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New types of connected, component spaces and maximal open set via s-operation

Sarhad F. Namiq [§]
Mathematics Department
College of Education
University of Garmian
Kurdistan- Region
Iraq

Ennis Rosas [†]
Departamento de Ciencias Naturales y Exactas
Universidad de la Costa
Barranquilla
Colombia

and

Departamento de Matemática
Universidad De Oriente
Nucleo De Sucre
Venezuela

Margot Salas-Brown *
Programa de Matemáticas
Escuela de Ciencias Exactas e Ingeniería
Universidad Sergio Arboleda
Bogotá 11021
Colombia

[§] *E-mail:* sarhad.faiq@garmian.edu.krd or sarhad1983@gmail.com
https://orcid.org/0000-0001-8747-2542
[†] *E-mail:* ennisrafael@gmail.com
https://orcid.org/0000-0001-8123-9344
* *E-mail:* margot.salas@usa.edu.co (Corresponding Author)
https://orcid.org/0000-0001-7202-8916

Abstract

This paper aims to introduce a kind of connected space called λ_{rc}-connected. We analyze the characterizations, properties, and relationships of λ_{rc}-connected spaces with other types of connected spaces including, λ_{rc}-locally connected spaces. Additionally, we define a class of maximal open sets which called maximal λ_{rc}-open sets. Some of its basic properties, characterization and relationships are studied and obtained.

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1. Introduction

The notion of operator associated to a topology τ on a space Z was introduced by Kasahara [5] in 1979, as an application λ from τ to $P(Z)$. Since then, many authors have generalized and extended this notion using weak structures, resulting in a fruitful research area in this field ([1], [2], [3], [13], [14], among others). Following this trend, in 2013 Khalaf and Namiq [6], defined an operator λ on the family of semi-open sets in a topological space, this operator was called semi-operation and was denoted by s-operation. Using semi-operation, Namiq [9], introduced the λ_{rc}-open sets, he also investigated several properties of λ_{rc}-derived, λ_{rc}-interior and λ_{rc}-closure points in topological spaces. In this article, a novel type of connected space, referred to as λ_{rc}-connected space, is presented and examined. We also investigate the relationship between λ-connected spaces and λ_{rc}-connected spaces, and find that the every λ-connected spaces is a λ_{rc}-connected spaces. We discuss some characterizations and properties of λ_{rc}-connected spaces, λ_{rc}-components and λ_{rc}-locally connected spaces in a similar way to S. Willard in [16], for connectedness. Finally, we will introduce and define a new class of maximal open sets which will be called maximal λ_{rc}-open sets, in a similar way as was done in [11] and [12]. In the context of this paper, we will represent a topological space as Z.

2. Preliminaries

To begin, let us review certain definitions and outcomes employed in this context. When considering a subset B of Z, we represent the closure of B as $Cl(B)$ and the interior of B as $Int(B)$. A subset B of a space Z is said to be semi-open [7], if $B \subseteq Cl(Int(B))$. A set F is said to be semi-closed if its complement $Z \setminus F$ is a semi-open set. We will use the notation $SO(Z)$ to represent the set of all semi-open sets in Z. A subset B of a topological space Z is said to be regular closed [15], if $B = Cl(Int(B))$. A function λ from $SO(Z)$ to the power set $P(Z)$ of Z is called a semi-operation (briefly s-operation) if $B \subseteq \lambda(B)$ for every semi-open set B. We made the assumption that $\lambda(Z) = Z$ and $\lambda(\varnothing) = \varnothing$. If for every pair of semi-open sets B_1 and B_2 containing $z \in Z$, there exists a semi-open set B_3 such that $z \in B_3$ and $\lambda(B_3) \subseteq \lambda(B_1) \cap \lambda(B_2)$ then λ is called s-regular [6]. If $A \subset Z$ and for all $z \in A$, there exists a semi-open set B containing z such that $\lambda(B) \subseteq A$, then A will be called λ-open. The family of all λ-open sets of Z will be denoted by $SO_\lambda(Z)$.

As mentioned in the introduction, the λ_{rc}-open sets were introduced in [9], and some properties of them were studied in the same work. We present below this definition and state some properties that will be used to achieve the main results of this work. If W is a λ-open and for every $z \in W$, there exists a regular closed set B containing z such that B is contained in W, then W is called a λ_{rc}-open set, a set X will be referred to as λ_{rc}-closed if $Z \backslash X$ is a λ_{rc}-open set. The collection of all λ_{rc}-open (respectively λ_{rc}-closed) sets in a space Z will be denoted by $SO_{\lambda_{rc}}(Z)$ (respectively $SC_{\lambda_{rc}}(Z)$. Given the context, it is intuitive to introduce the notions of closure and interior in relation to these sets. Thus, in [9], the following operators are defined.

