal{S}_{p_1}\mathcal{S}_{p_2}$ are mean-square integrable on $[\alpha, \beta]$.

If $\mathcal{S}_{p_1}$ is convex and positive on $\mathcal{I}$ and $\mathcal{S}_{p_2}$ is s -convex, then for all $s \in [0, 1]$ we have :

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathcal{S}_p(\tau,\cdot)\mathcal{S}_{p_2}(\tau,\cdot)d\tau\leq\frac{1}{(s+2)}\eta(\alpha;\beta)+\frac{1}{(s+1)(s+2)}\rho(\alpha;\beta)$$

and

$$2^s(\mathcal{S}_{p_1}\mathcal{S}_{p_2})\left(\frac{\alpha+\beta}{2};\cdot\right)\leq\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau;\cdot)d\tau+\frac{1}{(s+1)(s+2)}\eta(\alpha;\beta)+\frac{1}{(s+2)}\rho(\alpha;\beta)$$

Where: $\eta(\alpha;\beta) = \mathcal{S}_{p_1}(\alpha;\cdot)\mathcal{S}_{p_2}(\alpha;\cdot) + \mathcal{S}_{p_1}(\beta;\cdot)\mathcal{S}_{p_2}(\beta;\cdot)$ and $\rho(\alpha;\beta) = \mathcal{S}_{p_1}(\alpha;\cdot)\mathcal{S}_{p_2}(\beta;\cdot) + \mathcal{S}_{p_1}(\beta;\cdot)\mathcal{S}_{p_2}(\alpha;\cdot)$

Proof: Since $\mathcal{S}_{p_1}$ is convex and $\mathcal{S}_{p_2}$ is s -convex, then for all $\tau \in [0, 1]$, $\mathcal{S}_{p_1}$ and $\mathcal{S}_{p_2}$ are positives.

Then:

$$\begin{aligned} & (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau\alpha+(1-\tau)\beta;\cdot)\leq\tau^{s+1}(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\alpha;\cdot)+(1-\tau)^{s+1}(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\beta;\cdot) \\ & +\tau(1-\tau)^s\mathcal{S}_{p_1}(\alpha;\cdot)\mathcal{S}_{p_2}(\beta;\cdot)+\tau^s(1-\tau)\mathcal{S}_{p_1}(\beta;\cdot)\mathcal{S}_{p_2}(\alpha;\cdot) \end{aligned}$$

so,

$$\begin{aligned}
& \int_0^1 \mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) \mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) d\tau = \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{S}_{p_1}(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \\
& \leq \frac{1}{(s+2)} [\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)] \\
& + \frac{1}{(s+1)(s+2)} [\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)]
\end{aligned}$$

Also we have :

$$\begin{aligned}
& (\mathcal{S}_{p_1} \mathcal{S}_{p_2}) \left(\frac{\alpha+\beta}{2}; \cdot \right) = \mathcal{S}_{p_1} \left(\frac{\tau\alpha + (1-\tau)\beta}{2} + \frac{(1-\tau)\alpha + \tau\beta}{2}; \cdot \right) \mathcal{S}_{p_2} \left(\frac{\tau\alpha + (1-\tau)\beta}{2} + \frac{(1-\tau)\alpha + \tau\beta}{2}; \cdot \right) \\
& \leq \frac{1}{2^{s+1}} [\mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) + \mathcal{S}_{p_1}((1-\tau)\alpha + \tau\beta; \cdot)] [\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) \\
& + \mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta; \cdot)] \\
& \leq \frac{1}{2^{s+1}} [(\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\tau\alpha + (1-\tau)\beta; \cdot) + (\mathcal{S}_{p_1} \mathcal{S}_{p_2})((1-\tau)\alpha + \tau\beta; \cdot)] \\
& + \frac{1}{2^{s+1}} [\tau \mathcal{S}_{p_1}(\alpha; \cdot) + (1-\tau) \mathcal{S}_{p_1}(\beta; \cdot)] [(1-\tau)^s \mathcal{S}_{p_2}(\alpha; \cdot) + \tau^s \mathcal{S}_{p_2}(\beta; \cdot)] \\
& + \frac{1}{2^{s+1}} [(1-\tau) \mathcal{S}_{p_1}(\alpha; \cdot) + \tau \mathcal{S}_{p_1}(\beta; \cdot)] [\tau^s \mathcal{S}_{p_2}(\alpha; \cdot) + (1-\tau)^s \mathcal{S}_{p_2}(\beta; \cdot)] \\
& \leq \frac{1}{2^{s+1}} [(\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\tau\alpha + (1-\tau)\beta; \cdot) + (\mathcal{S}_{p_1} \mathcal{S}_{p_2})((1-\tau)\alpha + \tau\beta; \cdot)] \\
& + \frac{1}{2^{s+1}} (\tau(1-\tau)^s + (1-\tau)\tau^s) [\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)] \\
& + \frac{1}{2^{s+1}} (\tau^{s+1} + (1-\tau)^{s+1}) [\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)]
\end{aligned}$$

