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Commutativity degree of cyclic chains of some of finite groups

Hamid Darabi *
Department of Mathematics
Esfarayen University of Technology
Esfarayen 9661998195
Iran

Mahdi Imanparast
Department of Computer Science
University of Bojnord
Bojnord 9453155111
Iran

Mostafa Sajedi
Department of Mathematics
Esfarayen University of Technology
Esfarayen 9661998195
Iran

Abstract

For a finite group G, we introduce the concept of commutativity degree of its cyclic chains. We denote the commutativity degree of cyclic chains of a group G by $ccd(G)$, and compute this measure for some famous groups such as dihedral groups D_{2n}, quaternion groups Q_{2^n}, quasi-dihedral groups QD_{2^n}, and modular groups M_{p^n}.

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1. Introduction

The probability is one of the topics that has been considered in finite group theory recently. One of the most important probabilities in this regard is the probability of commutativity of two subgroups in a finite

* *E-mail:* **darabi@esfarayen.ac.ir**

group such as G. This measure is known as the commutativity degree and has been studied for some finite groups in the literature [1, 2, 3, 4, 5, 6, 7, 8]. The study of special classes in the structure of finite groups has applications in the theoretical and practical topics of algebra [9]. Calculating the number of chains of a group is also one of the issues that has been seriously studied in recent years. Since the number of these chains and the number of fuzzy subgroups are equal, this concept can also be useful in calculating the number of chains in a group. The number of chains are calculated for some finite groups. For example, the number of chains in finite cyclic groups and finite abelian p-groups are calculated in [10] and [11], respectively. Moreover, the number of chains for non-abelian groups D_{2n}, Q_{4n}, QD_{2^n} and M_{p^n} are calculated in [12].

In this paper, the concept of the commutativity degree of cyclic chains in a finite group is introduced and some preliminary properties of this measure is studied. The commutativity degree is computed for dihedral groups D_{2n}, quaternion groups Q_{2^n}, quasi-dihedral groups QD_{2^n}, and modular groups M_{p^n}. Note that, the groups D_{2^n}, Q_{2^n}, QD_{2^n} and M_{p^n} together with abelian p-groups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$ produce all finite p-groups of order p^n with a maximal cyclic subgroup of order p^{n-1}.

2. Preliminaries

For a finite group G, we denote the set of all subgroups of the group G by $L(G)$, and the set of all cyclic subgroups of the group G by $L_1(G)$. We call the ordered n-tuple $(A_1, A_2, \ldots, A_n)$ of subgroups G a chain of length n, whenever for each $1 \le i \le n-1$, we have $A_i \subset A_{i+1}$. We say a chain ends in G, if the last element of the chain be G. We denote the set of all the chains of subgroups of G that ends in G by $Ch(G)$. A chain $C_1(A_1, A_2, \ldots, A_n = G)$ is called a cyclic chain when the subgroups $A_1, A_2, \ldots, A_{n-1}$ are cyclic. We show the set of all cyclic chains of the group G by $C^*(G)$.

Definition 1: For two members $C_1(A_1, A_2, \ldots, A_n = G)$ and $C_2(B_1, B_2, \ldots, B_m = G)$ of $Ch(G)$, we define the product of these two chains as C_1C_2 $(A_1B_1,\ A_2B_2,\ \ldots,\ A_rB_r = G$ when $r = min\ \{i : A_iB_i = G\}$. Two chains C_1 and C_2 commute $(C_1C_2 = C_2C_1)$, when the product of these two chains be a member of $Ch(G)$, and this is true only if A_1B_1, A_2B_2, $\ldots$, $A_{r-1}\ B_{r-1}$ are subgroups of G.

Authors in [13] have defined the commutativity degree for chains of a group as follows:

$$cd(G) = \frac{1}{|Ch(G)|^2} |\{(C_1, C_2) \in Ch(G)^2 : C_1C_2 = C_2C_1\}|$$
$$= \frac{1}{|Ch(G)|^2} |\{(C_1, C_2) \in Ch(G)^2 : C_1C_2 \in Ch(G)\}|. \quad (1)$$

If we consider only cyclic chains instead of all chains of the group, we will have the following definition.