- The intersection of all λ_{rc}-closed sets of Z containing B is the λ_{rc}-closure of B.
- The union of all λ_{rc}-open sets of Z contained in B is the λ_{rc}-interior of B.
- $z \in Z$ is a λ_{rc}-limit point of B if $W \setminus \{z\} \cap B \neq \varnothing$ for all $W \in SO_{\lambda_{rc}}(Z)$ containing z.
- The set $\lambda_{rc}\text{-}Cl(B) \cap \lambda_{rc}\text{-}Cl(Z \setminus B)$ is the λ_{rc}-boundary of B.

The λ_{rc}-closure of B is represented by $\lambda_{rc}\text{-}Cl(B)$, the λ_{rc}-interior of B is represented by $\lambda_{rc}\text{-}Int(B)$, and the set of all λ_{rc}-limit points of B is referred to as the λ_{rc}-derived set of B, denoted $\lambda_{rc}\text{-}D(B)$.

The λ_{rc}-open set are stable under arbitrary unions and under certain conditions are stable under finite intersections. This allowed for the establishment of properties of λ_{rc}-closure and λ_{rc}-interior in [9]. Here are some of them stated below.

Proposition 2.1: [9] Assuming Z is a space, the following propositions are valid:

(1) If $W_\alpha \in SO_{\lambda_{rc}}(Z)$, then $\bigcup_\alpha W_\alpha \in SO_{\lambda_{rc}}(Z)$.

(2) If $W_1, W_2 \in SO_{\lambda_{rc}}(Z)$, then $W_1 \cap W_2 \in SO_{\lambda_{rc}}(Z)$ as long as λ is as s-regular.

(3) $z \in \lambda_{rc}\text{-}Cl(B)$ if and only if $B \cap W \neq \varnothing$, for all $W \in SO_{\lambda_{rc}}(Z)$ containing z.

(4) $\lambda_{rc}\text{-}Cl(B) = B \cup \lambda_{rc}\text{-}D(B)$.

(5) $z \in \lambda_{rc}\text{-}Int(B)$ if there exists $W \in SO_{\lambda_{rc}}(Z)$ such that $z \in W \subseteq B$.

(6) If $B_1 \subset B_2$, then $\lambda_{rc}\text{-}Cl(B_1) \subseteq \lambda_{rc}\text{-}Cl(B_2)$.

(7) If $B_1 \subset B_2$ then $\lambda_{rc}\text{-}Int(B_1) \subseteq \lambda_{rc}\text{-}Int(B_2)$.

(8) B is λ_{rc}-c1osed, if $B = \lambda_{rc}\text{-}Cl(B)$.

(9) B is λ_{rc}-open, if $B = \lambda_{rc}\text{-}Int(B)$.

(10) $Z \setminus \lambda_{rc}\text{-}Int(B) = \lambda_{rc}\text{-}Cl(Z \setminus B)$.

(11) $Z \setminus \lambda_{rc}\text{-}Cl(B) = \lambda_{rc}\text{-}Int(Z \setminus B)$

The following example demonstrates the importance of having an s-regular s-operation to guarantee that the intersection of two λ_{rc}-open sets remains a λ_{rc}-open set.

Example 2.2: Consider $Z = \{z_1, z_2, z_3\}$ with the discrete topology, and define λ an s-operation from $SO(Z)$ to $P(Z)$ as follow:

$$\lambda(B) = \begin{cases} B & B \notin \{\{z_1\}, \{z_2\}\} \\ Z & \text{otherwise} \end{cases}$$

Note that $\{z_1, z_2\}, \{z_2, z_3\} \in SO_{\lambda_{rc}}(Z)$, but $\{z_1, z_2\} \cap \{z_2, z_3\} \notin SO_{\lambda_{rc}}(Z)$.

Also, in [10], notions of connected spaces in terms of λ-open sets were established. That is, Z is considered λ-connected if there is no pair of non-empty and disjoint λ-open subsets W and V of Z such that $Z = W \cup V$. Conversely, if such a pair (W, V) exists, Z is referred to as λ-disconnected, and (W, V) will be called a λ-disconnection of Z.

3. λ_{rc}-Connected Spaces

Within this section, our focus lies on a new notion of connected spaces in terms λ_{rc}-open sets. We will define λ_{rc}-connected spaces and delve into the study of its properties and some characterizations.

Definition 3.1: A pair (W, V) is called a λ_{rc}-disconnection of Z if W, V are nonempty; $W, V \in SO_{\lambda_{rc}}(Z)$, are disjoint and satisfy $W \cup V = Z$.

Definition 3.2: A space Z is considered λ_{rc}-connected if there doesn't exist a λ_{rc}-disconnection; otherwise, Z is called λ_{rc}-disconnected.