Theorem 2.3: Let $\mathcal{S}_{p_1}, \mathcal{S}_{p_2} : \mathcal{I} \times \mathcal{E} \rightarrow \mathbb{R}$ are two stochastic processes where $\mathcal{I} \subset \mathbb{R}^+$ and $\alpha, \beta \in \mathcal{I}$ with $\alpha < \beta$, such that $\mathcal{S}_{p_1}$ and $\mathcal{S}_{p_2}$ are mean-square integrable on $[\alpha, \beta]$.

If $\mathcal{S}_{p_1}$ is s_1 -convex and $\mathcal{S}_{p_2}$ is s_2 -convex, then we have :

$$\begin{aligned}
& \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{S}_{p_1}(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \leq \frac{1}{s_1+s_2+1} \eta(\alpha; \beta) + B(s_1+1, s_2+1) \rho(\alpha; \beta) \\
& = \frac{1}{s_1+s_2+1} \left[\eta(\alpha; \beta) + s_1 s_2 \frac{\Gamma(s_1) \Gamma(s_2)}{\Gamma(s_1+s_2+1)} \rho(\alpha; \beta) \right]
\end{aligned}$$

Where : $\eta(\alpha; \beta) = \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)$ and $\rho(\alpha; \beta) = \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)$

Proof: Since $\mathcal{S}_{p_1}$ is s_1 -convex and $\mathcal{S}_{p_2}$ is s_2 -convex on $[\alpha, \beta]$, then :

$$\begin{aligned}
& \mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) \leq \tau^{s_1} \mathcal{S}_{p_1}(\alpha; \cdot) + (1-\tau)^{s_1} \mathcal{S}_{p_1}(\beta; \cdot) \\
& \mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) \leq \tau^{s_2} \mathcal{S}_{p_2}(\alpha; \cdot) + (1-\tau)^{s_2} \mathcal{S}_{p_2}(\beta; \cdot)
\end{aligned}$$

As $\mathcal{S}_{p_1}$ and $\mathcal{S}_{p_2}$ are positives. then for all $t \in [0, 1]$, we have :

$$\begin{aligned} (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\tau\alpha + (1-\tau)\beta; \cdot) &\leq t^{s_1+s_2} (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\alpha; \cdot) + \tau^{s_1} (1-\tau)^{s_2} \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) \\ &+ \tau^{s_2} (1-\tau)^{s_1} \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) + (1-\tau)^{s_1+s_2} (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\beta; \cdot) \end{aligned}$$

Then:

$$\begin{aligned} &\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_p(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \\ &\leq \frac{1}{s_1+s_2+1} [(\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\alpha; \cdot) + (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\beta; \cdot)] + \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) \int_0^1 \tau^{s_1} (1-\tau)^{s_2} d\tau \\ &\quad + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) \int_0^1 \tau^{s_2} (1-\tau)^{s_1} d\tau \\ &\leq \frac{1}{s_1+s_2+1} [(\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\alpha; \cdot) + (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\beta; \cdot)] + \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) B(s_1+1, s_2+1) \\ &\quad + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) B(s_2+1, s_1+1) = \frac{1}{s_1+s_2+1} [(\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\alpha; \cdot) + (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\beta; \cdot)] \\ &\quad + B(s_1+1, s_2+1) [\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)] \\ &= \frac{1}{s_1+s_2+1} [(\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\alpha; \cdot) + (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\beta; \cdot)] + s_1 s_2 \frac{\Gamma(s_1) \Gamma(s_2)}{\Gamma(s_1+s_2+1)} [\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) \\ &\quad + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)] \end{aligned}$$