Definition 2: For a group *G*, the commutativity degree of cyclic chains of *G* is defined as:

$$ccd(G) = \frac{1}{|C^*(G)|^2} |\{(C_1, C_2) \in C^*(G)^2 : C_1C_2 = C_2C_1\}|$$
$$= \frac{1}{|C^*(G)|^2} |\{(C_1, C_2) \in C^*(G)^2 : C_1C_2 \in Ch(G)\}|. \quad (2)$$

Remark 1: The set of all cyclic chains of length 2 in *G* that end in *G* is equivalent to $L_1(G)$ when *G* is non-cyclic. Therefore, the commutativity degree of chains of length 2 is the same as commutativity degree of cyclic subgroups of a group *G* as defined in [8] as follows:

$$csd(G) = \frac{1}{|L_1(G)|^2} |\{(H, K) \in L_1(G)^2 : HK = KH\}|$$
$$= \frac{1}{|L_1(G)|^2} |\{(H, K) \in L_1(G)^2 : HK \in L(G)\}|.$$

2.1 *Preliminary properties*

To calculate the number of chains of a given group *G*, we can use the lattice diagram of subgroups of group *G* used in [10]. In this case, we set $O(G) = 1$, and for each subgroup *K* of *G*, we have:

$$O(K) = \sum_{1 \leqslant i \leqslant n} O(K_i),$$

where $K_1, \dots, K_n$ being subgroups of *G* containing *K* properly. We know that the number of chains of *G* is $2O(\{e\})$, in which *e* is the identity element of a group *G*.

Lemma 1: *The number of chains in the group* $\mathbb{Z}_{p^n}$ *is* 2^n.

Proof: According to the lattice diagram of subgroups $\mathbb{Z}_{p^n}$, we have: $|Ch(\mathbb{Z}_{p^n})| = 1 + 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n$.

Obviously, for each finite group G, relation $0 < ccd(G) \leqslant 1$ is hold. The relation $ccd(G) = 1$ is equivalent to that all cyclic subgroups of the group commute together. Therefore, in the Dedekind groups, the relation $ccd(G) = 1$ is hold. For example $ccd(Q_8) = 1$.

If in a finite group G, all the cyclic subgroups are commute together, it follows that all the subgroups of G are commute together, and so $ccd(G) = 1$ is equivalent to $cd(G) = 1$.

Example 1: Consider a symmetric group $S_3 = <\alpha, \beta : \alpha^3 = 1, \beta^2 = 1, \beta\alpha = \alpha^2\beta>$. All subgroups of this group can be listed as follows:

$$L(S_3) = \{S_3, <\alpha>, <\beta>, <\alpha\beta>, <\alpha^2\beta>, 1\}.$$

The lattice diagram of the cyclic subgroups of this group is illustrated in Figure 1.

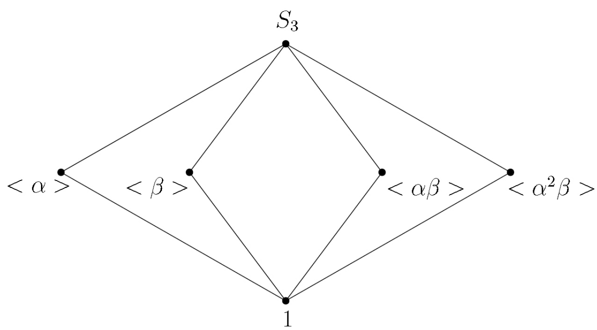

Figure 1

Lattice diagram of cyclic subgroups of S_3.