It is very simple to verify that if $SO_{\lambda_{rc}}(Z) = \{Z, \varnothing\}$, then Z is λ_{rc}-connected. Now, we provide a characterization of this new type of connected spaces, and the proof of this characterization is straightforward.

Theorem 3.3: *Let Z be a space, the following assertions hold true:*

(1) *Z is λ_{rc}-disconnected space if and only if there exists a subset $B \neq \varnothing$ of Z, such that $B \in SO_{\lambda_{rc}}(Z) \cap SC_{\lambda_{rc}}(Z)$.*
(2) *Z is λ_{rc}-connected space if and only if doesn't exist a nonempty proper subset B of Z, such that $B \in SO_{\lambda_{rc}}(Z) \cap SC_{\lambda_{rc}}(Z)$.*

The following result establishes the relationship between λ-connected spaces and λ_{rc}-connected spaces.

Theorem 3.4: *If Z isn't a λ_{rc}-connected space, then Z isn't a λ-connected space.*

Proof: Since every λ_{rc}-open set is also a λ-open set, if Z is a λ-connected space, there cannot exist a pair of nonempty disjoint λ-open subsets W and V of Z such that $Z = W \cup V$. Therefore, there is no λ_{rc}-disconnection (W, V), which implies that Z is λ_{rc}-connected.

In general, it is not true that the converse of Theorem 3.4 holds, as illustrated by considering the following example:

Example 3.5: Let us consider $Z = \{z_1, z_2, z_3\}$ with the topology defined as follows $\tau = \{\{z_1\}, \varnothing, \{z_2\}, Z, \{z_2, z_1\}, \{z_3, z_1\}\}$. Let λ be an s-operation from $SO(Z)$ to $P(Z)$ defined by:

$$\lambda(B) = \begin{cases} B & B = \{z_1\} \\ Z & \text{otherwise} \end{cases}$$

It is very simple to verify that $SO(Z) = \{\{z_2\}, Z, \{z_1\}, \varnothing, \{z_2, z_1\}, \{z_3, z_1\}\}$. $SO_\lambda(Z) = \{\{z_1\}, Z, \varnothing\}$ and therefore $SO_{\lambda_{rc}}(Z) = \{Z, \varnothing\}$. Hence, Z is λ_{rc}-connected, but it isn't λ-connected.

The following theorem establishes a necessary and sufficient condition for a space Z to be λ_{rc}-connected: Each nonempty proper subspace of Z must have a nonempty λ_{rc}-boundary.

Theorem 3.6: *If Z is λ_{rc} -connected, then each nonempty proper subspace has a nonempty λ_{rc} -boundary. The converse is also true.*

Proof: Suppose that there exists a proper subset $Y \neq \varnothing$ of Z such that Y has empty λ_{rc} -boundary, that is $\lambda_{rc}\text{-}Cl(Y) \cap \lambda_{rc}\text{-}Cl(Z-Y) = \varnothing$, then:

$$\lambda_{rc} - Cl(Y) \subset Z - \lambda_{rc} - Cl(Z-Y) = \lambda_{rc} - Int(Y)$$

Thus Y is a nonempty proper subset of Z such that $Y \in SO_{\lambda_{rc}}$ $(Z) \cap$ $SC_{\lambda_{rc}}$ (Z). By Theorem 3.3, Z is λ_{rc} -disconnected. Now, suppose that Z is λ_{rc} -disconnected. According to Theorem 3.3, Z has a proper subspace Y such that $Y \in SO_{\lambda_{rc}}$ (Z) $\cap SC_{\lambda_{rc}}$ (Z). Consequently, we have $\lambda_{rc}\text{-}Cl(Y) = Y$, $\lambda_{rc}\text{-}Cl(Z-Y)$ = $(Z-Y)$ and $\lambda_{rc}\text{-}Cl(Y) \cap \lambda_{rc}\text{-}Cl(Z-Y)$ = $\varnothing$. However, this implies that Y has an empty λ_{rc} -boundary, which contradicts our assumption. Therefore, we can conclude that Z must be λ_{rc} -connected.

Next, we will work with λ_{rc} -connected subspaces, for which the following definition is necessary.

Definition 3.7: Consider a space Z and a subset $Y \subseteq Z$. We naturally define the class of λ_{rc} -open sets in Y, denoted as $SO_{\lambda_{rc}}(Y)$, in the following manner:

$$SO_{\lambda_{rc}}(Y) = \{W \cap Y : W \in SO_{\lambda_{rc}}(Z)\}.$$

This implies that V is considered λ_{rc} -open in Y if and only if it is the intersection of Y with a λ_{rc} -open set W in Z. Consequently, Y can be regarded as a subspace of Z in terms of λ_{rc} -open sets.

