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Eight and ninth-order convergence iterative structures for obtaining nonlinear equations solution

O. Ogbereyivwe *
S. A. Ogumeyo †
Department of Mathematics
Delta State University of Science and Technology
Ozoro
Nigeria

E. J. Atajeromavwo §
Department of Software Engineering
Delta State University of Science and Technology
Ozoro
Nigeria

Abstract

An eighth and ninth-order fast convergence iterative structures for determining the solution of nonlinear equations is put forward in this manuscript. The iterative structures are modification of a three-step variants of the Newton method via the use of the divided deference and weight functions. The computational iterative structures possess the advantages that they, do not require evaluation of higher derivative and converge faster than compared iterative structures with same convergence order. The convergence analysis of the iterative structures was established via the method of Taylor series. The computational results obtained with the developed iterative structures are juxtaposed with those obtained from some contemporary existing methods, and they performed better in terms of fast convergence.

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* *E-mail:* **oghovese.ogbereyivwe@dspz.edu.ng (Corresponding Author)**
† *E-mail:* **simonogumeyo64@gmail.com**
§ *E-mail:* **edafejohn2006@yahoo.com**

1. Introduction

One common problem encountered in the field of scientific computing, engineering and applied mathematics is in obtaining solution of nonlinear equation (NLE) of the form $f(s)=0$. This is because, researchers in diverse fields, often modeled observed real life occurrences into NLE in order to properly investigates their behaviors. One can find examples of modeled real situations into NLE in the literature ,,,, and some reference therein. Unfortunately, only in rare cases exact solution methods exist in determining NLE solution, hence approximation techniques that are iterative in structure are resorted. The well known classical iterative structure (IS) for computing solution of NLE is the Newton's iterative structure (NIS) given as:

$$s_{j+1} = s_j - \Gamma; \tag{1.1}$$

where $\Gamma = \frac{f(s_j)}{f'(s_j)}$. It's convergence order (CO) is two. The Ostrowski index as defined in [2], states that the efficiency index (EI) of an IS is given by $\rho^{\frac{1}{T}}$, where ρ is the CO and T is the sum of all distinct functions evaluation in one complete iteration cycle. The NM requires evaluation of two distinct functions in one iteration cycle, consequently, it's EI is approximately 1.4142.

Several attempts have been made by researchers to modify the NIS with the aim of improving its CO and EI. Among these attempts, we are particularly interested in the concept put forward by Alfio et al. [10] with IS as

$$y_j = s_j - \Gamma, \qquad s_{j+1} = y_j - \frac{f(y_j)}{f'(s_j)}. \tag{1.2}$$

The IS in (1.2) can be regarded as an improvement of the IS in (1.1) because, it is of CO of 3 and $EI = 1.4417$ which are higher than that of the NIS. Further adoption of the concept used in (1.2) by extending it to three step IS yields

$$y_j = s_j - \Gamma, \qquad z_j = y_j - \frac{f(y_j)}{f'(s_j)}, \qquad s_{j+1} = z_j - \frac{f(z_j)}{f'(s_j)}. \tag{1.3}$$

Although the IS in (1.3) has CO 4, its major setback is that the EI remain as that of the NIS. Many researchers have identified this set back and applied techniques to improve not just its CO, but EI also. The weight function strategy have been utilized by many authors in improving the CO and EI of the IS in (1.3) as in the papers [6], [7], [11], [9], [8]. Whereas in the work [5], [1], some IS were put forward by combining the weight functions and divided difference techniques.

In this manuscript, the CO and EI of the three step IS in (1.3) is improved via the combination of the divided difference and weight function strategies. Consequently, two IS for obtaining solution of NLE with CO eight and nine are put forward. The rest structure of this manuscript is described next. In Section 2, the new IS formulation is presented, Section 3 considers the convergence analysis of the the proposed IS, while in Section 4, the numerical implementation of the the proposed IS to solving some NLE problems and results obtained were compared with the ones obtained from some existing IS of same CO. The last section of this manuscript was dedicated to conclusion remarks.