Theorem 2.4: Let $\mathcal{S}_{p_1}, \mathcal{S}_{p_2} : \mathcal{I} \subset \mathbb{R}^+ \times \mathcal{E} \rightarrow \mathbb{R}^+$ are two stochastic processes and $\alpha, \beta \in \mathcal{I}$ with $\alpha < \beta$. We suppose that $\mathcal{S}_{p_1}$ and $\mathcal{S}_{p_2}$ are mean-square integrable on $[\alpha, \beta]$. If $\mathcal{S}_{p_1}$ is m_1 -convex and decreasing on $\mathcal{I}$, and $\mathcal{S}_{p_2}$ is m_2 -convex then for every $m_1, m_2 \in]0, 1]$ we have :

$$\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_p(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \leq \min\{M_1, M_2\}$$

Where :

$$\begin{aligned} M_1 &= \frac{1}{6} \left[\mathcal{S}_{p_1}(\alpha; \cdot) + \mathcal{S}_{p_2}(\alpha; \cdot) + m_1 \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_1}\left(\frac{\beta}{m_1}; \cdot\right) + m_2 \mathcal{S}_{p_2}(\alpha; \cdot) \mathcal{S}_{p_2}\left(\frac{\beta}{m_2}; \cdot\right) \right. \\ &\quad \left. + m_1^2 \mathcal{S}_{p_1}\left(\frac{b}{m_1}; \cdot\right) + m_2^2 \mathcal{S}_{p_2}\left(\frac{b}{m_2}; \cdot\right) \right] \\ M_2 &= \frac{1}{6} \left[\mathcal{S}_{p_1}(\beta; \cdot) + \mathcal{S}_{p_2}(\beta; \cdot) + m_1 \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_1}\left(\frac{\alpha}{m_1}; \cdot\right) + m_2 \mathcal{S}_{p_2}(\beta; \cdot) \mathcal{S}_{p_2}\left(\frac{\alpha}{m_2}; \cdot\right) \right. \\ &\quad \left. + m_1^2 \mathcal{S}_{p_1}\left(\frac{a}{m_1}; \cdot\right) + m_2^2 \mathcal{S}_{p_2}\left(\frac{a}{m_2}; \cdot\right) \right] \end{aligned}$$

Proof: Since $\mathcal{S}_{p_1}$ is m_1 -convex and $\mathcal{S}_{p_2}$ is m_2 -convex, then we have :

$$X(\tau\alpha + (1-\tau)\beta; \cdot) \leq \tau\mathcal{S}_{p_1}(\alpha; \cdot) + m_1(1-\tau)\mathcal{S}_{p_1}\left(\frac{\beta}{m_1}; \cdot\right)$$

and

$$\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta; \cdot) \leq \tau\mathcal{S}_{p_2}(\alpha; \cdot) + m_2(1-\tau)\mathcal{S}_{p_2}\left(\frac{\beta}{m_2}; \cdot\right)$$

By using the inequality : $cd \leq \frac{1}{2}(c^2 + d^2)$, then :