Theorem 3.8: *Let (W, V) be a λ_{rc} -disconnection of Z. If Y is a λ_{rc} -connected subspace of Z, then $Y \subset W$ or $Y \subset V$.*

Proof: Under the given conditions, let's consider the case where Y is neither contained within V nor within W. Under these circumstances, we observe that $V \cap Y \neq \varnothing$, $W \cap Y \neq \varnothing$ and $V \cap Y$, $W \cap Y$ $\in$ $SO_{\lambda_{rc}}$ (Y). Furthermore, we observe that $(Y \cap V)$ $\cap$ $(Y \cap W) = \varnothing$ and $(Y \cap V)$ $\cup$ $(Y \cap W) = Y$. This implies that $(Y \cap V,\ Y \cap W)$ forms a λ_{rc} -disconnection of Y. By reaching this contradiction, we have established the proof of the theorem.

The subsequent theorem establishes that the union of non disjoint λ_{rc} -connected spaces is again a λ_{rc} -connected space.

Theorem 3.9: *If $Z = \bigcup_{\alpha \in I} Z_\alpha$, where every Z_α is λ_{rc} -connected and $\bigcap_{\alpha \in I} Z_\alpha \neq \varnothing$, then Z is a λ_{rc} -connected space.*

Proof: If Z isn't λ_{rc}-connected, then there exists a λ_{rc}-disconnection (W,V) of Z. According to Theorem 3.8, either $Z_\alpha \subseteq W$ or $Z_\alpha \subseteq V$. Since $\bigcap_{\lambda\in I} Z_\alpha \neq \varnothing$, it follows that Z is either a subset of W or a subset of V. If $Z \subseteq W$, then $V = \varnothing$ and if $Z \subseteq V$, then $W = \varnothing$. In either case, we obtain a contradiction.

By employing Theorem 3.9, we provide a necessary and sufficient condition for a space to be λ_{rc}-connected.

Theorem 3.10: *A space Z is λ_{rc}-connected if and only if for any pair of points $y,z \in Z$, there exists a λ_{rc}-connected subset Y of Z, such that $y,z \in Y$.*

Proof: The necessity is evident as the λ_{rc}-connected space inherently includes these two points. Now, suppose that for every pair of points y and z in Z, there exists a λ_{rc}-connected subspace $Y_{(y,z)}$ of Z that contains both y and z. Let b be a fixed point in Z. Consider the class of all λ_{rc}-connected subsets of Z that contain b for all y in Z, denoted by $\{Y_{(b,y)},\ y \in Z\}$. Note that b belongs to the union of all $Y_{(b,y)}$ for y in Z. By Theorem 3.9, the union of all $YC_{(b,y)}$ is λ_{rc}-connected and equal to Z.

If a subspace is λ_{rc}-connected, then adding any λ_{rc}-limit point to it will still result in an λ_{rc}-connected subspace.

Theorem 3.11: *If Y is a λ_{rc}-connected subset of a space Z and $Y \subseteq A \subseteq \lambda_{rc}$-$Cl(Y)$, then A is λ_{rc}-connected.*

Proof: It is enough to demonstrate that $\lambda_{rc}\text{-}Cl(Y)$ is λ_{rc}-connected. To that end, let's assume the contrary, meaning that $\lambda_{rc}\text{-}Cl(Y)$ is λ_{rc}-disconnected. This implies the existence of a λ_{rc}-disconnection (W,V) of $\lambda_{rc}\text{-}Cl(Y)$. As a consequence, we have $W \cap Y \in SO_{\lambda_{rc}}(Y)$, $V \cap Y \in SO_{\lambda_{rc}}(Y)$, $(W \cap Y) \cap (V \cap Y) = (W \cap V) \cap Y = \varnothing$, and $(W \cap Y) \cup (V \cap Y) = (W \cup V) \cap Y = Y$. Thus, we can conclude that $(W \cap Y, V \cap Y)$ is a λ_{rc}-disconnection of Y, which contradicts our assumption. Consequently, $\lambda_{rc}\text{-}Cl(Y)$ is λ_{rc}-connected.

4. λ_{rc}-components and λ_{rc}-locally connected spaces

This section focuses on introducing and examining new notions of components and sets with local connectivity using λ_{rc}-open sets.

Definition 4.1: A subset C of Z is called a λ_{rc}-component of Z if it is a largest λ_{rc}-connected subset of Z.

When Z is λ_{rc}-connected, it is the unique λ_{rc}-component of itself. Now, we proceed to examine the properties associated with λ_{rc}-components in the space Z.

