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On LA-Gamma-semihypergroups containing two-sided bases

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Abstract

Ideal theory and base theory of algebraic structures play an important role in studied. In this study, we focus on studying bases of ideals in algebraic structures. The algebraic structure LA-Γ -semihypergroups is very interesting to study. This structure is generalization of many algebraic structures, for example, commutative semigroups, commutative Γ -semigroups, LA-semigroups, LA-Γ -semigroups, commutative semihypergroups, etc. Many results in these algebraic structures were extended to results in LA-Γ -semihypergroups. On the other hand, the results in these structures are the special cases of results in LA-Γ

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-semihypergroups. The main purpose of this paper is to introduce the notion of two-sided base of an LA-Γ-semihypergroup with pure left identity. We prove that every two-sided base of an LA-Γ-semihypergroup with pure left identity is one-element base. Finally, we prove that the compliment of union of all two-sided base of an LA-Γ-semihypergroup with pure left identity is maximal proper Γ-hyperideal.

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1. Introduction

The algebraic hyperstructure notion was introduced in 1934 by Marty [4] at the 8th Congress of Scandinavian Mathematicians. In binary classical algebraic structures, the product of two elements must be an element, while in the algebraic hyperstructure, the product of two elements is not an element but it will be a non-empty set.

Let H be a non-empty set and $P^*(H) = P(H) \setminus \{\varnothing\}$ denotes the set of all non-empty subsets of H. The map $\circ : H \times H \to P^*(H)$ is called the hyperoperation or the join operation on the set H. A couple $(H, \circ)$ is called a hypergroupoid. This structure is one of generalization of groupoids. Let A and B be two non-empty subsets of a hypergroupoid H, we will denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad a \circ A = \{a\} \circ A \text{ and } a \circ B = \{a\} \circ B.$$

Fabrici [3] first introduced the concepts of two-sided bases on semigroups. The notion of Γ-semigroups was introduced by Sen [5]. This algebraic structure generalize semigroup. Many results in semigroups were extended to study in Γ-semigroups. Results in [3] have been extended to results on a Γ-semigroups by Changpas and Kummoon [2]. The aim of this paper is to extend the results of a two-sided bases to LA-Γ-semihypergroups. The algebraic hyperstructure of an LA-Γ-semihypergroup was defined by Yaqoob and Aslam [6] as a generalization of many algebraic structures, for example, commutative semigroups, commutative Γ-semigroups, LA-semigroups, LA-Γ-semigroup, commutative semihypergroups, etc. The rest of this section, we recall certain definitions and results needed for our purpose. These can be found in [1, 7, 8, 9].

Definition 1.1 : ([6]) Let H and Γ be two non-empty sets. The algebraic hyperstructure H is called a left almost G-semihypergroup (abbreviated as an LA-Γ-semihypergroup) if every $\gamma \in \Gamma$ is a hyperoperation on H, i.e., $x\gamma y \subseteq H$ for every $x, y \in H$, and for every $\gamma, \beta \in \Gamma$ and $x, y, z \in H$ we have $(x\gamma y)\beta\{z\} = (z\gamma y)\beta\{x\}$.

The law $(x\gamma y)\beta\{z\} = (z\gamma y)\beta\{x\}$ is called left invertive law. Through out the paper H will denote an LA-Γ-semihypergroup unless otherwise specified. Let A and B be two non-empty subsets of an LA-Γ-semihypergroup H. We define

$$A\gamma B = \bigcup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

Also,

$$A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B = \bigcup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

Every LA-Γ-semihypergroup satisfies the law

$$(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$$

for all $a, b, c, d \in H$ and $\alpha, \beta, \gamma \in \Gamma$. This law is known as Γ-hypermedial law (see [6]).

Definition 1.2 : ([6]) Let H be an LA-Γ-semihypergroup. A non-empty subset A of H is called a sub LA-Γ-*semihypergroup* of H if $A\Gamma A \subseteq A$.

Definition 1.3 : ([6]) A non-empty subset A of an LA-Γ-semihypergroup H is a *right* (*resp., left*) *G-hyperideal* of H if $A\Gamma H \subseteq A$ (resp., $H\Gamma A \subseteq A$), and it is called a Γ-hyperideal of H if it is both a right and a left Γ-hyperideal.

Definition 1.4 : A proper Γ-hyperideal M of an LA-Γ-semihypergroup H is said to be *maximal* if for any Γ-hyperideal A of $H, M \subseteq A \subseteq H$ implies $M = A$ or $A = H$.

Definition 1.5 : Let H be an LA-Γ-semihypergroup. An element $e \in H$ is called a *left identity* if $a \in e\gamma a$ for all $a \in H$ and $\gamma \in \Gamma$.

Definition 1.6 : Let H be an LA-Γ-semihypergroup. An element $e \in H$ is called a *pure left identity* if $\{a\} = e\gamma a$ for all $a \in H$ and $\gamma \in \Gamma$.

Obviously, every pure left identity is a left identity but the converse is not generally true.