According to this diagram, we have: $|Ch(S_3)| = 10$, and $|C^*(S_3)| = 9$. The cyclic chains of this group are:

$$<\alpha> \subset S_3, <\beta> \subset S_3, <\alpha\beta> \subset S_3, <\alpha^2\beta> \subset S_3, 1 \subset S_3,$$

$$1 \subset <\alpha> \subset S_3, 1 \subset <\beta> \subset S_3, 1 \subset <\alpha\beta> \subset S_3, 1 \subset <\alpha^2\beta> \subset S_3.$$

The subgroups $<\alpha>$ and 1 commute with all subgroups. Subgroups $<\beta>$, $<\alpha\beta>$, $<\alpha^2\beta>$ only commute with themselves. Therefore, chains $<\alpha> \subset S_3$ and $1 \subset S_3$ and $1 \subset <\alpha> \subset S_3$ commute with all chains. Chain $<\beta> \subset S_3$ commutes with all cyclic chains except $<\alpha\beta> \subset S_3$, $<\alpha^2\beta> \subset S_3$, and the chain $1 \subset <\beta> \subset S_3$ commutes with all cyclic chains except $1 \subset <\alpha\beta> \subset S_3$, $1 \subset <\alpha^2\beta> \subset S_3$.

In the same way, the chain $<\alpha\beta> \subset S_3$ commutes with all cyclic chains except $<\beta> \subset S_3$, $<\alpha^2\beta> \subset S_3$, the chain $1 \subset <\alpha\beta> \subset S_3$ commutes with all cyclic chains except $1 \subset <\beta> \subset S_3$, $1 \subset <\alpha^2\beta> \subset S_3$, the chain $<\alpha^2\beta> \subset S_3$ commutes with all cyclic chains except $<\beta> \subset S_3$, $<\alpha\beta> \subset S_3$, and the chain $1 \subset <\alpha^2\beta> \subset S_3$ commutes with all cyclic chains except $1 \subset <\beta> \subset S_3$, $1 \subset <\alpha\beta> \subset S_3$. Therefore

$$ccd(S_3) = \frac{(3\times 9)+(6\times 7)}{9\times 9} = \frac{23}{27}.$$

Example 2: Consider the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$. The lattice diagram of the cyclic subgroups of this group is as shown in Figure 2.

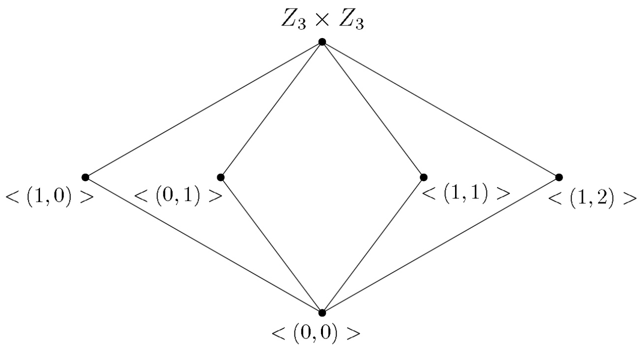

Figure 2

Lattice diagram of cyclic subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

According to this diagram, we have: $|Ch(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 10$ and $|C^*(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 9$. Due to the fact that the group is abelian, we have:

$$ccd(\mathbb{Z}_3 \times \mathbb{Z}_3) = 1.$$

Example 3: Consider a group $S_3 \times \mathbb{Z}_5$ (see Figure 3). Given that $S_3 = \{1, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$ and $Z_5 = \{e, x, x^2, x^3, x^4\}$ where $x^5 = e$, the cyclic subgroups of the group are

$$L_1(S_3 \times \mathbb{Z}_5) = \{<(\alpha,x)>, <(\alpha,e)>, <(1,x)>, <(\beta,e)>, <(\beta,x)>,$$
$$<(\alpha\beta,e)>, <(\alpha\beta,x)>, <(\alpha^2\beta,e)>, <(\alpha^2\beta,x)>, <(1,e)>\}.$$

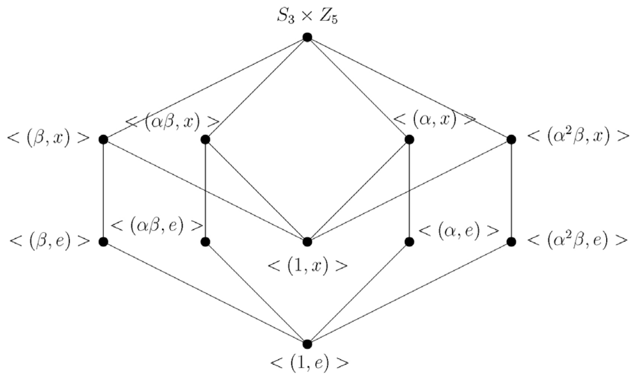

Figure 3

Lattice diagram of cyclic subgroups of $S_3 \times \mathbb{Z}_5$.

