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On the convergence of periodic continued fractions via linearly recurring sequences

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Abstract

In this paper, we give a new process to study the convergence of periodic continued fractions, making use of the approach on the linear difference equations with periodic coefficients, based on the analytic formula of the linearly recurring sequences. Some new explicit expressions of nth numerators and nth denominators of the convergents of periodic continued fractions lead to discuss their convergence. Moreover, we provide a new discussion on the closed link between quadratic irrational numbers and simple periodic continued fractions. Some applications of the Pell's equations are given and some numerical examples which illustrate the theory are supplied.

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1. Introduction

The theory of continued fractions is considered one of the areas of the number theory and has undergone tremendous development during recent decades. A continued fraction has the general form

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This article has been corrected with minor changes. These changes do not impact the academic content of the article.

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$$[b_0, a_1/b_1, a_2/b_2, \cdots] = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} \tag{1.1}$$

For any n, the nth convergent of (1.1) is given by $\frac{A_n}{B_n} = [b_0, a_1/b_1, a_2/b_2, \cdots, a_n/b_n]$. The quantities A_n and B_n are called the nth numerator and nth denominator of the continued fraction respectively and satisfy for every $n \geq 1$ the fundamental recurrence formulas

$$\begin{aligned} A_n &= b_n A_{n-1} + a_n A_{n-2} \\ B_n &= b_n B_{n-1} + a_n B_{n-2} \end{aligned} \tag{1.2}$$

where $A_{-1} = 1, A_0 = b_0, B_{-1} = 0, B_0 = 1.$

The continued fractions are interesting and useful in various areas of Mathematics (see [4, 7, 8, 9, 11]). Furthermore, several mathematical constants, such as Euler number e and π, are conjectured as continued fractions (listed in www.RamanujanMachine.com). In [5], Kadyrov and Mashurov proved some of these conjectures and provided a simple method (compute the first few terms of A_n and B_n and guess a closed form of them, then prove the closed-form expression using mathematical induction) to give rise to other continued fractions employing their series representations.

In this paper, we will focus on periodic continued fractions, where the sequence in (1.1) eventually repeats, i.e $a_{p+i} = a_i$, and $b_{p+i} = b_i$ for all $i \geq 1$. In this case, the fraction is usually written as

$$[b_0, \overline{a_1/b_1, a_2/b_2, \cdots, a_p/b_p}] = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_p}{b_p + \cdots}}} \tag{1.3}$$

The topic related to this theory is currently being studied intensively, we refer the reader to [1, 2].

In the case where $a_i = 1$ for all $i \geq 1$ and $\{b_i\}_{1 \leq i \leq p}$ being a sequence of positive integers, except perhaps b_0 which may be zero or negative integer, the continued fraction (1.3) is called simple periodic continued fraction and denoted by $[b_0, \overline{b_1, b_2, \cdots, b_p}]$.

In this paper, we examine the closed link between the convergents of the periodic continued fraction (1.3) and the linearly recurring sequences, namely the generalized Fibonacci sequences. More precisely, we remark that equation (1.2) represents a special case of order 2 of a generalized Fibonacci sequence $\{y_n\}_{n \geq 0}$ (of order $r \geq 2$) defined by:

$$n \geq r-1 \quad y_{n+1} = v_0 y_n + v_1 y_{n-1} + \cdots + v_{r-1} y_{n-r+1}, \tag{1.4}$$

where the two sequences of complex numbers $v_0, v_1, \cdots, v_{r-1}$ with $(v_{r-1} \neq 0)$ and $y_0, y_1, ..., y_{r-1}$ are given as the coefficients and initial values respectively. The linearly recurring sequences possess many properties that have been used in the resolution of different mathematical problems (see for instance[1, 2, 3, 6]). Yet, our main goal is to exploit the properties of these recurring sequences to study the periodic continued fractions and their convergence.

This paper is organized as follows. In section 2 we provide a new approach to establish some explicit formulas for the nth convergents of periodic continued fractions. By using some results on the linear difference equation with periodic coefficients established by R. Ben Taher and H. Benkhaldoun in [1], which are in turn arising from the properties of the linearly recurring sequences. Yet, based on the established formulas we discuss the convergence of nth convergents of periodic continued fractions in general. Section 3 is devoted to a detailed study of the simple periodic continued fractions. The tools employed here are simple which makes it efficient to apply for pointing up the relationship between quadratic irrational numbers and simple periodic continued fractions. In section 4, we focus on providing the expression of the solutions (when they exist) of the Pell's equation $x^2 - dy^2 = \pm 1$, using our method presented in previous sections. Furthermore, some numerical examples are given.