Theorem 4.2: *Let Z be a space, the following properties are satisfied:*

(1) *For every $z \in Z$, there exists only one λ_{rc}-component C of Z such that $z \in C$.*

(2) *If Y is a λ_{rc}-connected subset of Z, then there exists only one λ_{rc}-component C of Z such that $Y \subset C$.*

(3) *If λ is s-regular and Y is a λ_{rc}-connected set which is both λ_{rc}-closed and λ_{rc}-open, then Y is a λ_{rc}-component.*

(4) *If C is a λ_{rc}-component of Z, then $C \in SC_{\lambda_{rc}}(Z)$.*

Proof:

(1) Take $z \in Z$ and consider the collection $\mathcal{A} = \{C_\alpha : \alpha \in I\}$ where each C_α is a λ_{rc}-connected subsets of Z containing z. Then, by Theorem 3.9, $C = \cup_{\alpha \in I} C_\alpha$ is a λ_{rc}-connected and $z \in C$. Suppose that there exists C^* a λ_{rc}-connected subset of Z such that $C \subseteq \hat{C}$, then $z \in \hat{C}$ and hence $\hat{C} \in \mathcal{A}$, hence $\hat{C} \subseteq C$. Consequently $C = \hat{C}$ and this proves that C is the only one λ_{rc}-component of Z, such that $z \in C$.

(2) Let Y be a λ_{rc}-connected subset of Z and suppose that Y isn't a λ_{rc}-component of Z, then there exists C a λ_{rc}-component of Z such that $Y \subseteq C$. If $\hat{C}$ is other λ_{rc}-component of Z such that $Y \subseteq \hat{C}$, then $C \cup \hat{C}$ is another λ_{rc}-connected set that contains at C as well as $\hat{C}$, this contradicts the assumption that both C and $\hat{C}$ are λ_{rc}-components. Therefore, it can be concluded that Y is contained in only one λ_{rc}-component of Z.

(3) Suppose that λ is s-regular and Y is a λ_{rc}-connected subset of Z that is both λ_{rc}-closed and λ_{rc}-open. According to the previous item, Y is contained in just one λ_{rc}-component C of Z. If Y is a proper subset of C, then $C \cap Y$ and $C \cap (Z - Y)$ are λ_{rc}-open subsets of C. Since $C = (Y \cap C) \cup (C \cap (Z - Y))$ we have a λ_{rc}-disconnection $((C \cap Y), (C \cap (Z - Y))$ of C, which isn't possible. Therefore, Y must be equal to C.

(4) Let C be a λ_{rc}-component C of Z, according to Theorem 3.11, $\lambda_{rc}\text{-}Cl(C)$ is λ_{rc}-connected and $C \subset \lambda_{rc}\text{-}Cl(C)$. Since C is a λ_{rc}-component we obtain $C = \lambda_{rc}\text{-}Cl(C)$.

Connected spaces and locally connected spaces are related concepts in topology, but they capture different aspects of the topological structure of a space. Locally connected spaces also can be defined by λ_{rc}-open set.

Definition 4.3: A space Z will be called locally λ_{rc}-connected if for any point $z \in Z$ and any $W \in SO_{\lambda_{rc}}(Z)$ such that $z \in W$, there exists a λ_{rc}-connected $V \in SO_{\lambda_{rc}}(Z)$ containing z and such that $V \subseteq W$.

Next theorem states that λ_{rc}-local connectedness is inherited by λ_{rc}-open subsets.

Theorem 4.4: *If $Y \in SO_{\lambda_{rc}}(Z)$ and Z is a λ_{rc}-locally connected space, then Y is λ_{rc}-locally connected.*

Proof: Let Z be λ_{rc}-locally connected space and $Y \in SO_{\lambda_{rc}}(Z)$. Let $z \in Y$ and $V \in SO_{\lambda_{rc}}(Y)$ such that $z \in V \subset Y$. Then $V \in SO_{\lambda_{rc}}(Z)$ and due to Z is λ_{rc}-locally connected, there exists $W \in SO_{\lambda_{rc}}(Z)$ a λ_{rc}-connected in Z such that $z \in W \subseteq V$. So, $W \in SO_{\lambda_{rc}}(Y)$ is a λ_{rc}-connected and $z \in W \subseteq V \subseteq Y$.

5. About $MSO_{l_{rc}}$ Sets

In the subsequent section, we present and delve into the notion of maximal λ_{rc}-open sets, exploring several of their properties.

Definition 5.1: If $B \in SO_{\lambda_{rc}}$, we refer to B as a maximal λrc-open set if any λ_{rc}-open set that contains B is either B itself or the entire space Z.

The family of all maximal λrc-open sets of Z will be denoted by $MSO_{\lambda rc}(Z)$. It can be readily noticed that, every maximal λ_{rc}-open set is λ_{rc}-open and hence its semi-open. But not conversely, this can be illustrated through the following example. Also, through the subsequent example, we demonstrate the independence of the notions of maximal open sets and maximal λ_{rc}-open sets.

Example 5.2: Consider $Z = \{a, b, c\}$ with the discrete topology τ_d.