According to this diagram, we have: $|C^*(S_3 \times \mathbb{Z}_5)| = 35$ (See [14]). Subgroups $<(\alpha,x)>$, $<(\alpha,e)>$, $<(1,x)>$ and $<(1,e)>$ commute with all cyclic subgroups. Subgroups $<(\beta,e)>$ and $<(\beta,x)>$ commute together, subgroups $<(\alpha\beta,e)>$ and $<(\alpha\beta,x)>$ commute together, and subgroups $<(\alpha^2\beta,e)>$ and $<(\alpha^2\beta,x)>$ commute together. By direct calculation we obtain:

$$ccd(S_3 \times \mathbb{Z}_5) = \frac{(35\times35)-(3\times70)}{35\times35} = \frac{29}{35}.$$

Remark 2: If two groups G_1 and G_2 are isomorphic, it is obvious that $ccd(G_1) = ccd(G_2)$, but the opposite is not true. For example, $ccd(\mathbb{Z}_2 \times \mathbb{Z}_3) = ccd(\mathbb{Z}_5 \times \mathbb{Z}_3)$, but $\mathbb{Z}_2 \times \mathbb{Z}_3 \not\cong \mathbb{Z}_5 \times \mathbb{Z}_3$.

Remark 3: Examples 1 and 2 show that if the lattices of two groups G_1 and G_2 are isomorphic, $ccd(G_1)$ and $ccd(G_2)$ may not be equal.

Remark 4: The relation $ccd(G_1 \times G_2) = ccd(G_1)ccd(G_2)$ is not holding in general. On the other hand, for the finite groups G_1 and G_2, by using Proposition 2.3 of [8], if $|G_1|$ and $|G_2|$ are relatively prime, then the relation $csd(G_1 \times G_2) = csd(G_1)\ csd(G_2)$ is hold. But this is not valid for the commutativity degree of cyclic chains of a finite groups. For example, $(|S_3|,|\mathbb{Z}_5|) = 1$, but $ccd(S_3 \times \mathbb{Z}_5) = \frac{29}{35} \neq ccd(S_3)\ ccd(\mathbb{Z}_5) = \frac{23}{27} \times 1$. (see Examples 1 and 3)

The following theorem from [14], gives the number of cyclic chains of some non-abelian groups that are used to prove the results of the main results section.

Theorem 1: ([14]) *For every natural number n,*

(i) *The number of cyclic chains of the dihedral group* $D_{2n} = <a, b : a^n = b^2 = 1,\ a^b = a^{-1}>$ *is equal to* $|C^*(D_{2n})| = \sum_{k|n} |Ch(\mathbb{Z}_k)| + 2n$,

(ii) *The number of cyclic chains of the Quaternion group* $Q_{2^n} = <a, b : a^{2^{n-1}} = 1,\ a^{2^{n-2}} = b^2,\ a^b = a^{-1}>$ *is equal to* $|C^*(Q_{2^n})| = 2^{n+1} - 1$,

(iii) *The number of cyclic chains of the quasi-dihedral group* $QD_{2^n} = <a, b : a^{2^{n-1}} = b^2 = 1,\ a^b = a^{2^{n-2}-1}>$ *is equal to* $|C^*(QD_{2^n})| = 2^{n+1} - 1$,

3. Main results

In this section, we compute the commutativity degree for cyclic chains in some finite groups. As a first result, we compute this measure for the dihedral group D_{2n}.