2. Explicit formulas of the nth convergents of periodic continued fractions.

In this section, employing some results on the linear difference equations with periodic coefficients established by R.Ben Taher and H. Benkhaldoun in [1], we manage to obtain some new explicit expressions for the nth numerators A_n and nth denominators B_n of the convergents of the periodic continued fraction (1.3).

When the nth convergent of the given periodic continued fraction converges, then the limit, if it exists, is given.

Let us first consider the periodic continued fraction (1.3) of period p. We observe that the fundamental recurrence formulas (1.2) satisfied by the two sequences $\{A_n\}_{n\geq -1}$ and $\{B_n\}_{n\geq -1}$ can be viewed as two linear difference equations of order 2 with periodic coefficients of period p.

We get around our approach through recourse to the results established in [1]. That is, we set $C(k) = \begin{pmatrix} b_k & a_k \\ 1 & 0 \end{pmatrix}$ for all $1 \leq k \leq p$,

$C(0)=\begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix}$ and the matrix $B=C(p)C(p-1)...C(2)C(1).$

We observe that $BC(0)=\begin{pmatrix} A_p & B_p \\ A_{p-1} & B_{p-1} \end{pmatrix}$. Thence, $B=\begin{pmatrix} B_p & A_p-b_0B_p \\ B_{p-1} & A_{p-1}-b_0B_{p-1} \end{pmatrix}$.

Let $P_B(x)=x^2-c_0x-c_1$ $(c_1\neq 0)$ be the characteristic polynomial of the matrix B, then we get

$$\begin{aligned} c_0 &= B_p + A_{p-1} - b_0 B_{p-1} \\ c_1 &= A_p B_{p-1} - B_p A_{p-1} = (-1)^{p-1} a_1 a_2 \cdots a_p \end{aligned} \tag{2.1}$$

Pursuing the approach given in [1] (Subsection 3.2), it ensures that the expressions of the terms of the sequences $\{A_n\}_{n\geq -1}$ and $\{B_n\}_{n\geq -1}$ are determined by considering the characteristic roots of $P_B(X)$. There are only two cases to be discussed.

First, we suppose that the roots λ_1,λ_2 of $P_B(X)$ are distinct, namely $\lambda_1\neq\lambda_2$. Then, by results of Proposition 3.6 in [1], the terms A_n and B_n are given for every $n=kp+i$, $i=0,1,...,p-1$ by,

$$\begin{aligned} A_{kp+i} &= \frac{\lambda_1^k}{P'(\lambda_1)}\left(\frac{\alpha_i}{\lambda_1}+\frac{\alpha_{p+i}}{\lambda_1^2}\right)+\frac{\lambda_2^k}{P'(\lambda_2)}\left(\frac{\alpha_i}{\lambda_2}+\frac{\alpha_{p+i}}{\lambda_2^2}\right), \\ B_{kp+i} &= \frac{\lambda_1^k}{P'(\lambda_1)}\left(\frac{\beta_i}{\lambda_1}+\frac{\beta_{p+i}}{\lambda_1^2}\right)+\frac{\lambda_2^k}{P'(\lambda_2)}\left(\frac{\beta_i}{\lambda_2}+\frac{\beta_{p+i}}{\lambda_2^2}\right), \end{aligned} \tag{2.2}$$

where $\alpha_{p+i}=c_1A_{p+i}$, $\alpha_i=c_1A_i+c_0A_{p+i}$, $\beta_{p+i}=c_1B_{p+i}$ and $\beta_i=c_1B_i+c_0B_{p+i}$. Suppose that $|\lambda_2|<|\lambda_1|$. A straightforward calculation for $i=0,1,...,p-1$ leads to

$$\lim_{k\to\infty}\frac{A_{kp+i}}{B_{kp+i}}=\frac{\alpha_i\lambda_1+\alpha_{p+i}}{\beta_i\lambda_1+\beta_{p+i}}.$$

Hence, the periodic continued fraction (1.3) converges to $w\in\mathbb{R}$, if and only if

$$\omega=\lim_{k\to\infty}\frac{A_{kp+p-1}}{B_{kp+p-1}}=\lim_{k\to\infty}\frac{A_{kp+p-2}}{B_{kp+p-2}}=\cdots=\lim_{k\to\infty}\frac{A_{kp+1}}{B_{kp+1}}=\lim_{k\to\infty}\frac{A_{kp}}{B_{kp}}.$$

Thus, we obtain the following result.