2. The Iterative Structure Development

We modify the IS in (1.3) by estimating the function derivative $f'(s)$ using the backward difference operator as:

$$f'(s) \approx \frac{f(s) - f\left(s - \frac{f(s)}{f'(s)}\right)}{\frac{f(s)}{f'(s)}}, \tag{2.4}$$

and the introduction of two real value weight functions $G(\eta)$ and $H(\mu)$ (where $\eta = \frac{f(y)}{f(s)}$ and $\mu = \frac{f(z)}{f(y)}$), , to the third step of the IS. Consequently, a new estimation for Γ and denoted as Ω is obtained next as:

$$\Omega \approx \frac{f(s_j)^2}{f'(s_j)\left[f(s_j) - f\left(s_j - \frac{f(s)}{f'(s)}\right)\right]}, \tag{2.5}$$

On the basis of the result in (2.2), a modification to the IS in (1.3) is made and presented as:

$$y_j = s_j - \Omega, \tag{2.6}$$

$$z_j = y_j - \Omega \frac{f(y_j)}{f(s_j)}, \tag{2.7}$$

$$s_{j+1} = z_j - \Omega \frac{f(z_j)}{f(s_j)} \{H(\eta) \times G(\mu)\}. \tag{2.8}$$

Our interest here is to impose some suitable conditions on the weight functions $H(\eta)$ and $G(\mu)$ so as to enable the IS in (2.3) to determine the solution of NLE with a minimum CO of eight. When this is achieved, the

IS will have a minimum $EI \approx 1.5157$, which is better than the NM and the IS in (1.3) from which it was developed.

3. The IS Convergence Analysis

We begin this section by establishing the convergence of sequence $\{s_j\}_{j\geq 0}$ of approximations produced by the IS put forward in (2.3).

Theorem 3.1: *Let $f: I \subset R \to R$ be a differentiable function with a simple solution θ such that $f'(\cdot) \neq 0$. For a choice of an initial guess s_0 close to $\theta \in I$, the sequence of approximations $\{s_j\}_{j\geq 0}, (s_j \in I)$, generated by the IS in (2.3) will converges to θ with order eight so long the real value weight functions $G(\eta)$ and $H(\mu)$ satisfies the conditions $H(0)=1$, $H^0 = 0$, $G(0)=G'(0)=1$, $G''(0)=2$, $G'''(0)=12$ and $G^{iv}(0)<\infty$.*

Proof: Let $D_{j+1} = s_{j+1} - \theta$ and $c_m = \frac{1}{m!}\frac{f^{(m)}(\theta)}{f'(\theta)}$, $m \geq 2$. Consider replacing s with s_j in the Taylor expansion of $f(s)$ and $f'(s)$, then

$$f(s_j) = f'(\theta)\left[D_j + \sum_{m=2}^{5} c_m D_j^m + O(D_j^6)\right], \tag{3.9}$$

and

$$f'(s_j) = f'(\theta)\left[1 + \sum_{m=2}^{5} m c_m D_j^{m-1} + O(D_j^6),\right] j = 0,1,2,\cdots \tag{3.10}$$

By the use of the expressions in (3.9) and (3.10), the expansion for $f\left(s_j - \frac{f(s_j)}{f'(s_j)}\right)$ is obtained as:

$$f\left(s_j - \frac{f(s_j)}{f'(s_j)}\right) =$$

$$f'(\theta)\left[\begin{array}{l} c_2 D_j^2 + 2(c_3 - c_2^2)D_j^3 + (7c_2c_3 - 9c_2^3)D_j^4 \\ -2(5c_2c_4 + 3c_3^2 + 6c_2^4 - 2c_5 - 12c_2^2)D_j^5 \\ +(28c_2^5 + 37c_2c_3^2 + 34c_2^2c_4 - 73c_2^3 - 17c_3c_4 - 13c_2c_5 + 5c_6)D_j^6 \\ +2(32c_2^6 - 103c_8^4c_3 - 9c_3^3 + 52c_2^3c_4 \ldots + (8c_6 - 52c_3c_4 - 3c_7)D_j^7 \\ +(144c_2^7 - 552c_2^5c_3 + 297c_4^2c_4 + 75c_2^3c_4 + \ldots) \\ \qquad\qquad + c_2(73c_4^2 + 134c_2c_5 - 147c_3^3 - 19c_7) + 7c_8)D_j^8 \\ -2(160c_2^8 - 712c_2^6c_3 + \ldots 2c_2^4(476c_2^3 - 95c_5) - \ldots 4c_9)D_j^9 \\ +O(D_j^{10}) \end{array}\right]. \tag{3.11}$$