Example 1.7 : Let $H=\{x,y,z,t\}$ and $\Gamma=\{\beta,\gamma\}$ be the sets of binary hyperoperations defined below:

β	x	y	z	t
x	$\{x\}$	$\{x\}$	$\{x\}$	$\{x\}$
y	$\{x\}$	$\{z, t\}$	$\{z\}$	$\{z, t\}$
z	$\{x\}$	$\{z\}$	$\{z\}$	$\{z\}$
t	$\{x\}$	$\{y, t\}$	$\{z\}$	$\{z, t\}$

γ	x	y	z	t
x	$\{x\}$	$\{x\}$	$\{x\}$	$\{x\}$
y	$\{x\}$	$\{y, z, t\}$	$\{z\}$	$\{y, z, t\}$
z	$\{x\}$	$\{z\}$	$\{z\}$	$\{z\}$
t	$\{x\}$	$\{y, t\}$	$\{z\}$	$\{y, t\}$

Clearly H is not a Γ-semihypergroup because $\{y,z,t\}=(t\beta t)\gamma\{y\}\neq\{t\}\beta(t\gamma y)=\{y,t\}$ Thus H is an LA-Γ-semihypergroup because it satisfies left invertive law. It is easy to see that t is a left identity of H, but t is not a pure left identity of H.

Example 1.8 : Let $H=\{a,b,c,d,e\}$ and $\Gamma=\{\beta\}$ be the sets of binary hyperoperations defined as follows:

β	a	b	c	d	c
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b, c\}$	$\{c\}$	$\{a, d\}$	$\{c\}$
c	$\{a\}$	$\{b\}$	$\{b, c\}$	$\{a, d\}$	$\{b\}$
d	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$
c	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{c\}$

Then H is an LA-Γ-semihypergroup. It is easy to see that e is a left identity and e is a pure left identity of H.

Lemma 1.9 : *Let H be an LA-Γ-semihypergroup with pure left identity e. Then the following statements hold.*

(1) For all $x,y,z\in H$ and $\gamma,\beta\in\Gamma$,

$$\{x\}\gamma(y\beta z)=\{y\}\gamma(x\beta z).$$

(2) For all $x,y,z,w\in H$ and $\gamma,\beta,\alpha\in\Gamma$,

$$(x\gamma y)\beta(z\alpha w)=(w\gamma z)\beta(y\alpha x).$$

Proof: (1) Consider

$$\begin{aligned}\{x\}\gamma(y\beta z) &= (e\alpha x)\gamma(y\beta z) \quad \text{(by pure left identity } e)\\ &= (e\alpha y)\gamma(x\beta z) \quad \text{(by } \Gamma-\text{hypermedial law)}\\ &= \{y\}\gamma(x\beta z).\end{aligned}$$

Thus $\{x\}\gamma(y\beta z)=\{y\}\gamma(x\beta z)$.

(2) Consider

$$\begin{aligned}(x\gamma y)\beta(z\alpha w) &= (\{x\}\gamma\{y\})\beta(\{z\}\alpha\{w\})\\ &= ((e\alpha x)\gamma\{y\})\beta((e\gamma z)\alpha\{w\}) && \text{(by pure left identity } e)\\ &= ((y\alpha x)\gamma\{e\})\beta((w\gamma z)\alpha\{e\}) && \text{(by left invertive law)}\\ &= ((y\alpha x)\gamma(w\gamma z))\alpha(\{e\}\alpha\{e\}) && \text{(by } \Gamma-\text{hypermedial law)}\\ &= ((\{e\}\alpha\{e\})\gamma(w\gamma z))\beta(y\alpha x) && \text{(by left invertive law)}\\ &= (\{e\}\gamma(w\gamma z))\beta(y\alpha x)=(w\gamma z)\beta(y\alpha x).\end{aligned}$$

Thus $(x\gamma y)\beta(z\alpha w)=(w\gamma z)\beta(y\alpha x)$. □

The law $(x\gamma y)\beta(z\alpha w)=(w\gamma z)\beta(y\alpha x)$ is called a Γ-paramedial law.

Lemma 1.10 : *If H is an LA-Γ-semihypergroup with pure left identity e, then $H\Gamma H=H$ and $H=H\Gamma\{e\}=\{e\}\Gamma H$.*

Proof: Clearly, $H\Gamma H\subseteq H$. Let $x\in H$. Then for any $\gamma\in\Gamma$, we have $x\in\{x\}=e\gamma x\subseteq H\Gamma H$ and so $H\subseteq H\Gamma H$. Hence $H=H\Gamma H$. Next, we will show that $H=H\Gamma\{e\}=\{e\}\Gamma H$. Since *e* is a pure left identity of *H*, so for any $\gamma\in\Gamma$, it is obvious that $\{e\}\Gamma H=H$. Consider

$$H\Gamma\{e\}=(H\Gamma H)\Gamma\{e\}=(\{e\}\Gamma H)\Gamma H=H\Gamma H=H.$$

Hence $H=H\Gamma\{e\}=\{e\}\Gamma H$. □

Lemma 1.11 : *Let H be an LA-Γ-semihypergroup and A_i be a Γ-hyperideal of H for all $i\in I$. If $\bigcap_{i\in I}A_i\neq\varnothing$, then $\bigcap_{i\in I}A_i$ is a Γ-hyperideal of H.*

Proof: Assume that $\bigcap_{i\in I}A_i\neq\varnothing$. We will show that $\bigcap_{i\in I}A_i$ is a Γ-hyperideal of *H*. First, to show that $H\Gamma(\bigcap_{i\in I}A_i)\subseteq\bigcap_{i\in I}A_i\cdot$ Let $x\in H\Gamma(\bigcap_{i\in I}A_i)$. Then $x\in h\gamma a_1$ for some $h\in H,\gamma\in\Gamma$ and $a_1\in\bigcap_{i\in I}A_i$. Since $a_1\in\bigcap_{i\in I}A_i$, we obtain $a_1\in A_i$ for all $i\in I$. Since A_i is a Γ-hyperideal of *H* for all $i\in I$. Then $x\in h\gamma a_1\subseteq H\Gamma A_i\subseteq A_i$, for all $i\in I$. So we obtain $x\in\bigcap_{i\in I}A_i$. Thus $H\Gamma(\bigcap_{i\in I}A_i)\subseteq\bigcap_{i\in I}A_i$. Next, to show that $(\bigcap_{i\in I}A_i)\Gamma H\subseteq\bigcap_{i\in I}A_i$, let $x\in(\bigcap_{i\in I}A_i)\Gamma H$.