Proposition 2.1: Let $[b_0,\overline{a_1/b_1,a_2/b_2,\cdots,a_p/b_p}]$ be the p-periodic continued fraction (1.3), and $\frac{A_n}{B_n}$ its nth convergent. Set the matrix $B=\begin{pmatrix} B_p & A_p-b_0B_p \\ B_{p-1} & A_{p-1}-b_0B_{p-1} \end{pmatrix}$ and $P_B(x)=x^2-c_0x-c_1$ its characteristic polynomial. Suppose that P_B

admits the roots λ_1, λ_2 such that $|\lambda_2| < |\lambda_1|$. Then the periodic continued fraction (1.3) converges if and only if for every $1 \le i \le p-1$ we have

$$\frac{c_1 A_i + \lambda_1 A_{p+i}}{c_1 B_i + \lambda_1 B_{p+i}} = \frac{c_1 A_0 + \lambda_1 A_p}{c_1 B_0 + \lambda_1 B_p}.$$

Furthermore, its limit ω is given by

$$\omega = \frac{c_1 b_0 + \lambda_1 A_p}{c_1 + \lambda_1 B_p} = \frac{b_0 \lambda_2 - A_p}{\lambda_2 - B_p}.$$

To illustrate Proposition 2.1, we consider the following example.

Example 2.2: We aim to find the limit of the 4-periodic continued fraction $[2, \overline{1/1, 1/1, 1/1, 1/4}]$. We have $C(1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $C(2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$,

$C(3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $C(4) = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$, Then $B = \begin{pmatrix} 14 & 9 \\ 3 & 2 \end{pmatrix}$, and its characteristic polynomial $P_B(x) = x^2 - 16x + 1$ owns two distinct roots $\lambda_1 = 8 + 3\sqrt{7}$ and $\lambda_2 = 8 - 3\sqrt{7}$.

A straightforward calculation shows that

$$\frac{c_1 A_0 + \lambda_1 A_4}{c_1 B_0 + \lambda_1 B_4} = \frac{c_1 A_1 + \lambda_1 A_5}{c_1 B_1 + \lambda_1 B_5} = \frac{c_1 A_2 + \lambda_1 A_6}{c_1 B_2 + \lambda_1 B_6} = \frac{c_1 A_3 + \lambda_1 A_7}{c_1 B_3 + \lambda_1 B_7} = \sqrt{7}.$$

Hence, we have

$$\sqrt{7} = [2, \overline{1/1, 1/1, 1/1, 1/4}].$$

Remark 2.3: We point out that if the discriminant of the polynomial $P_B(x) = x^2 - c_0 x - c_1$ is negative, thence the above periodic continued fraction (1.3) diverges because the roots λ_1 and λ_2 are not real and $|\lambda_1| = |\lambda_2|$. Indeed, if we put $\lambda_1 = r\exp(i\theta)$, we get $\lambda_2 = r\exp(-i\theta)$. Thus, we have

$$\lim_{k \to \infty} \frac{A_{kp+i}}{B_{kp+i}} = \lim_{k \to \infty} \frac{\frac{1}{P'(\lambda_1)}\left(\frac{\alpha_i}{\lambda_1} + \frac{\alpha_{p+i}}{\lambda_1^2}\right) + \frac{\exp(-2ik\theta)}{P'(\lambda_2)}\left(\frac{\alpha_i}{\lambda_2} + \frac{\alpha_{p+i}}{\lambda_2^2}\right)}{\frac{1}{P'(\lambda_1)}\left(\frac{\beta_i}{\lambda_1} + \frac{\beta_{p+i}}{\lambda_1^2}\right) + \frac{\exp(-2ik\theta)}{P'(\lambda_2)}\left(\frac{\beta_i}{\lambda_2} + \frac{\beta_{p+i}}{\lambda_2^2}\right)}.$$

Since the quantity $\exp(-2ik\theta)$ do not approach a limit, the continued fraction (1.3) diverges.