Using (3.9) -(3.11), the expansion for Ω is gotten as:

$$\begin{aligned}\Omega = {} & D_j - c_2^2 D_j^3 + 3c2(c_2^2 - c_3)D_j^4 - 2(3c_2^4 - 6c_2^2 c_3 + c_3^2 + 2c_2 c_4)D_j^5 \\ & +(9c_2^5 - 29c_2^3 c_3 + 14c_2 c_3^2 + 16c_2^2 c_4 - 5c_3 c_4 - 5c_2 c_5)D_j^6 \\ & +(-9c_2^6 + 48c_2^4 c_3 + 4c_3^3 - 40c_2^3 c_4 - 3c_4^2 - \cdots + 6c_2(6c_3 c_4 - 6c_6))D_j^7 \\ & +(-42c_2^5 c_3 + 73c_2^4 + 14c_3^2 + \cdots + c_2(-17c_3^3 + 23c_4^2 + 44c_3 c_5 - 7c_7))D_j^8 \\ & +(27c_2^8 - 54c_2^6 c_3 - 92c_2^5 c_4 + + \cdots + c_2(56c_4 c_5 - 7c_3^2 + 52c_3 c_6 - 8c_8))D_j^9 \\ & +O(D_j^{10}). \end{aligned} \tag{3.12}$$

Using (3.12), we have

$$\begin{aligned} y_j = {} & \theta + c_2^2 D_j^3 + 3(c_2 c_3 - c_2^3)D_j^4 + 2(3c_2^4 - 6c_2^2 c_3 + c_3^2 + 2c_2 c_4)D_j^5 \\ & +(29c_2^3 c_3 - 9c_2^5 - 14c_2 c_3^2 - 16c_2^2 + 5c_3 c_4 + 5c_2 c_5)D_j^6 \\ & +(9c_2^6 - 48c_2^4 c_3 - 4c_3^3 + 40c_2^3 c_4 + \cdots + 3c_4^2 \cdots + 6c_3 c_5 + 6c_2(c_6 - 6c_3 c_4))D_j^7 \\ & +2(42c_2^5 - 73c_2^4 c_4 \cdots + c_4(17c_3^3 - 23c_4^2 - 44c_3 c_5 + 7c_7))D_j^8 \\ & +(54c_2^6 c_3 - 27c_2^8 - \cdots + c_2(74c_3^2 c_4 - 52c_3 c_6 + 8(c + 8 - 7c_4 c_5)))D_j^9 \\ & +O(D_j^{10}). \end{aligned} \tag{3.13}$$

The expression for y_j in (3.13) is used to obtain the expansion for $f(y_j)$ as

$$\begin{aligned} f(y_j) = {} & c_2^2 D_j^3 + 3(c_2 c_3 - c_2^3)D_j^4 + 2(3c_2^4 - 6c_2^2 c_3 + c_3^2 + 2c_2 c_4)D_j^5 \\ & +(29c_2^3 c_3 + 5c_3 c_4 + 5c_2 c_5 - 8c_2^5 - 14c_2 c_3^2 - 16c_2^2 c_4)D_j^6 \\ & +(3c_2^6 - 42c_2^4 c_3 - 4c_3^3 - 16c_2^2 c_4 + \ldots + 6c_2(c_6 - 6c_3))D_j^7 \\ & +(21c_2^7 - 65c_2^4 c_4 + c_2^3(50c_5 - 61c_2^3) + \ldots \\ & +c_2(17c_3^3 - 23c_4^2 - 44c_3 c_5 + 7c_7)D_j^8 \\ & +(-81c_2^8 + 221c_2^6 c_3 - 6c_3^4 + 36c_2^5 c_4 - 16c_3^2 c_5 \ldots \\ & +c_2(74c_3^2 - 52c_3 c_6 + 8(c_8 - 7c_4 c_5))D_j^9 + O(D_j^{10}). \end{aligned} \tag{3.14}$$