$$\begin{aligned} & \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_p(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \\ & \leq \frac{1}{2} \int_0^1 [\{\mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta; \cdot)\}^2 + \{\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta; \cdot)\}^2] d\tau \\ & \leq \frac{1}{2} \int_0^1 \left[\left(\tau\mathcal{S}_{p_1}(\alpha; \cdot) + m_1(1-\tau)\mathcal{S}_{p_1}\left(\frac{\beta}{m_1}; \cdot\right) \right)^2 + \left(\tau\mathcal{S}_{p_2}(\alpha; \cdot) + m_2(1-\tau)\mathcal{S}_{p_2}\left(\frac{\beta}{m_2}; \cdot\right) \right)^2 \right] d\tau \\ & = \frac{1}{6} \left[\mathcal{S}_{p_1}(\alpha; \cdot) + \mathcal{S}_{p_2}(\alpha; \cdot) + m_1\mathcal{S}_{p_1}(\alpha; \cdot)\mathcal{S}_{p_1}\left(\frac{\beta}{m_1}; \cdot\right) + m_2\mathcal{S}_{p_2}(\alpha; \cdot)\mathcal{S}_{p_2}\left(\frac{\beta}{m_2}; \cdot\right) \right. \\ & \quad \left. + m_1^2\mathcal{S}_{p_1}\left(\frac{\beta}{m_1}; \cdot\right) + m_2^2\mathcal{S}_{p_2}\left(\frac{\beta}{m_2}; \cdot\right) \right] \end{aligned}$$

Then :

$$\begin{aligned} & \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_p(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \\ & \leq \frac{1}{6} \left[\mathcal{S}_{p_1}(\beta; \cdot) + \mathcal{S}_{p_2}(\beta; \cdot) + m_1\mathcal{S}_{p_1}(\beta; \cdot)\mathcal{S}_{p_1}\left(\frac{\alpha}{m_1}; \cdot\right) + m_2\mathcal{S}_{p_2}(\beta; \cdot)\mathcal{S}_{p_2}\left(\frac{\alpha}{m_2}; \cdot\right) \right] \\ & \quad + \frac{1}{6} \left[m_1^2\mathcal{S}_{p_1}\left(\frac{\alpha}{m_1}; \cdot\right) + m_2^2\mathcal{S}_{p_2}\left(\frac{\alpha}{m_2}; \cdot\right) \right] \end{aligned}$$

Theorem 2.5: Let $\mathcal{S}_{p_1}$ is h_1 -convex and $\mathcal{S}_{p_2}$ h_2 -convex, $\alpha, \beta \in \mathcal{I}$ with $\alpha < \beta$. If $\mathcal{S}_{p_1}\mathcal{S}_{p_2}$ are mean-square integrable on $[\alpha, \beta]$ and $h_1, h_2 \in L[a, b]$, Then :

$$\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau; \cdot) d\tau \leq \eta(\alpha; \beta) \int_0^1 h_1(\tau) h_2(\tau) d\tau + \rho(\alpha; \beta) \int_0^1 h_1(\tau) h_2(1-\tau) d\tau$$

and

$$\begin{aligned} & \frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} (\mathcal{S}_{p_1}\mathcal{S}_{p_2})\left(\frac{\alpha+\beta}{2}; \cdot\right) - \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau; \cdot) d\tau \leq \eta(\alpha; \beta) \int_0^1 h_1(\tau) h_2(\tau) d\tau \\ & \quad + \rho(\alpha; \beta) \int_0^1 h_1(\tau) h_2(1-\tau) d\tau \end{aligned}$$

Where : $\eta(\alpha; \beta) = \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)$ and $\rho(\alpha; \beta) = \mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot) \mathcal{S}_{p_2}(\alpha; \cdot)$

Proof: Since $\mathcal{S}_{p_1}$ is h_1 -convex and $\mathcal{S}_{p_2}$ is h_2 -convex, then :

$$\begin{aligned} & \mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) \mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) \leq h_1(\tau)h_2(\tau)(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\alpha; \cdot) \\ & + h_1(1-\tau)h_2(1-\tau)(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\beta; \cdot) + h_1(\tau)h_2(1-\tau)\mathcal{S}_{p_1}(\alpha; \cdot)\mathcal{S}_{p_2}(\beta; \cdot) \\ & + h_1(1-\tau)h_2(\tau)\mathcal{S}_{p_1}(\beta; \cdot)\mathcal{S}_{p_2}(\alpha; \cdot) \end{aligned}$$