Now, we suppose that $P_B(x) = x^2 - c_0x - c_1$ has one root λ. Turning back to Proposition 3.8 in [1], we derive the expressions of $\{A_n\}_{n\geq -1}$ and $\{B_n\}_{n\geq -1}$, for every $n = kp + i$ for $k \in \mathbb{N}$ and $0 \leq i \leq p-1$, as follows

$$A_{kp+i} = \lambda^k [A_i(1-k) + A_{i+p} k\lambda^{-1}]$$

$$B_{kp+i} = \lambda^k [B_i(1-k) + B_{i+p} k\lambda^{-1}]$$

That is, we have $\lim_{k\to\infty} \frac{A_{kp+i}}{B_{kp+i}} = \frac{-A_i\lambda + A_{i+p}}{-B_i\lambda + B_{i+p}}$. We state the following proposition.

Proposition 2.4: Under the same hypothesis above in Proposition (2.1) suppose that the polynomial $P_B(x) = x^2 - c_0x - c_1$ has one root λ. Then, the continued fraction (1.3) converges to the limit $w = \frac{-b_0\lambda + A_p}{-\lambda + B_p}$, if and only if

$$\frac{-A_i\lambda + A_{i+p}}{-B_i\lambda + B_{i+p}} = \cdots = \frac{-A_1\lambda + A_{1+p}}{-B_1\lambda + B_{1+p}} = \frac{-A_0\lambda + A_p}{-B_0\lambda + B_p}.$$

Example 2.5: Consider the 2-periodic continued fraction

$$[1, \overline{(-\sqrt{3}-\sqrt{2})/-\sqrt{2}, 3(\sqrt{2}-\sqrt{3})/(2-\sqrt{6})}].$$

We have $C(1) = \begin{pmatrix} -\sqrt{2} & -\sqrt{3}-\sqrt{2} \\ 1 & 0 \end{pmatrix}$, $C(2) = \begin{pmatrix} 2-\sqrt{6} & 3(\sqrt{2}-\sqrt{3}) \\ 1 & 0 \end{pmatrix}$ and the matrix $B = C(2)C(1) = \begin{pmatrix} -\sqrt{3}+\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{3}-\sqrt{2} \end{pmatrix}$, with the characteristic polynomial $P_B(x) = x^2 + 2\sqrt{3}x + 3 = (x+\sqrt{3})^2$.

Using the proposition 2.4, we get $\frac{-A_0\lambda + A_2}{-B_0\lambda + B_2} = \frac{-A_1\lambda + A_3}{-B_1\lambda + B_3} = 2$. Thus, the continued fraction converges to 2.

3. Detailed study of the simple periodic continued fractions and application to the quadratic irrationals.

In this section, we are interested in studying the simple periodic continued fractions by using the same previous approach based on the properties of generalized Fibonacci sequences, which leads us to emphasize the close connection between quadratic irrational numbers and simple periodic continued fractions.

We first start by defining a simply periodic continued fraction by

$$\left[b_0,\overline{b_1,\ldots,b_p}\right]=b_0+\cfrac{1}{b_1+\cfrac{1}{b_2+\ddots+\cfrac{1}{b_p+\cfrac{1}{b_1+\cdots}}}} \tag{3.1}$$

with $b_0\in\mathbb{Z}$ and $b_1,\cdots,b_p$ are nonnegative integers and $b_{p+i}=b_i$ for all $i\geq 1$.

It is well known in the literature that there exists a closed link between quadratic irrationals and periodic continued fractions. Notably, the fundamental theorem of J-L Lagrange (see [10, p. 378]) states that the infinite simple continued fraction of an irrational number is periodic if and only if this number is a quadratic irrational. We aim to prove that a periodic continued fraction represents a quadratic irrational. To that matter, we first start by proving that any periodic continued fraction (3.1) is convergent. This assertion is known in the literature and has been already proved, the novel here is to apply our approach reposed on Propositions 2.1-2.4.

Let us consider $B=C(p)C(p-1)\ldots C(2)C(1)$, we set $C(0)=\begin{pmatrix} b_0 & 1\\ 1 & 0\end{pmatrix}$ and $C(k)=\begin{pmatrix} b_k & 1\\ 1 & 0\end{pmatrix}$, for which b_0 is an integer and b_k's are nonegative integers for all $1\leq k\leq p$. Set $P_B(x)=x^2-c_0x-c_1$ the characteristic polynomial of the matrix B, making use of expression (2.1) we obtain,

$$\begin{aligned} c_0 &= B_p+A_{p-1}-b_0B_{p-1} \qquad (c_0\in\mathbb{N}^\star)\\ c_1 &= A_pB_{p-1}-B_pA_{p-1}=(-1)^{p-1} \qquad (c_1\in\{-1,1\}) \end{aligned} \tag{3.2}$$

Thence $P_B(x)=x^2-c_0x-c_1$ admits two real roots λ_1 and λ_2, we set $|\lambda_1|>|\lambda_2|$, with $\lambda_1=\frac{c_0+\sqrt{c_0^2+4(-1)^{p-1}}}{2}$ and $\lambda_2=\frac{c_0-\sqrt{c_0^2+4(-1)^{p-1}}}{2}$, owing to the fact that $\Delta=c_0^2+4(-1)^{p-1}>0$.