By combining the Equations (3.9), (3.12), (3.13) and (3.14), the Taylor,s expansion for z_j is obtained as

$$\begin{aligned} z_j = {} & \theta + c_2^3 D_j^4 + (4c_2^2 c_3 - 3c_2^4)D_j^5 + c_2(2c_2^4 - 14c_2^2 c_5 + 5c_2 c_4)D_j^6 \\ & +(16c_2^6 + 4c_2^4 c_3 + 2c_3^3 - 17c_2^3 + 12c_2 c_3 c_4 + c_2^2(6c_5 - 22c_2^2))D_j^7 \end{aligned}$$

$$
\begin{aligned}
&+(-81c_2^7+145c_2^5c_3+2c_2^4+7c_3^2c_4-\cdots+c_2^2(7c_6-51c_3c_4))D_j^8\\
&+(234c_2^8-736c_2^6c_3+\cdots-c_2^2(56c_3^3+29c_4^2+58c_3c_5-8c_7))D_j^9+O(D_j^{10}).
\end{aligned} \tag{3.15}
$$

Equation (3.15) is then used in the expansion of $f(z_j)$ and we obtained

$$
f(z_j)=
$$

$$
f'(\theta)\left[\begin{array}{l}
(c_2^3D_j^4+(4c_2^2c_3-3c_2^4)D_j^5+c_2(2c_2^4-14c_2^2c_5+5c_2c_4)D_j^6\\
+(16c_2^6+4c_2^4c_3+2c_3^3-17c_2^3+12c_2c_3c_4+c_2^2(6c_5-22c_2^2))D_j^7\\
+(-81c_2^7+145c_2^5c_3+2c_2^4+7c_3^2c_4-\cdots+c_2^2(78c_6-51c_3c_4))D_j^8\\
+(234c_2^8-728c_2^6c_3+\\
\quad\cdots-c_2^2(56c_3^3+29c_4^2+58c_3c_5-8c_7))D_j^9+O(D_j^{10}).
\end{array}\right] \tag{3.16}
$$

Using Equations (3.9), (3.12) and (3.16), we have

$$
\begin{aligned}
\Omega\frac{f(z)}{f(s_j)}=&\ c_2^3D_j^4+(4c_2^2c_3-4c_2^4)D_j^5+c_2(5c_2^4-19c_2^2c_3+5c_2c_3+5c_2c_4)D_j^6\\
&+(17c_2^6+20c_2^4c_3+2c_3^3-23c_2^3c_4+12c_2c_3c_4+c_2^2(6c_5-31c_2^2))D_j^7\\
&+(-114c_2^7+167c_2^5c_3+20c_2^4c_4+7c_3^2+\cdots+c_2^2(7c_6-72c_3c_4))D_j^8\\
&+(389c_2^8-1053c_2^6c_3+\\
&\qquad\cdots+c_2^2(30c_3^3+41c_4^2+82c_3c_5-8c_7))D_j^9+O(D_j^{10}).
\end{aligned} \tag{3.17}
$$

To expand η and μ, the equations (3.9) with (3.14) and (3.14) with (3.16) respectively, were utilized to obtain the following expressions.

$$
\begin{aligned}
\eta=&\ c_2^2D_j^2+(3c_2c_3-4c_2^3)D_j^3+2(5c_2^4-8c_2^2c_3+c_2^3+2c_2c_4)D_j^4\\
&+(-18c_2^5+49c_2^3c_3-19c_2c_3^2-21c_2^2c_4+5c_3c_4+5c_2c_5)D_j^5\\
&+(21c_2^6-101c_2^4c_3-6c_3^3+65c_2^3+\cdots+6c_2(-8c_3c_4+c_6))D_j^6\\
&+(119c_2^5c_3-140c_2^4c_4+\cdots+c_2(42c_3^3-30c_4^2-58c_3c_5+7c_7))D_j^7\\
&+(81c_2^6c_3-81c_2^8+194c_2^5c_4\cdots+2c_2(81c_3^2c_4-34c_3c_6+4(-9c_4c_5+c_8)))D_j^8\\
&+(270c_2^9-909c_2^7+\ldots\\
&\quad+c_2(78c_3^4+215c_3c_4^2+186c_3^2c_5-43c_5^2-84c_4c_6-78c_3c_7+9c_7))D_j^9\\
&+O(D_j^{10}).
\end{aligned} \tag{3.18}
$$

and

$$\begin{aligned}\mu &= c_2D_j + c_3D_j^2 + (c_2c_3 - 4c_2^3 + c_4)D_j^3 + 2(12c_2^4 - 16c_2^2c_3 + c_3^2 + 2c_2c_4 + c_5)D_j^4 \\ &+(-23c_2^5 + 57c_2^3c_3 - 22c_4c_2^2 + 3c_3c_4 + 3c_2(6c_3^2 + c_5) + c_6)D_j^5 \\ &+(34c_2^6 - 127c_2^4c_3 + \cdots + c_2(-47c_3c_4 + c_6) + c_7)D_j^6 \\ &+(-44c_2^7 - \cdots + c_2(42c_3^3 - 58c_3c_5 + 5(-6c_4^2 + c_3)) + c_8)D_j^7 + O(D_j^8).\end{aligned} \tag{3.19}$$