Then

$$\begin{aligned} & \int_0^1 \mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) \mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) d\tau \leq [(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\alpha; \cdot) \\ & + (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\beta; \cdot)] \int_0^1 h_1(\tau)h_2(\tau) d\tau + [\mathcal{S}_{p_1}(\alpha; \cdot)\mathcal{S}_{p_2}(\beta; \cdot) \\ & + \mathcal{S}_{p_1}(\beta; \cdot)\mathcal{S}_{p_2}(\alpha; \cdot)] \int_0^1 h_1(\tau)h_2(1-\tau) d\tau \end{aligned}$$

so, we obtain the first inequality :

$$\begin{aligned} & \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_{p_1}(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \leq [(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\alpha; \cdot) + (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\beta; \cdot)] \int_0^1 h_1(\tau)h_2(\tau) d\tau \\ & + [\mathcal{S}_{p_1}(\alpha; \cdot)\mathcal{S}_{p_2}(\beta; \cdot) + \mathcal{S}_{p_1}(\beta; \cdot)\mathcal{S}_{p_2}(\alpha; \cdot)] \int_0^1 h_1(\tau)h_2(1-\tau) d\tau \end{aligned}$$

Also we have

$$\begin{aligned} & \mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2}, \cdot\right) \mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2}, \cdot\right) = \mathcal{S}_{p_1}\left(\frac{\tau\alpha+(1-\tau)\beta}{2} + \frac{(1-\tau)\alpha+\tau\beta}{2}, \cdot\right) \mathcal{S}_{p_2}\left(\frac{\tau\alpha+(1-\tau)\beta}{2} + \frac{(1-\tau)\alpha+\tau\beta}{2}, \cdot\right) \\ & \leq h_1\left(\frac{1}{2}\right) [\mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) + \mathcal{S}_{p_1}((1-\tau)\alpha + \tau\beta, \cdot)] \\ & \times h_2\left(\frac{1}{2}\right) [\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) + \mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta, \cdot)] \\ & \leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau\alpha + (1-\tau)\beta; \cdot) + (\mathcal{S}_{p_1}\mathcal{S}_{p_2})((1-\tau)\alpha + \tau\beta; \cdot) \\ & + h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) [h_1(\tau)\mathcal{S}_{p_1}(\alpha; \cdot) + h_1(1-\tau)\mathcal{S}_{p_1}(\beta; \cdot)] [h_2(1-\tau)\mathcal{S}_{p_2}(\alpha; \cdot) + h_2(\tau)\mathcal{S}_{p_2}(\beta; \cdot)] \\ & + h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) [h_1(1-\tau)\mathcal{S}_{p_1}(\alpha; \cdot) + h_1(\tau)\mathcal{S}_{p_1}(\beta; \cdot)] [h_2(\tau)\mathcal{S}_{p_2}(\alpha; \cdot) + h_2(1-\tau)\mathcal{S}_{p_2}(\beta; \cdot)] \end{aligned}$$

Then we obtain:

$$\begin{aligned} & \frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} (\mathcal{S}_{p_1}\mathcal{S}_{p_2})\left(\frac{\alpha+\beta}{2}, \cdot\right) - \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\tau; \cdot) d\tau \\ & \leq [(\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\alpha; \cdot) + (\mathcal{S}_{p_1}\mathcal{S}_{p_2})(\beta; \cdot)] \int_0^1 h_1(\tau)h_2(\tau) d\tau + [\mathcal{S}_{p_1}(\alpha; \cdot)\mathcal{S}_{p_2}(\beta; \cdot) \\ & + \mathcal{S}_{p_1}(\beta; \cdot)\mathcal{S}_{p_2}(\alpha; \cdot)] \int_0^1 h_1(\tau)h_2(1-\tau) d\tau \end{aligned}$$