In order to prove the convergence of the continued fraction (3.1) of period p, its convergents $\frac{A_i}{B_i}$ need to satisfy the condition $\frac{c_1A_i+\lambda_1A_{p+i}}{c_1B_i+\lambda_1B_{p+i}}=\frac{c_1A_0+\lambda_1A_p}{c_1B_0+\lambda_1B_p}$, for every $1\leq i\leq p-1$. In fact, we observed that it is easy to show by recurrence that for every positive integer $n\geq 0$ we have

$$\begin{aligned} A_nB_{p+n-1}+A_{p+n}B_{n-1}-B_nA_{p+n-1}-B_{p+n}A_{n-1} &= (-1)^{n-1}(B_p+A_{p-1}-b_0B_{p-1})\\ &= (-1)^{n-1}c_0. \end{aligned}$$

Furthermore, it was demontrated in [4] that $A_nB_{n-1}-B_nA_{n-1}=(-1)^{n-1}$ for every $n\geq 0$. Thus we get

$$\frac{c_1A_{p-1}+\lambda_1A_{2p-1}}{c_1B_{p-1}+\lambda_1B_{2p-1}}=\cdots=\frac{c_1A_1+\lambda_1A_{p+1}}{c_1B_1+\lambda_1B_{p+1}}=\frac{c_1A_0+\lambda_1A_p}{c_1B_0+\lambda_1B_p}. \tag{3.3}$$

Consequently, in view of proposition (2.1), we manage to state the following result.

Proposition 3.1: Let $[b_0;\overline{b_1,\dots,b_p}]$ $(\{b_i\}_{1\le i\le p}\in\mathbb{N}^\star$ and $b_0\in\mathbb{Z})$ the simply periodic continued fraction (3.1) of period p. Set $B=C(p)C(p-1)\cdots C(2)C(1)$ with $C(i)=\begin{pmatrix} b_i & 1\\ 1 & 0\end{pmatrix}$, for $1\le i\le p$, and P_B being its characteristic polynomial $P_B(x)=x^2-c_0x-c_1$, where $c_0=B_p+A_{p-1}-b_0B_{p-1}$ and $c_1=(-1)^{p-1}$.

Then, the continued fraction (3.1) converges and its limit ω is a quadratic irrational given by

$$w=\frac{b_0\left(\frac{c_0-\sqrt{c_0^2+4(-1)^{p-1}}}{2}\right)-A_p}{\left(\frac{c_0-\sqrt{c_0^2+4(-1)^{p-1}}}{2}\right)-B_p}.$$

Proof: By Proposition 2.1 and formula (3.1), the simple periodic continued fraction (3.1) converges and its limit $\omega=\frac{b_0\lambda_2-A_p}{\lambda_2-B_p}=\frac{b_0\left(\frac{c_0-\sqrt{c_0^2+4(-1)^{p-1}}}{2}\right)-A_p}{\left(\frac{c_0-\sqrt{c_0^2+4(-1)^{p-1}}}{2}\right)-B_p}$. Thus, ω is an irrational number since $b_0,c_0,A_p,B_p\in\mathbb{Z}$ and $\Delta=c_0^2+4(-1)^{p-1}$ is not a perfect square. Furthermore, it is easy to verify that ω is a solution of the quadratic equation $B_{p-1}\omega^2+(B_p-A_{p-1}-b_0B_{p-1})\omega+(-A_p+b_0A_{p-1})=0$.

Example 3.2: Let us consider the simply periodic continued fraction $[1;\overline{2,1,3}]$.

We have $C(1)=\begin{pmatrix}2&1\\1&0\end{pmatrix}$, $C(2)=\begin{pmatrix}1&1\\1&0\end{pmatrix}$, $C(3)=\begin{pmatrix}3&1\\1&0\end{pmatrix}$. Thus, the characteristic polynomial of the matrix $B=C(3)C(2)C(1)=\begin{pmatrix}11&4\\3&1\end{pmatrix}$ is $P_B(x)=x^2-12x-1$. Hence, we have $A_3=15$, $B_3=11$, $\lambda_1=6+\sqrt{37}$ and $\lambda_2=6-\sqrt{37}$. It turns out that the continued fraction $[1;\overline{2,1,3}]$ converges to $\frac{\sqrt{37}-2}{3}$.