The Taylor's expansion of $H(\eta)$ and $G(\mu)$ around zero are

$$\begin{aligned}H(\eta) &= H(0) + H'(0)c_2^2D_j^2 + H'(0)(-4c_2^2 + 3c_2c_3)D_j^3 \\ &+(2(5c_2^4 - 8c_2^3 + c_3^2 + 2c_2c_4)H'(0) + \tfrac{1}{2}H''(0)c_2^4)D_j^4 \\ &+(-21H'(0)c_2^2c_4 + 5H'(0)c_3c_4 + c_2(-19H'(0)c_3^2 + 5H'(0)c_5) \\ &-2c_2^5(9H'(0) + 2H''(0)) + c_2^3c_3(19H'(0) + 3H''(0)))D_j^5 \\ &+\cdots + (-27c_3^2c_6H'(0) + c_3c_6(293H'(0) + 25H''(0)) \\ &+\cdots + \tfrac{1}{2}(-1818H'(0) + 2728H''(0) + 285H'''(0) + H^{iv}(0)))D_j^9 + O(D_j^{10}),\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}G(\mu) &= G(0) + G'(0)c_2D_j + (\tfrac{1}{2}c_2^2 + c_3G'(0))D_j^2 \\ &+(c_4G'(0) + c_2c_3(G'(0) + G''(0)) + c_2^3(-4G'(0) + \tfrac{1}{6}G'''(0)))D_j^3 \\ &+\cdots + \tfrac{1}{6}(3(2c_3c_6(5G'(0) + G''(0)) + 2(c_8G'(0) + c_4c_5(5G'(0) + G''(0))) \\ &+\cdots + c_2^3(c_5(576G'(0) - 192G''(0) + 9G'''(0 + G^{iv}(0)) \\ &+c_3^2(-1398G'(0) + 74G''(0) - 159G'''(0) + 4G^{iv}(0)))))D_j^7 + O(D_j^8)\end{aligned} \tag{3.21}$$

respectively.

When the expressions in (3.15), (3.16), (3.17), (3.20) and (3.21) are substituted in s_{j+1}, the last step of (2.8), we get

$$\begin{aligned}s_{j+1} &= \theta + c_2^3(1 - H(0)G(0))D_j^4 \\ &+c_2^2(4c_3(1 - H(0)G(0)) + \cdots + c_2^2(4H(0)G(0) - G'(0) - 3)))D_j^5 \\ &+c_2(5c_3^2(G(0)H(0) - 1) + \cdots + c_2^4(2 - \cdots - G(0)(5H(0) + H'(0))))D_j^6 \\ &+(2c_2^3(G(0)H(0) - 1) + \cdots + c_2^6(16 + \cdots + G(0)(8H'(0) - 17H(0))))D_j^7 \\ &+\tfrac{1}{24}(7c_3^2c_4(G(0)H(0) - 1) + \cdots + 12G(0)(62H'(0) + H''(0) - 228H(0))))D_j^8\end{aligned}$$

$$+\frac{1}{6}(2c_2(-8c_4c_5(G(0)H(0)-1)+\cdots+G(0)(1584H'(0)+\cdots-6318H(0)))D_j^9$$
$$+O(D_j^{10}). \tag{3.22}$$

For the IS in (2.3) to converge to θ with a minimum CO of eight, the coefficients of the error terms $D_j^i,(0\le i\le 7)$ in (3.14) must be decimated. This implies solving the set of equations:

$$1-G(0)H(0)=0, \quad G(0)-G'(0), \quad -2G(0)+G''(0)+2G(0)^2H'(0)=0$$
$$-2G(0)+G''(0)=0, \quad 30G(0)-9G''(0)-G'''(0)=0. \tag{3.23}$$

The solutions of the equations in (3.15) are:

$$H''(0)<\infty G(0)=G'(0)=1,\ G''(0)=2,\ G'''(0)=12,\ G^{iv}(0)<\infty.$$

Consequently, the error equation in (3.14) reduces to

$$s_{j+1}=\theta-c_2^7\left(\frac{(2+2G^{iv}(0)+H''(0))}{2}\right)D_j^8+O(D_j^9). \tag{3.24}$$

The error equation in (3.17) implies that the family of IS in (2.8) has minimum CO eight. This ends the proof.