(1) Let λ be an s-operation from $SO(Z)$ to $P(Z)$ defined by:

$$\lambda(B) = \begin{cases} B & B \in \{\{b,a\},\{a\}\} \\ Z & \text{otherwise} \end{cases}$$

We obtain that $\{a\} \in SO_{\lambda_{rc}}(Z)$ but $\{a\} \notin MSO_{\lambda_{rc}}(Z)$.

(2) Let λ be an s-operation from $SO(Z)$ to $P(Z)$ defined by:

$$\lambda(B)=\begin{cases} B & B\in\{\{a\},\varnothing\} \\ Z & \text{otherwise} \end{cases}$$

Then the set $\{a\}\in MSO_{\lambda_{rc}}(Z)$ but $\{a\}\notin SO_{\lambda_{rc}}(Z)$. However, the set $\{b,a\}$ is a maximal open set but $\{b,a\}\notin MSO_{\lambda_{rc}}(Z)$.

Proposition 5.3: Let $B\in MSO_{\lambda_{rc}}(Z)$, the following assertions are valid:

(1) If $A\in SO_{\lambda_{rc}}(Z)$, then either $A\subseteq B$ or $A\cup B=Z$.

(2) If $A\in MSO_{\lambda_{rc}}(Z)$, then either $A\cup B=Z$ or $A=B$.

Proof:

(1) As $A,B\in SO_{\lambda_{rc}}(Z)$, according to Proposition 2.1, their union $A\cup B\in SO_{\lambda_{rc}}(Z)$ and $A\subseteq A\cup B$. Therefore, based on Definition 5.1, we can conclude that either $A\cup B=A$ or $A\cup B=Z$. In other words, it is either $A\subseteq B$ or $B\cup A=Z$.

(2) It follows from item 1.

Proposition 5.4: Let $A,B,C\in MSO_{\lambda_{rc}}(Z)$ such that $A\neq B$. If $A\cup B\subseteq C$, then either $C=A$ or $C=B$.

Proof: Since A and C are maximal λ_{rc}-open, then by item 2 of the Proposition 5.3 either $A\cup C=Z$ or $A=C$. If $C\cup A=Z$, then $B=Z\cap B$ $=(C\cup A)\cap B=(C\cap B)\cup(A\cap B)\subseteq C$. Since B is a maximal λ_{rc}-open set, then by the item 2 of the Proposition 5.3, either $B=C$ or $Z=C$. Due to $C\neq Z$ we obtain $B=C$. Consequently, we can deduce that either C is equal to A or C is equal to B.

Proposition 5.5: If $A\in MSO_{\lambda_{rc}}(Z)$ and $z\in A$, then

$$A=\cup\{B:z\in B, B\in SO_{\lambda_{rc}}(Z) \text{ and } A\cup B\neq Z\}.$$

Proof: Let $C=\cup\ \{B:z\in B,\ B\in SO_{\lambda_{rc}}(Z)$ and $A\cup B\neq Z\}$. Since $A\in SO_{\lambda_{rc}}(Z)$ and $z\in A\cup A=A\neq Z$, then we have $A\subseteq C$. Conversely, for each $x\in C$ there exists a λ_{rc}-open set B such that $x\in B$ and $A\cup B\neq Z$. By item 1 of Proposition 5.3, we have $B\subseteq A$, so $x\in A$ and therefore $C\subseteq A$. Hence, $C=A$.

Proposition 5.6: If $B\in MSO_{\lambda_{rc}}(Z)$ and $b\notin B$, then $B\cup W=Z$ for each $W\in SO_{\lambda_{rc}}(Z)$ such that $b\in W$.

Proof: Let $W \in SO_{\lambda_{rc}}(Z)$ such that $b \in W$. Due to B is a maximal λ_{rc}-open, then by item 1 of the Proposition 5.3, either $B \cup W = Z$ or $W \subseteq B$. But since $W \nsubseteq B$, so $B \cup W = Z$.

Corollary 5.7: *If $B \in MSO_{\lambda_{rc}}(Z)$ and $b \notin B$, then:*

- $Z \setminus B \subseteq W$, *for each* $W \in SO_{\lambda_{rc}}(Z)$ *such that* $b \in W$.
- $Z \setminus W \subseteq B$, *for each* $W \in SO_{\lambda_{rc}}(Z)$ *such that* $b \in W$.
- $F \subseteq B$, *for each* $F \in SC_{\lambda_{rc}}(Z)$ *such that* $b \in Z - F$.

Proof: It follows from Proposition 5.6.

Proposition 5.8: Let $A \in SO_{\lambda_{rc}}(Z)$ be a non-empty proper subset of Z such that each point of $Z \setminus A$ is contained in a finite λ_{rc}-closed set, then there exists $B \in MSO_{\lambda_{rc}}(Z)$ such that $A \subseteq B$ and $Z \setminus B$ is finite.