4. Some applications on Pell's equations

Relying on the results established in the previous sections, we are interested in giving the explicit solutions, when they exist, of the Pell's equation defined by

$$x^2 - dy^2 = \pm 1, \tag{4.1}$$

with d is a given positive nonsquare integer. In 1963, C.D. Olds proved in [9] that we can always find the particular solutions of the equation $x^2 - dy^2 = 1$, and sometimes particular solutions of the equation $x^2 - dy^2 = -1$, using the continued fraction expansion of $\sqrt{d}$. Furthermore, if each of these equations has solutions, we can get the two smallest integers $x_1 > 0, y_1 > 0$ such that $x_1^2 - dy_1^2 = 1$ or $x_1^2 - dy_1^2 = -1$. Once the least (minimal) positive solution is obtained, we can systematically generate all the other positive solutions. (See Theorem 4.4 and Theorem 4.5 in [9]).

4.1 *Solutions of Pell's equation* $x^2 - dy^2 = 1$

Let us first consider the Pell's equation

$$x^2 - dy^2 = 1 \tag{4.2}$$

The complete set of solutions of equation (4.2) was given in [4], using important characteristics of continued fractions. That is, it was provided that the minimal solution to Pell's equation (4.2) is given by

$$(x_1, y_1) = \begin{cases} (A_{p-1}, B_{p-1}) & \text{if } p \text{ is even} \\ (A_{2p-1}, B_{2p-1}) & \text{if } p \text{ is odd} \end{cases}$$

where p is the minimal period of the continued fraction of $\sqrt{d}$. Furthermore, the set of (positive) solutions to Pell's equation would be for all $k = 1, 2, 3, \cdots$

$$(x_k, y_k) = \begin{cases} (A_{kp-1}, B_{kp-1}) & \text{if } p \text{ is even} \\ (A_{2kp-1}, B_{2kp-1}) & \text{if } p \text{ is odd} \end{cases} \tag{4.3}$$

Here our goal is to give explicitly the solutions (x_k, y_k) for all $k = 1, 2, 3, \cdots$ based on our results in the previous sections.

First, we keep all the notations used before, and we put

$$\sqrt{d} = \left[b_0; \overline{b_1, \ldots, b_p}\right] = \left[b_0; \overline{b_1, b_2, \ldots, b_2, b_1, 2b_0}\right],$$

where $b_1, b_2, \ldots, b_2, b_1$ is a palindrome. Let the matrix $B = C(p)C(p-1)\cdots C(2)C(1)$, where $C(i) = \begin{pmatrix} b_i & 1 \\ 1 & 0 \end{pmatrix}$ for all $1 \le i \le p$, and $P_B(x) = x^2 - c_0 x - c_1$ be its characteristic polynomial $(c_1 = (-1)^{p-1})$.

Making use of proposition 3.1, it is easy to show that the two distinct roots λ_1 and λ_2 of P_B are given explicitly by

$$\begin{aligned}\lambda_1 &= \frac{dB_p - b_0A_p - (b_0B_p - A_p)\sqrt{d}}{(d - b_0^2)}\\ \lambda_2 &= \frac{dB_p - b_0A_p + (b_0B_p - A_p)\sqrt{d}}{(d - b_0^2)}\end{aligned} \tag{4.4}$$

Therefore, this enables us to express the solutions of equation (4.2) using formulas (2.2) and (4.3). Indeed, we get the following result.

Proposition 4.1: Let $\sqrt{d} = [b_0; \overline{b_1, \ldots, b_p}]$ be the simply periodic continued fraction of period p. The positive solutions of Pell's equation (4.2) are given as follows.