Theorem 2.6: Let $\mathcal{S}_{p_1}, \mathcal{S}_{p_2} : I \times \mathcal{E} \rightarrow]0, +\infty[$ two log-convex stochastic processes on $\mathcal{I}$ and $\alpha; \beta \in \mathcal{I}$ with $\alpha < \beta$. Then we have the following inequality:

$$\begin{aligned} & \mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2}; \cdot\right) \mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2}; \cdot\right) \\ & \leq \frac{1}{2} \left\{ \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} [\mathcal{S}_p(\tau, \cdot) \mathcal{S}_{p_1}(a+b-\tau; \cdot) + \mathcal{S}_{p_2}(\tau, \cdot) \mathcal{S}_{p_2}(a+b-\tau; \cdot)] d\tau \right\} \\ & \leq \frac{\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_1}(\beta; \cdot) + \mathcal{S}_{p_2}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)}{2} \end{aligned}$$

Proof: We have :

$$\begin{aligned} & \mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2}; \cdot\right) \mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2}; \cdot\right) \leq \frac{1}{2} \left[\mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2}; \cdot\right) + \mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2}; \cdot\right) \right] \\ & = \frac{1}{2} \mathcal{S}_{p_1}\left(\frac{\tau\alpha+(1-\tau)\beta}{2} + \frac{(1-\tau)t\beta}{2}; \cdot\right) + \frac{1}{2} \mathcal{S}_{p_2}\left(\frac{\tau\alpha+(1-\tau)\beta}{2} + \frac{(1-\tau)t\beta}{2}; \cdot\right) \\ & \leq \frac{1}{2} \left[(\mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot))^{\frac{1}{2}} \right]^2 \left[(\mathcal{S}_{p_1}((1-\tau)\alpha + \tau\beta; \cdot))^{\frac{1}{2}} \right]^2 \\ & + \frac{1}{2} \left[(\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot))^{\frac{1}{2}} \right]^2 \left[(\mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta; \cdot))^{\frac{1}{2}} \right]^2 \\ & \leq \frac{1}{2} [\mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) \mathcal{S}_{p_1}((1-\tau)\alpha + \tau\beta; \cdot) + \mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) \mathcal{S}_{p_2}((1-\tau)\alpha + \tau\beta; \cdot)] \\ & \leq \frac{\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_1}(\beta; \cdot) + \mathcal{S}_{p_2}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)}{2} \end{aligned}$$

Then,

$$\begin{aligned} & \mathcal{S}_{p_1}\left(\frac{\alpha+\beta}{2}; \cdot\right) \mathcal{S}_{p_2}\left(\frac{\alpha+\beta}{2}; \cdot\right) \\ & \leq \frac{1}{2} \left[\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} [\mathcal{S}_p(\tau, \cdot) \mathcal{S}_{p_1}(a+b-\tau; \cdot) + \mathcal{S}_{p_2}(\tau, \cdot) \mathcal{S}_{p_2}(a+b-\tau; \cdot)] d\tau \right] \\ & \leq \frac{\mathcal{S}_{p_1}(\alpha; \cdot) \mathcal{S}_{p_1}(\beta; \cdot) + \mathcal{S}_{p_2}(\alpha; \cdot) \mathcal{S}_{p_2}(\beta; \cdot)}{2} \end{aligned}$$

Theorem 2.7: Let $\mathcal{S}_{p_1}, \mathcal{S}_{p_2} : [a, b] \times \mathcal{E} \rightarrow]0, +\infty[$ are two r -convex and 0 -convex stochastic processes respectively on $[\alpha, \beta]$ and $r > 0$, Then we have the following inequality :

$$\begin{aligned}
& \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} (\mathcal{S}_{p_1} \mathcal{S}_{p_2})(\tau; \cdot) d\tau \\
& \leq \frac{1}{2} \left(\frac{r}{r+2} \right) \left(\frac{\mathcal{S}_{p_1}(\beta; \cdot)^{r+2} - \mathcal{S}_{p_1}(\alpha; \cdot)^{r+2}}{\mathcal{S}_{p_1}(\beta; \cdot)^r - \mathcal{S}_{p_1}(\alpha; \cdot)^r} \right) + \frac{1}{4} [\mathcal{S}_{p_2}(\beta; \cdot)^2 - \mathcal{S}_{p_2}(\alpha; \cdot)^2] \ln \left(\frac{\mathcal{S}_{p_2}(\beta; \cdot)}{\mathcal{S}_{p_2}(\alpha; \cdot)} \right) \\
& (\mathcal{S}_{p_1}(\alpha; \cdot) \neq \mathcal{S}_{p_1}(\beta; \cdot))
\end{aligned}$$