Remark 3.2: For any two functions $H(\eta)$ and $G(\mu)$ that satisfy the conditions in (3.16) and used in (2.3), will produces an IS with minimum CO eight. Consider the real value functions, $H(\eta)=1+\varphi\eta^2$ and $G(\mu)=1+\mu+\mu^2+2\mu^3+\sigma\mu^4$, we put forward a new family of IS as:

$$y_j=s_j-\Omega,$$
$$z_j=y_j-\Omega\frac{f(y_j)}{f(s_j)},$$
$$s_{j+1}=z_j-\Omega\frac{f(z_j)}{f(s_j)}\{(1+\varphi\eta^2)(1+\mu+\mu^2+2\mu^3+\sigma\mu^4)\}, \tag{3.25}$$

where $\varphi,\sigma\in R$, with $\varphi+\sigma\neq-1$. Consequently, its error equation is

$$D_{j+1}=\theta-((1+\varphi+\sigma)c_2^7)D_j^8$$
$$+c_2^4((21+11\varphi+4\sigma)c_2^4-(11+10\varphi+8\sigma)c_2^2c_3-c_2^2+c_2c_4)D_j^9+D_j^{10}. \tag{3.26}$$

For the case where $\varphi+\sigma=-1$ in (3.19), the IS in (3.18) will attain CO nine.

Remark 3.3: The IS in (3.18) require evaluation of five distinct function in an iteration cycle . Therefore, in the sense of Ostrowski index, its $EI\approx 1.5157$ which is better than the NIS and the IS in (1.3). Although the developed IS is not optimal, its fast convergence and precision will often compensate this setback.

It is important to note that, from the error equation in (3.17), the family of IS in (2.3) can attain CO nine and EI=1.5518, when $2G^{iv}(0)+H''(0)+2=0$.

Remark 3.5: For instance, when $H''(0)=-2$ and $G^{iv}(0)=0$ in (3.17), a concrete form of the IS in (2.3) with CO nine is formed and it is obtained as:

$$\begin{aligned} y_j &= s_j-\Omega, \\ z_j &= y_j-\Omega\frac{f(y_j)}{f(s_j)}, \\ s_{j+1} &= z_j-\Omega\frac{f(z_j)}{f(s_j)}\{(1-\eta^2)(1+\mu+\mu^2+2\mu^3)\}. \end{aligned} \tag{3.28}$$

with error equation as

$$D_{j+1}=\theta+c_2^4(10c_2^4-c_2^2c_3-c_3^2+c_2c_4)D_j^9+O(D_j^{10}). \tag{3.29}$$

4. Numerical Implementation

To demonstrate the effectiveness of the methods put forward herein, three concrete forms of the IS in (3.18) with $\varphi=-3,\sigma=0$ (M_1^8), $\varphi=\frac{1}{2},\sigma=-\frac{29}{18}$ (M_2^8), $\varphi=1,\sigma=-2$ (M_1^9) and the IS in (3.20) (M_2^9) were applied to solve some NLEs. The computation results obtained when the concrete forms of the developed IS were applied to solve some NLE, were compared to those from some existing IS in Liu and Wang [6] (LWM), Petkovic et.al [8] (PNPDM) and Kong-ied [4] (KM) that are of CO eight and Naseem et. al [3] (NSM) with CO nine. The stopping condition for the written Maple software program with 2000 significant digits is $|s_j-s_{j+1}|<10^{-2000}$. The measures used for comparison includes the absolute values of function of various IS results $|f(s_j)|$. number of iterations required for the IS to converge to solution *NIT* and computational CO (ρ_{co}) due to Jay , given by

$$\rho_{co}=\frac{log|f(s_{j+1})|}{log|f(s_j)|}. \tag{4.30}$$

The following examples were utilised for implementation.

$$f_1(s)=s(s^2-1)+3, s_0=-1.7,$$

$$f_2(s) = s^2 - e^s - 3s + 2, s_0 = -2.6,$$
$$f_3(s) = sin(2cos(s)) - 1 - s^2 + e^{sin(s^3)}, s_0 = -1.2 \tag{4.31}$$
$$f_4(s) = s^3 + 4s^2 - 10, s_0 = 0.5$$