- If p is even and $k = 1, 2, 3\cdots$

$$A_{kp-1} = \left(\frac{\lambda_1^{2k}-1}{\lambda_1^{k-1}(\lambda_1^2-1)}\right)A_{p-1} - \left(\frac{\lambda_1^{2k-2}-1}{\lambda_1^{k-2}(\lambda_1^2-1)}\right).$$

$$B_{kp-1} = \left(\frac{\lambda_1^{2k}-1}{\lambda_1^{k-1}(\lambda_1^2-1)}\right)B_{p-1}.$$

- If p is odd and $k = 1, 2, 3\cdots$

$$A_{2kp-1} = \left[\frac{\lambda_1^{4k}-1}{\lambda_1^{2k-1}(\lambda_1^2+1)}\right]A_{p-1} + \left[\frac{\lambda_1^{4k-2}+1}{\lambda_1^{2k-2}(\lambda_1^2+1)}\right].$$

$$B_{2kp-1} = \left[\frac{\lambda_1^{4k}-1}{\lambda_1^{2k-1}(\lambda_1^2+1)}\right]B_{p-1}.$$

where λ_1 and λ_2 are given by (4.4).

Proof: When p is even, we have $c_1 = -1$, $\lambda_1 = \frac{1}{\lambda_2}$, $A_{2p-1} = c_0A_{p-1} - 1$ and $B_{2p-1} = c_0B_{p-1}$. By using the explicit expressions (2.2) of the convergents $\{A_n\}_{n\geq -1}$ and $\{B_n\}_{n\geq -1}$, we get

$$\begin{aligned}A_{kp-1} &= A_{(k-1)p+p-1}\\ &= \frac{\lambda_1^{k-1}}{P'(\lambda_1)}\left(\frac{\alpha_{p-1}}{\lambda_1} + \frac{\alpha_{2p-1}}{\lambda_1^2}\right) + \frac{\lambda_2^{k-1}}{P'(\lambda_2)}\left(\frac{\alpha_{p-1}}{\lambda_2} + \frac{\alpha_{2p-1}}{\lambda_2^2}\right).\end{aligned}$$

We have $\alpha_{2p-1} = c_1A_{2p-1}$ and $\alpha_{p-1} = c_1A_{p-1} + c_0A_{2p-1}$. Then, a direct computation leads to

$$A_{kp-1} = \frac{\lambda_1^{k-1}}{\lambda_1^2P'(\lambda_1)}(-\lambda_1A_{p-1} + \lambda_1c_0A_{2p-1} - A_{2p-1}) + \frac{\lambda_2^{k-1}}{\lambda_2^2P'(\lambda_2)}(-\lambda_2A_{p-1} + \lambda_2c_0A_{2p-1} - A_{2p-1}).$$

Hence,

$$A_{kp-1} = \left(\frac{\lambda_1^{2k} - 1}{\lambda_1^{k-1}(\lambda_1^2 - 1)} \right) A_{p-1} - \left[\frac{\lambda_1^{2k-2} - 1}{\lambda_1^{k-2}(\lambda_1^2 - 1)} \right].$$

Similarly, it is easy to check that

$$B_{kp-1} = \left(\frac{\lambda_1^{2k} - 1}{\lambda_1^{k-1}(\lambda_1^2 - 1)} \right) B_{p-1}.$$

When p is odd, we have $c_1 = 1$, $\lambda_1 = \frac{-1}{\lambda_2}$, $A_{2p-1} = c_0 A_{p-1} + 1$ and $B_{2p-1} = c_0 B_{p-1}$. A straightforward computation yields

$$\begin{aligned} A_{2kp-1} &= A_{(2k-1)p+p-1} \\ &= \frac{\lambda_1^{2k-1}}{\lambda_1^2 P'(\lambda_1)} (\lambda_1 A_{p-1} + \lambda_1^2 A_{2p-1}) + \frac{\lambda_2^{2k-1}}{\lambda_2^2 P'(\lambda_2)} (\lambda_2 A_{p-1} + \lambda_2^2 A_{2p-1}) \\ &= \left[\frac{\lambda_1^{4k} - 1}{\lambda_1^{2k-1}(\lambda_1^2 + 1)} \right] A_{p-1} + \left[\frac{\lambda_1^{4k-2} + 1}{\lambda_1^{2k-2}(\lambda_1^2 + 1)} \right] \end{aligned}$$

Similarly, one can check that

$$B_{2kp-1} = \left[\frac{\lambda_1^{4k} - 1}{\lambda_1^{2k-1}(\lambda_1^2 + 1)} \right] B_{p-1}.$$

4.2 *Solutions of Pell's equation* $x^2 - dy^2 = -1$

Let us consider the negative Pell's equation

$$x^2 - dy^2 = -1. \tag{4.5}$$

K. Rosen has established in [10, p. 404] (see theorem 11.5), that the positive solutions of (4.5) are given by

$$\begin{aligned} x &= A_{(2k-1)p-1} \qquad k = 1, 2, 3, \cdots \\ y &= A_{(2k-1)p-1} \qquad k = 1, 2, 3, \cdots \end{aligned} \tag{4.6}$$

when p is odd, and there is no solutions when p is even, where p is the period length of the simply periodic continued fraction $\sqrt{d} = [b_0; \overline{b_1, \ldots, b_p}]$.