Then $x \in a_1\gamma h$ for some $a_1 \in \bigcap_{i\in I} A_i, \gamma \in \Gamma$ and $h \in H$. Since $a_1 \in \bigcap_{i\in I} A_i$, we obtain $a_1 \in A_i$ for all $i \in I$. Since A_i is a Γ-hyperideal of H for all $i \in I$, thus $x \in a_1\gamma h \subseteq A\Gamma H \subseteq A_i$, for all $i \in I$. So $x \in \bigcap_{i\in I} A_i$. Then $H\Gamma(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} A_i$. Therefore $\bigcap_{i\in I} A_i$ is a Γ-hyperideal of H. □

Definition 1.12 : Let A be a non-empty subset of an LA-Γ-semihypergroup H. The intersection of all Γ-hyperideal of H containing A is the smallest Γ-hyperideal of H generated by A and is denoted by $(A)_T$.

Lemma 1.13 : *Let A be a non-empty subset of an LA-Γ-semihypergroup H with pure left identity. Then*

$$(A)_T = A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H.$$

Proof: Let $B = A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H.$ First, we consider

$$\begin{aligned}
&H\Gamma B \\
&= H\Gamma(A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H) \\
&= H\Gamma A \cup H\Gamma(H\Gamma A) \cup H\Gamma(A\Gamma H) \cup H\Gamma((H\Gamma A)\Gamma H) \\
&= H\Gamma A \cup (H\Gamma H)\Gamma(H\Gamma A) \cup A\Gamma(H\Gamma H) \cup (H\Gamma A)\Gamma(H\Gamma H) \\
&= H\Gamma A \cup (A\Gamma H)\Gamma(H\Gamma H) \cup A\Gamma H \cup (H\Gamma A)\Gamma H \\
&= H\Gamma A \cup (A\Gamma H)\Gamma H \cup A\Gamma H \cup (H\Gamma A)\Gamma H \\
&= H\Gamma A \cup (H\Gamma H)\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H \\
&= H\Gamma A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H \subseteq B
\end{aligned}$$

and

$$\begin{aligned}
&B\Gamma H \\
&= (A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H)\Gamma H \\
&= A\Gamma H \cup (H\Gamma A)\Gamma H \cup (A\Gamma H)\Gamma H \cup (H\Gamma A)\Gamma H)\Gamma H \\
&= A\Gamma H \cup (H\Gamma A)\Gamma H \cup (H\Gamma H)\Gamma A \cup (H\Gamma H)\Gamma(H\Gamma A) \\
&= A\Gamma H \cup (H\Gamma A)\Gamma H \cup H\Gamma A \cup (A\Gamma H)\Gamma(H\Gamma H) \\
&= A\Gamma H \cup (H\Gamma A)\Gamma H \cup H\Gamma A \cup (A\Gamma H)\Gamma H \\
&= A\Gamma H \cup (H\Gamma A)\Gamma H \cup H\Gamma A \cup (H\Gamma H)\Gamma A \\
&= A\Gamma H \cup (H\Gamma A)\Gamma H \cup H\Gamma A \cup H\Gamma A \subseteq B.
\end{aligned}$$

So $H\Gamma B \subseteq B$ and $B\Gamma H \subseteq B$. Thus B is a Γ-hyperideal of H containing A. Next, let C be a Γ-hyperideal of H containing A. Since $A \subseteq C$, we have $H\Gamma A \subseteq H\Gamma C \subseteq C, A\Gamma H \subseteq C\Gamma H \subseteq C$ and $(H\Gamma A)\Gamma H \subseteq (H\Gamma C)\Gamma H \subseteq C\Gamma H \subseteq C$. So $B = (A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H) \subseteq C$. Thus B is a smallest Γ-hyperideal of H containing A. Thus $(A)_T = (A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H)$. □

For an element $a \in H$, we write $(\{a\})_T$ by $(a)_T$ which is called the principal Γ-hyperideal of H generated by a. By Lemma 1.4, we have that

$$(a)_T = \{a\} \cup H\Gamma\{a\} \cup \{a\}\Gamma H \cup (H\Gamma\{a\})\Gamma H.$$

Corollary 1.14 : *Let H be an LA-Γ-semihypergroup with pure left identity. Then $H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$ is a Γ-hyperideal of H for all $b \in H$.*

2. Bases on LA-Γ-semihypergroups

We begin this section with the definition of two-sided base of an LA-Γ-semihypergroup with pure left identity as follows:

Definition 2.1 : Let H be an LA-Γ-semihypergroup with pure left identity. A non-empty subset A of H is called a two-sided base of H if it satisfies the following two conditions:

(1) $H = A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H$;

(2) if B is subset of A such that $H = B \cup H\Gamma B \cup B\Gamma H \cup (H\Gamma B)\Gamma H$, then $B = A$.

We now provide some examples. By Example 1.7., it is easy to see that the two-sided base of H is $A = \{e\}$.