Proof: Since A is a non-empty proper subset of Z, then for each $x \in Z \setminus A$ there exists a finite set $F \in SC_{\lambda_{rc}}(Z)$ such that $x \in F$. Thus, $A \cup (Z \setminus F) \in SO_{\lambda_{rc}}(Z)$ and $A \subset A \cup (Z \setminus F)$. If A is a maximal λ_{rc}-open set, then by the item 1 of the Proposition 5.3 either $A \cup (Z \setminus F) = Z$ or $Z \setminus F \subseteq A$. Due to there exists $x \in (Z \setminus A) \cap F$, then $Z \setminus F \subseteq A$. Hence, $Z \setminus A$ is finite and $B = A$ is the required set. If $A \cup (Z \setminus F)$ is maximal λ_{rc}-open, then $B = (Z \setminus F) \cup A$ is the required set. If not, there exists a λ_{rc}-open set A_0, such that $A \cup (Z \setminus F) \subsetneqq A_0 \subsetneqq Z$. If A_0 is maximal λ_{rc}-open set, then the set $B = A_0$ is the required set (since $Z \setminus B = Z \setminus A_0 \subseteq F$ which is finite). If A_0, is not a maximal λ_{rc}-open, then there exists a λ_{rc}-open set A_1 such that $A \cup (Z \setminus F) \subsetneqq A_0 \subsetneqq A_1 \subsetneqq Z$.

By following this process, a sequence of λ_{rc}-open sets A_m, where $m \in N$, is obtained and

$$A \cup (Z \setminus F) \subsetneqq A_0 \subsetneqq A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_m \subsetneqq \cdots \subsetneqq Z.$$

Due to $(Z \setminus A) \cap F \subseteq F$ and F is a finite set, this sequence can not be infinite. That is, there exists $n \in N$ such that $A_m = A_n$, if $m \geq n$. So the set $B = A_n$ is the required set.

Corollary 5.9: *Let $A \in SO_{\lambda_{rc}}(Z)$ be a non-empty proper co-finite subset of Z. Then there exists a co-finite $B \in MSO_{\lambda_{rc}}(Z)$ contains A.*

Proof: As a consequence of Proposition 5.8.

The existence of a unique maximal λ_{rc}-open set, as indicated in Proposition 5.8 and Corollary 5.9, is not assured, as exemplified by the following example.

Example 5.10: Let (Z, τ_d) be a space of Example 5.2 and let λ be an s-operation from $SO(Z)$ to $P(Z)$ given by

$$\lambda(B) = \begin{cases} B & B \in \{\{a,c\},\{a\},\varnothing,\{a,b\}\} \\ Z & \text{otherwise} \end{cases}$$

Then $\{c,a\}$ and $\{b,a\}$ are maximal λ_{rc}-open sets and both containing the co-finite set $\{a\}$.

Proposition 5.11: If $B \in MSO_{\lambda_{rc}}(Z)$, then either $B = \lambda_{rc}\text{-}Cl(B)$ or $Z = \lambda_{rc}\text{-}Cl(B)$.

Proof: If $B = \lambda_{rc}\text{-}Cl(B)$, then there is nothing to prove. Suppose that $B \subsetneq \lambda_{rc}\text{-}Cl(B)$ and suppose that there exists $z \in Z \setminus \lambda_{rc}\text{-}Cl(B)$, so we can choose $W \in SO_{\lambda_{rc}}(Z)$ such that $z \in W$ and $W \cap B = \varnothing$. Due to B is a maximal λ_{rc}-open set, then by the item 1 of the Proposition 5.3 either $B \cup W = Z$ or $W \subseteq B$, since $B \cap W = \varnothing$ and $W \neq \varnothing$, we have $B \cup W = Z$. This implies that $B = Z \setminus W$. By Theorem 2.1, $\lambda_{rc}\text{-}Cl(B) = \lambda_{rc}\text{-}Cl(Z \setminus W) = Z \setminus W = B$ which is impossible. Hence $\lambda_{rc} - Cl(B) = Z$.

Proposition 5.12: Let $B \in MSO_{\lambda_{rc}}(Z)$ and let D be any non-empty subset of Z such that $D \cap B = \varnothing$, then $\lambda_{rc}\text{-}Cl(D) = Z \setminus B$.