4.1 *Results discussion*

The obtained computation results using various IS are presented in Table 1. For brevity, computation results are presented in the form $A.Be - C$, which represents $A.B \times 10^{-C}$, where $A, B, C \in \Re$. Observe from Table 1 that, the IS put forward in this study solved all the examples with fast convergence. In fact, in all the examples used for the test, the developed IS of CO eight (OM_1^8 and OM_2^8) converges faster than the compared (PNPDM and LWN) both of CO eight, even when they are optimal. In same vein, in all the tested examples, the developed CO nine IS (OM_1^9 and OM_2^9), out performed the compared order nine IS (NSM) that requires second derivative of function.

Table 1

Iterative structures (IS) computational output comparison

$f_i(s)$	IS	$\|f(s_1)\|$	$\|f(s_2)\|$	$\|f(s_3)\|$	$\|f(s_4)\|$	NIT	ρco
	PNPDM	1.22e-11	1.99e-093	1.03e-747		3	8.0
	LWM	1.01e-12	3.71e-103	1.17e-826		3	8.0
	KM	5.15e-13	8.03e-106	2.80e-848		3	8.0
$f_1(s)$	OM^8_1	3.81e-13	4.95e-107	4.02e-858		3	8.0
	OM^8 _2	8.83e-14	2.28e-113	4.55e-910		3	8.0
	NSM	5.60e-14	4.77e-127	1.13e-1144		3	9.0
	OM^9_1	7.74e-14	1.15e-125	3.91e-1132		3	9.0
	OM^9_2	3.43e-14	3.21e-129	1.82e-1164		3	9.0
	PNPDM	5.52e-02	6.99e-20	4.87e-163	2.70e-1308	4	8.0
	LWM	9.18e-03	1.43e-25	4.94e-208	1.01e-1667	4	8.0
	KM	7.62e-03	1.31e-27	1.04e-225	1.56e-1810	4	8.0
$f_2(s)$	OM^8_1	1.56e-02	4.05e-26	8.25e-215	2.43e-1724	4	8.0
	OM^8 _2	1.75e-02	4.19e-27	6.02e-224	1.08e-1798	4	8.0
	NSM	2.54e-02	6.95e-25	6.26e-228		3	9.0
	OM^9_1	1.87e-02	2.74e-27	8.34e-251		3	9.0
	OM^9_2	1.12e-02	2.29e-29	1.36e-269		3	9.0
	PNPDM	4.00e-05	7.78e-30	1.60e-299		3	8.0
	LWM	4.21e-05	3.24e-38	4.02e-303		3	8.0
	KM	1.20e-05	2.87e-43	3.11e-344		3	8.0

Contd...

$f_3(s)$	OM^8_1	1.43e-06	2.80e-51	6.13e-409		3	8.0
	OM^8 _2	7.17e-08	6.29e-63	2.20e-503		3	8.0
	NSM			Div			
	OM^9_1	4.53e-07	4.44e-61	3.71e-547		3	9.0
	OM^9_2	1.54e-06	9.09e-57	8.01e-509		3	9.0
	PNPDM	2.93	1.56	8.36	4.47	19	8.0
	LWM	82.35	7.92e-01	1.42e-11	2.26e-097	5	8.0
	KM	2.46	2.91e-08	3.23e-71	7.29e-575	4	8.0
$f_4(s)$	OM^8_1	1.70	5.52e-09	3.52e-77	9.55e-623	4	8.0
	OM^8 _2	7.80e-01	1.12e-12	5.75e-108	2.70e-870	4	8.0
	NSM	8.17e-01	1.32	716939.55	37442.10	10	9.0
	OM^9_1	1.16	3.05e-11	3.10e-106	3.63e-0961	4	9.0
	OM^9_2	3.17e-01	2.18e-16	6.44e-153	1.11e-1381	4	9.0

5. Conclusion

The presented families of iterative structures in this manuscript have been established to converge with order eight or nine with efficiency index of 1.5157 or 1.5518 respectively. The numerical experience with the developed iterative structures shows that, on the average, they possess the advantage of producing a more accurate solution than the compared existing iterative structures.

Disclosure statement

Regarding this study, the authors have no any form of conflicting interest.

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