Example 2.2 : Let $H = \{x, y, z, w\}$ and $\Gamma = \{\gamma\}$ be the sets of binary hyperoperations defined as follows:

γ	x	y	z	w
x	$\{x\}$	$\{y\}$	$\{z\}$	$\{w\}$
y	$\{z\}$	H	H	$\{w\}$
z	$\{y\}$	H	H	$\{w\}$
w	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$

Then H is an LA-Γ-semihypergroup with pure left identity x. It is easy to see that the two-sided bases of H are $A_1 = \{x\}, A_2 = \{y\}$ and $A_3 = \{z\}$.

Definition 2.3 : Let H be an LA-Γ-semihypergroup with pure left identity. We define a quasi-ordering on H by for any $a,b \in H$,

$$a \leq_I b \Leftrightarrow (a)_T \subseteq (b)_T.$$

We write $a <_I b$ if $a \leq_I b$ but $a \neq b$.

First, we have the following useful lemma:

Lemma 2.4 : *Let A be a two-sided base of an LA-Γ-semihypergroup H with pure left identity and $a,b \in A$. If $a \in H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$, then $a = b$.*

Proof: Let $a,b \in A$ be such that $a \in H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$ and suppose that $a \neq b$. Let $B = A \setminus \{a\}$. Then $B \subset A$. Since $a \neq b$, we have that $b \in B$. We will show that $(A)_T \subseteq (B)_T$. Let $x \in (A)_T = A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H$. There are four cases to consider.

Case 1: $x \in A$. If $x \neq a$, then $x \in B \subseteq (B)_T$. So $x \in (B)_T$. If $x = a$, then by assumption we have

$$x = a \in H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H \subseteq H\Gamma B \cup B\Gamma H \cup (H\Gamma B)\Gamma H \subseteq (B)_T.$$

So $x \in (B)_T$. Thus $A \subseteq (B)_T$.

Case 2: $x \in H\Gamma A$. Then $x \in h\gamma c$ for some $h \in H, \gamma \in \Gamma$ and $c \in A$. If $c \neq a$, then $x \in h\gamma c \subseteq H\Gamma B \subseteq (B)_T$. So $x \in (B)_T$. If $c = a$, then $x \in h\gamma a \subseteq H\Gamma(H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H)$. Since $H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$ is a Γ-hyperideal of H for all $b \in H$, then $x \in H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H \subseteq (B)_T$. So $x \in (B)_T$. Thus $H\Gamma A \subseteq (B)_T$.

Case 3 : $x \in A\Gamma H$. Then $x \in c\gamma h$ for some $c \in A, \gamma \in \Gamma$ and $h \in H$. If $c \neq a$, then $x \in c\gamma h \subseteq B\Gamma H \subseteq (B)_T$. So $x \in (B)_T$. If $c = a$, then $x \in c\gamma h \subseteq (H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H)\Gamma H$. Since $H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$ is a Γ-hyperideal of H for all $b \in H$, then $x \in H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H \subseteq (B)_T \cdot$ So $x \in (B)_T$. Thus $A\Gamma H \subseteq (B)_T$.

Case 4 : $x \in (H\Gamma A)\Gamma H$. Then $x \in (h_1\gamma c)\beta\{h_2\}$ for some $h_1, h_2 \in H$, $\gamma, \beta \in \Gamma$ and $c \in A$. If $c = a$, then $x \in (h_1\gamma c)\ \beta\{h_2\} \subseteq (H\Gamma(H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H))\Gamma H$. Since $H\Gamma\{b\} \cup \{b\}\ \Gamma H \cup (H\Gamma\{b\})\Gamma H$ is a Γ-hyperideal of H for all $b \in H$, then $x \in (h_1\gamma c)\ \beta\{h_2\} \subseteq (H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H)\Gamma H \subseteq H\Gamma\{b\} \cup \{b\}\Gamma H \cup H\Gamma\{b\} \subseteq (B)_T$. So $x \in (B)_T$. If $c \neq a$, then $x \in (h_1\gamma c)\ \beta\{h_2\} \subseteq (H\Gamma B)\Gamma H \subseteq (B)_T$. So $x \in (B)_T$. Thus $(H\Gamma A)\Gamma H \subseteq (B)_T$.

Hence $(A)_T = A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\ \Gamma H \subseteq (B)_T$. By $H = (A)_T \subseteq (B)_T \subseteq H$, this implies that $(B)_T = H$. This is a contradiction. Therefore, $a = b$. □

Lemma 2.5 : *Let A be a two-sided base of an LA-Γ-semihypergroup H with pure left identity. If $a,b \in A$ such that $a \neq b$ then neither $a \leq_I b$ nor $b \leq_I a$.*

Proof: Let $a,b \in A$ be such that $a \neq b$. Suppose that $a \leq_I b$. Then $a \in (a)_T \subseteq (b)_T$. Since $a \in (b)_T$ and $a \neq b$, we have that $a \in H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$. By Lemma 2.1, $a = b$. This is a contradiction. Thus $a \leq_I b$ is false. The case $b \leq_I a$ can be proved similarly. □

3. Main results

In this part, the remarkable results of two-sided bases in an LA-Γ-semihypergroup with pure left identity containing two-sided base will be presented.

Theorem 3.1 : *A non-empty subset A of an LA-Γ-semihypergroup H with pure left identity, is a two-sided base if and only if A satisfies the following two condition.*

(1) *For any $x \in H$ there exists $a \in A$ such that $x \leq_I a$.*

(2) *For any $a,b \in A$, if $a \neq b$, then neither $a \leq_I b$ nor $b \leq_I a$.*

Proof: Assume that A is a two-sided base of H. Then $H = (A)_T$. First, to show that the condition (1) holds, we let $x \in H$. Then $x \in A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H$. There are four cases to consider.