Proof: Due to $Z \setminus B$ is λ_{rc}-closed and $D \subseteq Z \setminus B$, then by Theorem 2.1, we have $\lambda_{rc}\text{-}Cl(D) \subseteq Z \setminus B$. Now, let $d \notin \lambda_{rc}\text{-}Cl(D)$, then there exists $W \in SO_{\lambda_{rc}}(Z)$ such that $d \in W$ and $W \cap D = \varnothing$. Since B is a maximal λ_{rc}-open set, then by item 1 of the Proposition 5.3, either $B \cup W = Z$ or $W \subseteq B$. If $B \cup W = Z$, then $D = D \cap Z = D \cap (B \cup W) = (D \cap B) \cup (D \cap W) = \varnothing$, which is a contradiction. So, we have $W \subseteq B$ and this implies that $d \in B$. Therefore, $d \notin Z \setminus B$. Thus $\lambda_{rc}\text{-}Cl(D) = Z \setminus B$.

Proposition 5.13: If $B \in MSO_{\lambda_{rc}}(Z)$, then $B = \lambda_{rc}\text{-}Int(A)$, for each proper set A such that $B \subseteq A$.

Proof: Due to $B \subseteq A$, then according Theorem 2.1, $B = \lambda_{rc}\text{-}Int(B) \subseteq \lambda_{rc}\text{-}Int(A)$. Since A is a proper subset of Z and $\lambda_{rc}\text{-}Int(A) \subseteq A$, then λ_{rc}-

$Int(A)$ is a proper λ_{rc} -open set contains B. Since B is maximal λ_{rc} -open, then λ_{rc} - $Int(A) = B$.

Proposition 5.14: Let $B, B_\alpha \in MSO_{\lambda_{rc}}(Z)$ for each $\alpha \in \Gamma$. If $\bigcap_{\alpha\in\Gamma} B_\alpha$ is contained in B, then there exists $\alpha_1 \in \Gamma$ such that $B \subseteq B_{\alpha_1}$.

Proof: If $B \neq B_\alpha$ for each $\alpha \in \Gamma$, then by item 2 of the Proposition 5.3, $B \cup B_\alpha = Z$ for each $\alpha \in \Gamma$. That is $\bigcap_{\alpha\in\Gamma} (B \cap B_\alpha) = Z$, so that $B \cap (\bigcap_{\alpha\in\Gamma} B_\alpha)$ $= Z$. It follows that $B = Z$ which isn't possible. Hence, $B = B_{\alpha_1}$, for some $\alpha_1 \in \Gamma$.

Corollary 5.15: *Let* $\{B_\alpha\}_{\alpha\in\Gamma}$ *be a collection of maximal* λ_{rc} *-open sets of Z. If* $A \in MSO_{\lambda_{rc}}(Z)$ *and* $A \neq B_\alpha$, *for each index* $\alpha \in \Gamma$, *then* $Z \setminus \bigcap_{\alpha\in\Gamma} B_\alpha \subseteq A$ *and the sets A and* $\bigcap_{\alpha\in\Gamma} B_\alpha$ *are disjoint.*

Proof: Both followed from Proposition 5.14.

Proposition 5.16: Let $\{B_\alpha\}_{\alpha\in\Gamma}$ be a collection of maximal λ_{rc} -open sets of Z and suppose that Γ has more than one element. If $B_\alpha \neq B_\beta$ for each index $\alpha \neq \beta$ in G, then $B_\sigma = (\bigcap_{\alpha\in\Gamma} B_\alpha) \cup (Z \setminus \bigcap_{\alpha\in\Gamma\setminus\{\beta\}} B_\alpha)$, for all $\beta \in \Gamma$.

Proof: Let $\beta \in \Gamma$, , then by Corollary 5.15, $Z \setminus \bigcap_{\alpha\in\Gamma\setminus\{\beta\}} B_\alpha \subseteq B_\beta$, , so

$$\begin{aligned} B_\beta = B_\beta \cap Z &= B_\beta \cap \left(\left(\bigcap_{\alpha\in G\setminus\{\beta\}} B_\alpha \right) \cup \left(Z \setminus \bigcap_{\alpha\in G\setminus\{\beta\}} B_\alpha \right) \right) \\ &= \left(B_\beta \cap \left(\bigcap_{\alpha\in G\setminus\{\beta\}} B_\alpha \right) \right) \cup \left(B_\beta \cap \left(Z \setminus \bigcap_{\alpha\in G\setminus\{\beta\}} B_\alpha \right) \right) \\ &= \bigcap_{\alpha\in G} B_\alpha \cup \left(Z \setminus \bigcap_{\alpha\in G\setminus\{\beta\}} B_\alpha \right). \end{aligned}$$

Corollary 5.17: *If* $\{B_\alpha\}_{\alpha\in\Gamma}$ *is a collection of maximal* λ_{rc} *-open subset of Z such that* Γ *has more than two elements and* $B_\alpha \neq B_\beta$ *for each* $\alpha \neq \beta$ *in* Γ, *then* $B_\alpha \cap B_\beta \neq \emptyset$ *for each* $\alpha, \beta \in \Gamma$.

Proof: It follows from Proposition 5.16.

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