Proof: Since $\mathcal{S}_{p_1}$ is r -convex and $\mathcal{S}_{p_2}$ 0-convex, for all $\tau \in [0, 1]$ we have :

$$\begin{aligned}
\mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) & \leq (\tau\mathcal{S}_{p_1}(\alpha; \cdot)^r + (1-\tau)\mathcal{S}_{p_1}(\beta; \cdot)^r)^{\frac{1}{r}} \\
\mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) & \leq [\mathcal{S}_{p_2}(\alpha; \cdot)]^{\tau} [\mathcal{S}_{p_2}(\beta; \cdot)]^{1-\tau}
\end{aligned}$$

Then :

$$\begin{aligned}
& \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathcal{S}_p(\tau, \cdot) \mathcal{S}_{p_2}(\tau, \cdot) d\tau \\
& = \int_0^1 \mathcal{S}_{p_1}(\tau\alpha + (1-\tau)\beta, \cdot) \mathcal{S}_{p_2}(\tau\alpha + (1-\tau)\beta, \cdot) d\tau \\
& \leq \int_0^1 [\tau\mathcal{S}_{p_1}(\alpha; \cdot)^r + (1-\tau)\mathcal{S}_{p_1}(\beta; \cdot)^r]^{\frac{1}{r}} [\mathcal{S}_{p_2}(\alpha; \cdot)]^{\tau} [\mathcal{S}_{p_2}(\beta; \cdot)]^{1-\tau} d\tau
\end{aligned}$$

Using the Cauchy inequality:

$$\begin{aligned}
& \int_0^1 [\tau\mathcal{S}_{p_1}(\alpha; \cdot)^r + (1-\tau)\mathcal{S}_{p_1}(\beta; \cdot)^r]^{\frac{1}{r}} [\mathcal{S}_{p_2}(\alpha; \cdot)]^{\tau} [\mathcal{S}_{p_2}(\beta; \cdot)]^{1-\tau} d\tau \\
& \leq \frac{1}{2} \int_0^1 [\tau\mathcal{S}_{p_1}(\alpha; \cdot)^r + (1-\tau)\mathcal{S}_{p_1}(\beta; \cdot)^r]^{\frac{2}{r}} d\tau + \frac{1}{2} \int_0^1 [\mathcal{S}_{p_2}(\alpha; \cdot)]^{2\tau} [\mathcal{S}_{p_2}(\beta; \cdot)]^{2-2\tau} d\tau
\end{aligned}$$

Let $S = t(\mathcal{S}_{p_1}(\alpha; \cdot)^r - \mathcal{S}_{p_1}(\beta; \cdot)^r) + \mathcal{S}_{p_1}(\beta; \cdot)^r$, then :

$$\int_0^1 [\tau\mathcal{S}_{p_1}(\alpha; \cdot)^r + (1-\tau)\mathcal{S}_{p_1}(\beta; \cdot)^r]^{\frac{2}{r}} d\tau = \left(\frac{r}{r+2} \right) \left(\frac{\mathcal{S}_{p_1}(\beta; \cdot)^{r+2} - \mathcal{S}_{p_1}(\alpha; \cdot)^{r+2}}{\mathcal{S}_{p_1}(\beta; \cdot)^r - \mathcal{S}_{p_1}(\alpha; \cdot)^r} \right)$$

and

$$\int_0^1 [\mathcal{S}_{p_2}(\alpha; \cdot)]^{2\tau} [\mathcal{S}_{p_2}(\beta; \cdot)]^{2-2\tau} d\tau = \frac{1}{2} [\mathcal{S}_{p_2}(\beta; \cdot)^2 - \mathcal{S}_{p_2}(\alpha; \cdot)^2] \ln \left(\frac{\mathcal{S}_{p_2}(\beta; \cdot)}{\mathcal{S}_{p_2}(\alpha; \cdot)} \right)$$

Conclusion

In this paper, we obtained a crucial inequalities of Hermite-Hadamard for the product of two convex stochastic processes. within the future new

inequalities for the other convex stochastic process can be obtained using similar methods during this study.