Case 1 : $x \in A$. Then $x \leq_I x$.

Case 2 : $x \in H\Gamma A$. Then $x \in h\gamma a$ for some $h \in H, \gamma \in \Gamma$ and $a \in A$. Since $x \in h\gamma a \subseteq H\Gamma\{a\} \subseteq (a)_T$, so $x \in (a)_T$. It follows that $(x)_T \subseteq (a)_T$. Thus $x \leq_I a$ where $a \in A$.

Case 3 : $x \in A\Gamma H$. Then $x \in b\gamma h$ for some $h \in H, \gamma \in \Gamma$ and $b \in A$. Since $x \in b\gamma h \subseteq \{b\}\Gamma H \subseteq (b)_T$, so $x \in (b)_T$. It follows that $(x)_T \subseteq (b)_T$. Thus $x \leq_I b$ where $b \in A$.

Case 4 : $x \in (H\Gamma A)\Gamma H$. Then $x \in (h_1\gamma c)\beta\{h_2\}$ for some $h_1, h_2 \in H, \gamma, \beta \in \Gamma$ and $c \in A$. Since $x \in (h_1\gamma c)\beta\{h_2\} \subseteq (H\Gamma\{c\})\Gamma H \subseteq (c)_T$, so $x \in (c)_T$. It follows that $(x)_T \subseteq (c)_T$. Thus $x \leq_I c$ where $c \in A$.

Hence the condition (1) holds. The validity of condition (2) follows from Lemma 2.2.

Conversely, assume that (1) and (2) hold. We will show that A is a two-sided base of H. First, to show that $H=(A)_T$, we let $x \in H$. By (1) there exists $a \in A$ such that $(x)_T \subseteq (a)_T$. Then $x \in (x)_T \subseteq (a)_T \subseteq (A)_T$. So $H \subseteq (A)_T$. Thus $H=(A)_T$. Next, to show that A is a minimal subset of H with the property $H=(A)_T$, we let $B \subset A$ such that $H=(B)_T$. Since $B \subset A$, there exists $a \in A$ and $a \notin B$. Since $a \notin B$ and $a \in A \subseteq H=(B)_T$, we have that $a \in H\Gamma B \cup B\Gamma H \cup (H\Gamma B)\Gamma H$. There are three cases to consider:

Case 1 : $a \in H\Gamma B$. Then $a \in h\gamma b$ for some $h \in H, \gamma \in \Gamma$ and $b \in B$. By $a \in h\gamma b \subseteq H\Gamma\{b\} \subseteq (b)_T$, it follows that $(a)_T \subseteq (b)_T$. Thus $a \leq_I b$ where $a,b \in A$. This is a contradiction.

Case 2 : $a \in B\Gamma H$. Then $a \in b_1\gamma h$ for some $h \in H, \gamma \in \Gamma$ and $b_1 \in B$. By $a \in b_1\gamma h \subseteq \{b_1\}\Gamma H \subseteq (b_1)_T$, it follows that $(a)_T \subseteq (b_1)_T$. Thus $a \leq_I b_1$ where $a, b_1 \in A$. This is a contradiction.

Case 3 : $a \in (H\Gamma B)\Gamma H$. Then $a \in (h_1\gamma b_2)\beta\{h_2\}$ for some $h_1,h_2 \in H, \gamma, \beta \in \Gamma$ and $b_2 \in B$. By $a \in (h_1\gamma b_2)\beta\{h_2\} \subseteq (H\Gamma\{b_2\})\Gamma H \subseteq (b_2)_T$, it follows that $(a)_T \subseteq (b_2)_T$. Thus $a \leq_I b_2$ where $a, b_2 \in A$. This is a contradiction.

From both cases, we conclude that A is a two-sided base of H. □

Theorem 3.2 : *Let A be a two-sided base of an LA-Γ-semihypergroup of H with pure left identity, such that $(a)_T=(b)_T$ for some $a \in A$ and $b \in H$. If $a \neq b$, then H contains at least two two-sided bases.*

Proof: Assume that $a \neq b$ and suppose that $b \in A$. Since $(a)_T=(b)_T$ and $a \neq b$, it follows that $a \in H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$. By Lemma 2.1, we have that $a=b$. This is a contradiction. Thus $b \in H \setminus A$. Setting $B=(A\setminus\{a\})\cup\{b\}$. Since $b \in B$ and $b \notin A$, then $B \not\subseteq A$. Thus $B \neq A$. We will show that B is a two-sided base of H. First, to show that B satisfies the condition (1) of Theorem 3.1. Let $x \in H$. Since A is a two-sided base of H, there exists $c \in A$ such that $x \leq_I c$. If $c \neq a$ then $c \in B$. If $c=a$, then $(c)_T=(a)_T$. Since $(a)_T=(b)_T$, we have $(c)_T=(b)_T$ i.e., $c \leq_I b$. So $x \leq_I c \leq_I b$. Thus $x \leq_I b$ where $b \in B$. Next, to show that B satisfies the condition (2) of Theorem 3.1. Let $c_1, c_2 \in B$ such that $c_1 \neq c_2$. We will show that neither $c_1 \leq_I c_2$ nor $c_2 \leq_I c_1$. Since $c_1 \in B$ and $c_2 \in B$, we have $c_1=b$ or $c_1 \neq b$ and $c_2=b$ or $c_2 \neq b$. There are four cases to consider.