In light of this result and using the explicit expressions (2.2) of the convergents $\{A_n\}_{n \geq -1}$ and $\{B_n\}_{n \geq -1}$, we state the following proposition.

Proposition 4.2: Let $\sqrt{d} = [b_0; \overline{b_1, \ldots, b_p}]$ be the simply periodic continued fraction of period p, and let λ_1 and λ_2 given by (4.4) be the two distinct roots of P_B. Then, the positive solutions of Pell's equation (4.5) are given explicitly by

$$A_{(2k-1)p-1} = \left[\frac{\lambda_1^{4k-2}+1}{\lambda_1^{2k-2}(\lambda_1^2+1)}\right]A_{p-1} + \left[\frac{\lambda_1^{4k-4}-1}{\lambda_1^{2k-3}(\lambda_1^2+1)}\right]$$
$$B_{(2k-1)p-1} = \left[\frac{\lambda_1^{4k-2}+1}{\lambda_1^{2k-2}(\lambda_1^2+1)}\right]B_{p-1},$$

when p is odd and $k = 1,2,3,\cdots,$ and this equation has no solutions when p is even.

Remark 4.3: The efficiency of our method lies in the possibility of generating all solutions of the equation (4.1) from the penultimate partial numerator A_{p-1} and penultimate partial denominator B_{p-1}.

Example 4.4: We are looking for solving the Pell's equation $x^2 - 13y^2 = \pm 1$. The length period of $\sqrt{13} = [3;\overline{1,1,1,1,6}]$ is odd, and we have $A_5 = 119$, $B_5 = 33$ and $\lambda_1 = 18 + 5\sqrt{13}$.

- Positive solutions of $x^2 - 13y^2 = 1$

  By using the proposition 4.1, we provide the first two solutions of this equation.

  - For $k = 1$, we have $A_9 = c_0 A_{p-1} + 1$ and $B_9 = c_0 B_{p-1} + 1$. After a simple calculation we find that $A_9 = 649$ and $B_9 = 180$. Thereafter, $649^2 - 13 \times 180^2 = 1$.
  - For $k = 2$, we have $A_{19} = \frac{(\lambda_1^2-1)(\lambda_1^4+1)}{\lambda_1^3} A_{p-1} + \frac{\lambda_1^6+1}{\lambda_1^2(\lambda_1^2+1)}$ and $B_{19} = \frac{(\lambda_1^2-1)(\lambda_1^4+1)}{\lambda_1^3} B_{p-1}$. Then $A_{19} = 842401$ and $B_{19} = 233640$.

  That is $842401^2 - 13 \times 233640^2 = 1$.

- Positive solutions of $x^2 - 13y^2 = -1$

  To obtain positive solutions of the equation $x^2 - 13y^2 = -1$, we use the proposition 4.2 and we give the first two solutions of this equation.

  - For $k = 1$, we have $A_4 = 18$ and $B_4 = 5$. Therefore $18^2 - 13 \times 5^2 = -1$.
  - For $k = 2$, and after a straightforward computation, we get $A_{14} = \frac{\lambda_1^6+1}{\lambda_1^2(\lambda_1^2+1)} A_{p-1} + \frac{\lambda_1^2-1}{\lambda_1}$ and $B_{14} = \frac{\lambda_1^6+1}{\lambda_1^2(\lambda_1^2+1)} B_{p-1}$. Thus $A_{14} = 23382$ and $B_{14} = 6485$.

Indeed, we have $23382^2 - 13 \times 6485^2 = -1$.

5. Concluding Remarks

Unlike other works that study the convergence of periodic continued fractions, our new approach shows a meticulous consideration towards the nth convergents $\frac{A_n}{B_n}$ of the continued fraction (1.3). The convergents satisfy two recursive formulas, which are linear difference equations of order 2 with periodic coefficients. First, we provided the explicit expressions of the general terms A_n and B_n with the aid of the properties of generalized Fibonacci sequences. Then, we apply our method to finding solutions of Pell's equation (4.1) using the roots of polynomial P_B of the matrix $B = C(p)C(p-1)...C(2)C(1)$. Finally, some examples are given to clarify the results.

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