References

- [1] D. Kotrys, Hermite–Hadamard inequality for convex stochastic processes, *Aequationes mathematicae*, 83, pages 143–151 (2012).
- [2] K. Nikodem, On convex stochastic processes, *Aequationes Math.*, 20, No. 1, 184–197 (1980).
- [3] Tunc M, On Some Hadamard Type Inequalities for Product of Different Kinds of Convex Functions, *Applied Mathematics Letters* (2010). <http://rgmia.vu.edu.au/SSDragomirWeb.html>.
- [4] Wang, JinRong and Fečkan, Michal. Fractional Hermite-Hadamard Inequalities, Berlin, Boston: De Gruyter (2018). <https://doi.org/10.1515/9783110523621>.
- [5] Dragomir SS and Pearce CEM, Selected topics on Hermite-Hadamard inequalities and applications, Science direct working paper (2003). <https://rgmia.org/papers/monographs/Master.pdf>.
- [6] Hernandez Hernandez, Jorge Gómez, Juan. Hermite Hadamard type inequalities for Stochastic Processes whose Second Derivatives are $(m; h_1; h_2)$ -Convex using Riemann-Liouville Fractional Integral. *Revista Matua - Universidad del Atlántico*. 5. 13-28 (2018).
- [7] Lackovic I. B. and Mitrinovic D. S., “Hermite and convexity”, *Aequationes Math.*, 28, No. 1, 229-232 (1985).
- [8] Yuming Feng. Refining Hermite-Hadamard integral inequality by two parameters, *Journal of Interdisciplinary Mathematics*, 21:3, 743-746 (2018), DOI: 10.1080/09720502.2018.1424093.
- [9] Kumar P., “Hermite-Hadamard inequalities and applications in estimating moments”, in *Inequality Theory and Applications*, Volume 2, (Edited by Y.J. Cho, J. K. Kim and S. S. Dragomir), Nova Science, (2003).
- [10] Zeng Lin Jin Rong Wang. New Riemann-Liouville fractional Hermite-Hadamard inequalities via two kinds of convex functions, *Journal of Interdisciplinary Mathematics*, 20:2, 357-382 (2017), DOI: 10.1080/09720502.2014.914281.
- [11] D. Barraez L. Gonzalez N. Merentes, On h -convex stochastic processes, *Mathematica Aeterna*, Vol. 5, no. 4, 571 - 581 (2015).

- [12] S. Ozcan, Hermite-Hadamard inequalities for m -convex and (α, m) -convex stochastic processes, *International Journal of Analysis and Applications*, V.17, No 5, 793-802 (2019).
- [13] E. Set, M. Tomar, S. Maden. Hermite Hadamard Inequalities for s-Convex Stochastic Processes in the Second Sense, *Turkish Journal of Analysis and Number Theory*, 2, No. 6, 202-207 (2014).
- [14] J.J. Shynk, Probability Random Variables and Random Processes, Theory and Signal Processing Applications, Wiley (2013).
- [15] W. Ul-Haq, N. Rehman ; Z. Al-Hussain, Hermite-Hadamard inequalities for r-convex positive stochastic processes, *Journal of Taibah University for Science*, V 13, N 1 (2019).
- [16] Kee Qiu Jin Rong Wang. A fractional integral identity and its application to fractional Hermite-Hadamard inequalities, *Journal of Interdisciplinary Mathematics*, 21:1, 1-16 (2018), DOI: 10.1080/09720502.2017.1400795.
- [17] Jamal El-Achky, Driss Gretete Mohammed Barmaki: Inequalities of Hermite-Hadamard for stochastic process whose fourth derivatives absolute are quasi-convex, P-convex, s-convex and h-convex., *Journal of Interdisciplinary Mathematics* (2021), DOI: 10.1080/09720502.2021.1887607.

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