Case 1 : $c_1 \neq b$ and $c_2 \neq b$. Then $c_1, c_2 \in A$. This implies neither $c_1 \leq_I c_2$ nor $c_2 \leq_I c_1$.

Case 2 : $c_1 \neq b$ and $c_2 = b.$ Then $(c_2)_T = (b)_T.$ If $c_1 \leq_I c_2,$ then $(c_1)_T \subseteq (c_2)_T = (b)_T = (a)_T.$ Thus $c_1 \leq_I a$ where $c_1, a \in A.$ This is contradiction. If $c_2 \leq_I c_1,$ then $(a)_T = (b)_T = (c_2)_T \subseteq (c_1)_T.$ Thus $a \leq_I c_1$ where $c_1, a \in A.$ This is a contradiction.

Case 3 : $c_1 = b$ and $c_2 \neq b.$ Then $(c_1)_T = (b)_T.$ If $c_1 \leq_I c_2,$ then $(a)_T = (b)_T = (c_1)_T \subseteq (c_2)_T.$ Thus $a \leq_I c_2$ where $c_2, a \in A.$ This is contradiction. If $c_2 \leq_I c_1$ then $(c_2)_T \subseteq (c_1)_T = (b)_T = (a)_T.$ Thus $c_2 \leq_I a$ where $c_2, a \in A.$ This is a contradiction.

Case 4 : $c_1 = b$ and $c_2 = b.$ This is impossible.

Therefore B is a two-sided base of H. □

The next corollary is a consequence of Theorem 3.2.

Corollary 3.3 : *Let A be a two-sided base of an LA-*Γ*-semihypergroup H with pure left identity and let* $a \in A.$ *If* $(x)_T = (a)_T$ *for some* $x \in H, x \neq a,$ *then x belongs to some two-sided base of H, which is different from A.*

Theorem 3.4 : *Let A and B be any two-sided bases of an LA-*Γ*-semihypergroup H with pure left identity. Then A and B have the same cardinality.*

Proof : Let A and B be two-sided bases of H. Let $a \in A.$ Since B is a two-sided base of H, by Theorem 3.1(1), then there exists $b \in B$ such that $a \leq_I b.$ Similarly, since A is a two-sided base of H, there exists $a^* \in A$ such that $b \leq_I a^*.$ So $a \leq_I b \leq_I a^*$ and $a \leq_I a^*.$ By Theorem 3.1(2), $a = a^*.$ Hence $(a)_T = (b)_T.$ Now, define a mapping

$$\varphi : A \rightarrow B; \varphi(a) = b$$

for all $a \in A.$ First, to show that φ is well-defined, we let $a_1, a_2 \in A$ such that $a_1 = a_2, \varphi(a_1) = b_1,$ and $\varphi(a_2) = b_2$ for some $b_1, b_2 \in B.$ Then $(a_1)_T = (b_1)_T$ and $(a_2)_T = (b_2)_T.$ Since $a_1 = a_2,$ then $(a_1)_T = (a_2)_T.$ Thus $(a_1)_T = (a_2)_T = (b_1)_T = (b_2)_T,$ so $b_1 \leq_I b_2$ and $b_2 \leq_I b_1.$ By Theorem 3.1(2), $b_1 = b_2.$ Hence $\varphi(a_1) = \varphi(a_2).$ Therefore φ is well-defined. Next, to show that φ is one-to-one, we let $a_1, a_2 \in A$ such that $\varphi(a_1) = \varphi(a_2).$ Then $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B.$ We have $(a_1)_T = (a_2)_T = (b)_T.$ Since $(a_1)_T = (a_2)_T,$ so $a_1 \leq_I a_2$ and $a_2 \leq_I a_1.$ Thus $a_1 = a_2.$ Therefore φ is one-to-one. Finally, to show that φ is onto, we let $b \in B,$ then there exists $a \in A$ such that $b \leq_I a.$ Similarly, there exists $b^* \in B$ such that $a \leq_I b^*.$ Then

$b \leq_I a \leq_I b^*$ i.e., $b \leq_I b^*$. By Theorem 3.1(2), $b = b^*$, so $b \leq_I a$ and $a \leq_I b$ i.e., $(b)_T \subseteq (a)_T$ and $(a)_T \subseteq (b)_T$. Thus $(a)_T = (b)_T$. Therefore φ is onto. □

If a two-sided base of an LA-Γ-semihypergroup H with pure left identity, is a Γ-hyperideal of H, then

$$H = A \cup H\Gamma A \cup A\Gamma H \cup (H\Gamma A)\Gamma H = A \cup A \cup A \cup A = A.$$

Thus $H = A$. The converse statement is clear. So we conclude that.

Remark 3.5 : It is observed that a two-sided base A of an LA-Γ-semihypergroup H with pure left identity is Γ-hyperideal of H if and only if $A = H$.

Theorem 3.6 : *Let A be a two-sided base of an LA-Γ-semihypergroup H with pure left identity. If A is a sub LA-Γ-semihypergroup of A, then $A = \{a\}$ with $a \in a\gamma a$ for all $\gamma \in \Gamma$.*

Proof: Assume that A is a sub LA-Γ-semihypergroup of H with pure left identity. Let $a, b \in A$, and let $\gamma \in \Gamma$. By assumption, $a\gamma b \subseteq A$. Setting $c \in a\gamma b$, so we have $c \in a\gamma b \subseteq H\Gamma\{b\} \subseteq H\Gamma\{b\} \cup \{b\}\Gamma H \cup (H\Gamma\{b\})\Gamma H$. By Lemma 2.1, we have $c = b$. Similarly, $c \in a\gamma b \subseteq \{a\}\Gamma H \subseteq H\Gamma\{a\} \cup \{a\}\Gamma H \cup (H\Gamma\{a\})\Gamma H$. By Lemma 2.1, we have $c = a$. Thus $a = b$. Therefore $A = \{a\}$ with $a \in a\gamma a$. □

The converse statement is not valid general. By Example 2.1, we have $A_2 = \{y\}$ is a two-sided base of an LA-Γ-semihypergroup H with pure left identity, such that $y \in y\gamma y$ for all $\gamma \in \Gamma$. But A_2 is not a sub LA-Γ-semihypergroup H. The union of all two-sided bases of an LA-Γ-semihypergroup H with pure left identity, is denoted by A.

Theorem 3.7 : *Let H be an LA-Γ-semihypergroup with pure left identity. Then $H \setminus A$ is either empty set or a Γ-hyperideal of H.*

Proof : Assume that $H \setminus A \neq \varnothing$. We will show that $H \setminus A$ is a Γ-hyperideal of H. Let $a \in H, \gamma \in \Gamma$ and $b \in H \setminus A$. To show that $a\gamma b \subseteq H \setminus A$ and $b\gamma a \subseteq H \setminus A$. Suppose that $a\gamma b \not\subseteq H \setminus A$. Then there exists $c \in a\gamma b$ and $c \notin H \setminus A$ i.e., $c \in A$. Since $c \in A$, then $c \in A_1$ for some a two-sided base A_1 of H. Since $c \in a\gamma b \subseteq H\Gamma\{b\} \subseteq (b)_T$, then $c \in (b)_T$. So we have that $(c)_T \subseteq (b)_T$. Next, we will show that $(c)_T \subset (b)_T$. Suppose that $(c)_T = (b)_T$. Since $b \in H \setminus A$ and $c \in A_1$, then $b \neq c$. Since $(c)_T = (b)_T$, by Corollary 3.3, we have $b \in A$. This is a contradiction. Thus $(c)_T \subset (b)_T$ i.e., $c <_I b$.

Since A_1 is a two-sided bases of H, there exists $c_1 \in A_1$ such that $b \leq c_1$. Since $c <_I b \leq_I c_1$, so $c \leq_I c_1$ and $c, c_1 \in A_1$. This is a contradiction. Thus $a\gamma b \subseteq H \setminus A$. Similarly, we can proof that $b\gamma a \subseteq H \setminus A$. Therefore $H \setminus A$ is a Γ-hyperideal of H. □

Let M^* be a proper Γ-hyperideal of an LA-Γ-semihypergroup H with pure left identity containing every proper Γ-hyperideal of H.

Theorem 3.8 : *Let H be an LA-Γ-semihypergroup with pure left identity and $\emptyset \neq A \subset H$. Then $H \setminus A = M^*$ if and only if every two-sided base of H is a one-element base.*

Proof: Assume that $H \setminus A = M^*$. We will show that every two-sided base of H is a one-element base. To show that $A \subseteq (a)_T$ for all $a \in A$. Let $a \in A$. Suppose $A \not\subset (a)_T$. Then $(a)_T \neq H$, and so $(a)_T$ is a proper Γ-hyperideal of H. Thus $a \in (a)_T \subseteq M^* = H \setminus A$, and so $a \in H \setminus A$ i.e., $a \notin A$. This is a contradiction. Hence $A \subseteq (a)_T$ for all $a \in A$. We claim that $H \setminus A \subseteq (a)_T$ for all $a \in A$. Suppose that $H \setminus A \not\subseteq (a_1)_T$ for some $a_1 \in A$. Then $(a_1)_T \neq H$, and so $(a_1)_T$ is a proper Γ-hyperideal of H. This implies $a_1 \in (a_1)_T \subseteq M^* = H \setminus A$, and so $a_1 \in H \setminus A$ i.e., $a_1 \notin A$. This is a contradiction. Hence $H \setminus A \subseteq (a)_T$ for all $a \in A$. Since $H \setminus A \subseteq (a)_T$ and $A \subseteq (a)_T$ for all $a \in A$, then $H = (H \setminus A) \cup A \subseteq (a)_T \subseteq H$. It follows that $(a)_T = H$ for all $a \in A$. Thus $\{a\}$ is a two-sided base of H. Next, let B be a two-sided base of H, and let $a, b \in B$. We will show that $a = b$. Suppose that $a \neq b$. Since $B \subseteq A$ and $a \in B$, then $a \in A$. So we have $H = (a)_T$. Since $a \neq b$, and $b \in H = (a)_T$, then $b \in H\Gamma\{a\} \cup \{a\}\Gamma H \cup (H\Gamma\{a\})\Gamma H$. By Lemma 2.1, $a = b$. This is a contradiction. Therefore, every two-sided base of H is one-element base.

Conversely, assume that every two-sided base of H is a one-element base. Then $(a)_T = H$ for all $a \in A$. We will show that $H \setminus A = M^*$. Since $a \in A$ i.e., $a \notin H \setminus A$, and $A \subset H$, then $\emptyset \neq H \setminus A \subset H$. By Theorem 3.6, $H \setminus A$ is a proper Γ-hyperideal of H. Next, let M_1 be a Γ-hyperideal of H such that $H \setminus A \subseteq M_1 \subseteq H$. Suppose that $H \setminus A \neq M_1$. Then $H \setminus A \subset M_1$, there exists $x \in M_1$ such that $x \notin H \setminus A$ i.e., $x \in A$. So $x \in A \cap M_1$ and $A \cap M_1 \neq \emptyset$. Let $c \in A \cap M_1$. Since $c \in M_1$, $H\Gamma\{c\} \subseteq H\Gamma M_1 \subseteq M_1$, $\{c\}\Gamma H \subseteq M_1\Gamma H \subseteq M_1$ and $(H\Gamma\{c\})\Gamma H \subseteq (H\Gamma M_1)\Gamma H \subseteq M_1\Gamma H \subseteq M_1$, then $\{c\} \cup H\Gamma\{c\} \cup \{c\}\Gamma H \cup (H\Gamma\{c\})\Gamma H \subseteq M_1$. So $(c)_T = \{c\} \cup H\Gamma\{c\} \cup \{c\}\Gamma H \cup (H\Gamma\{c\})\Gamma H \subseteq M_1$. Since $c \in A$, by assumption we have $(c)_T = H$. Since $(c)_T = H$ and $(c)_T \subseteq M_1$, then $H = (c)_T \subseteq M_1 \subseteq H$. Thus $H = M_1$. Hence

$H \setminus A$ is a maximal proper Γ-hyperideal of H. Next, let B be a Γ-hyperideal of H is not contained in $H \setminus A$. Then there exists $x \in B$ such that $x \notin H \setminus A$ i.e., $x \in A$. So $B \cap A \neq \varnothing$. Let $c \in B \cap A$. Then $c \in B$ and $c \in A$. Since $c \in B$, then it follows that $(c)_T \subseteq B$. Since $c \in A$, by assumption $H = (c)_T$. So we have $H = (c)_T \subseteq B \subseteq H$. Thus $H = B$. Therefore $H \setminus A = M^*$. □

In Example 1.2, we have e is a pure left identity of H and $\{e\}$ is a two-sided base of H. Similarly, in Example 2.1, we have x is a pure left identity of H and $\{x\}$ is a two-sided base of H. This leads to prove the following theorem.

Theorem 3.9 : *Let H be an LA-Γ-semihypergroup with pure left identity. If e is a pure left identity of H, then {e} is a two-sided base of H.*

Proof: Assume that e is pure left identity of H. Let $A = \{e\}$. We will show that A is a two-sided base of H using Definition 2.1. To show that $H = (A)_T$. Since e is a pure left identity of H, by Lemma 1.2, we have $H = \{e\}\Gamma H = H\Gamma\{e\}$. Since $H = H\Gamma\{e\}$, we have $(H\Gamma\{e\})\Gamma H = (H\Gamma\{e\})\Gamma(H\Gamma\{e\}) = (H\Gamma H)\Gamma(\{e\}\Gamma\{e\}) = H\Gamma\{e\}$. So $(A)_T = \{e\} \cup H\Gamma\{e\} \cup \{e\}\Gamma H \cup (H\Gamma\{e\})\Gamma\{e\} = H$. Thus $(A)_T = H$. Obvious, A is a minimal subset of H with the property $H = (A)_T$. Therefore $A = \{e\}$ is a two-sided base of H. □

In Example 1.2 and Example 2.1, we have e and a is a pure left identity of H such that $\{e\}$ and $\{a\}$ are two-sided bases of H. Since $\{e\}$ and $\{a\}$ are one-element base, by Theorem 3.4, we obtain set of two-sided bases in Example 1.2 and Example 2.1, are only one-element base. Thus, the combining Theorem 3.4, and Theorem 3.8, we obtain the following corollary.

Corollary 3.10 : *Let H be an LA-Γ-semihypergroup with pure left identity. Then every two-sided base of H is one-element base.*

In Example 1.2, we have the of all two-sided base of H is $\{e\}$. Since $\{e\}$ is a one-element base, by Theorem 3.7, we have $H \setminus A = \{a, b, c, d\}$ is a maximal proper Γ-hyperideal of H containing ever proper Γ-hyperideal of H. Thus, by Theorem 3.7, and Corollary 3.9, we conclude the following theorem.

Theorem 3.11 : *Let H be an LA-Γ-semihypergroup with pure left identity. Then $H \setminus A$ is a maximal proper Γ-hyperideal of H containing every proper Γ-hyperideal of H.*

Proof: Let H be an LA-Γ-semihypergroup with pure left identity. By Corollary 3.9, we have every two-sided base of H is one-element base. Since every two-sided base of H is one-element base, by Theorem 3.7, we have that $H \setminus A = M^*$. Therefore $H \setminus A$ is a maximal proper Γ-hyperideal of H containing every proper Γ-hyperideal of H. □

4. Conclusions

In this paper, we study two-sided bases ideals of LA-Γ-semihypergroups and investigate their remarkable properties. We show in Corollary 3.9 that every two-sided base of an LA-Γ-semihypergroup with pure left identity is one-element base. Finally, we prove in Theorem 3.10 that the compliment of union of all two-sided base of an LA-Γ-semihypergroup with pure left identity is maximal proper Γ-hyperideal. This structure of LA-Γ-semihypergroups is generalization of many algebraic structures, for example, commutative semigroups, commutative Γ-semigroups, LA-semigroups, LA-Γ-semigroup, commutative semihypergroups, etc. The results in this paper are true in commutative semigroups, commutative Γ-semigroups, LA-semigroups, LA-Γ-semigroup, commutative semihypergroups, etc.

In the future work, we can study other results in this algebraic structures.

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