Theorem 2: *For a dihedral group* D_{2n}, *we have:*

(a) *if n is odd,*

$$ccd(D_{2n}) = \frac{(\Sigma_{k|n}|Ch(\mathbb{Z}_k)|)^2 + 4n(\Sigma_{k|n}|Ch(\mathbb{Z}_k)|) + 2n^2 + 2n}{(\Sigma_{k|n}|Ch(\mathbb{Z}_k)| + 2n)^2}.$$

(b) *if n is even,*

$$ccd(D_{2n}) = \frac{(\Sigma_{k|n}|Ch(\mathbb{Z}_k)|)^2 + 4n(\Sigma_{k|n}|Ch(\mathbb{Z}_k)|) + 2n^2 + 4n}{(\Sigma_{k|n}|Ch(\mathbb{Z}_k)| + 2n)^2}.$$

Proof: The cyclic subgroups of this group are: $<a^k>$ and $<a^i b>$, where $k | n$ and $0 \le i \le n-1$.

Therefore, the cyclic chains of this group are as follows:

(1) Chains of the form $\cdots \subset <a^{n/k}> \subset D_{2n}$,

(2) Chains of the form $<a^i b> \subset D_{2n}$ for $0 \le i \le n-1$,

(3) Chains of the form $1 \subset <a^i b> \subset D_{2n}$ for $0 \le i \le n-1$.

According to Theorem 1, number of cyclic chains of a dihedral group $G = D_{2n}$ that ending in G is:

$$|C^*(D_{2n})| = \sum_{k|n} |Ch(\mathbb{Z}_k)| + 2n.$$

The number of chains in case (1) is $\Sigma_{k|n} |Ch(\mathbb{Z}_k)|$. On the other hand, we know that cyclic subgroups $< a^k >$ where $k | n$, commute with all cyclic subgroups of the group. Moreover, if n is odd, for each $0 \le i \le n-1$, the subgroup $< a^i b >$ commutes with the subgroups $< a^k >$ in which $k | n$, and the subgroup $< a^i b >$. If n is even, for each $0 \le i \le n-1$, the subgroup $< a^i b >$ commutes with the subgroups $< a^k >$ in which $k | n$, and the subgroups $< a^i b >$ and $< a^{i+\frac{n}{2}} b >$.

Therefore, the chains in case (1) commute with all cyclic chains of the group. If n is odd, each chain in case (2) commutes with all chains in cases (1) and (3), and the chain $< a^i b > \subset D_{2n}$ in case (2). Also, each chain in case (3) commutes with all chains in cases (1) and (2), and chain $1 \subset < a^i b > \subset D_{2n}$ in case (3). So, we have:

$$ccd(D_{2n}) = \frac{\Sigma_{k|n} |Ch(\mathbb{Z}_k)| (\Sigma_{k|n} |Ch(\mathbb{Z}_k)| + 2n) + (2n(\Sigma_{k|n} |Ch(\mathbb{Z}_k)| + n + 1))}{(\Sigma_{k|n} |Ch(\mathbb{Z}_k)| + 2n)^2}.$$

Similarly, if n is even, each chain in case (2) with all the chains of cases (1) and (3) and two chains $< a^i b > \subset D_{2n}$ and $< a^{i+\frac{n}{2}} b > \subset D_{2n}$ of the case (2) commute together. And so, each chain in case (3) commutes with all the chains in cases (1) and (2) and two chains $1 \subset < a^i b > \subset D_{2n}$ and $1 \subset < a^{i+\frac{n}{2}} b > \subset D_{2n}$ in case (3). So, we have:

$$ccd(D_{2n}) = \frac{\Sigma_{k|n} |Ch(\mathbb{Z}_k)| (\Sigma_{k|n} |Ch(\mathbb{Z}_k)| + 2n) + (2n(\Sigma_{k|n} |Ch(\mathbb{Z}_k)| + n + 2))}{(\Sigma_{k|n} |Ch(\mathbb{Z}_k)| + 2n)^2}.$$

Therefore, the proof is complete.

Likewise, for a dihedral group D_{2p^m}, we will have the following two corollaries.

Corollary 1: *For a dihedral group D_{2p^m} where p is an odd prime number, the commutativity degree of cyclic chains of D_{2p^m}, is as follows:*

$$ccd(D_{2p^m}) = \frac{(2^{m+1}-1)(2^{m+1}-1+2p^m) + (2p^m(2^{m+1}+p^m))}{(2^{m+1}-1+2p^m)^2}.$$

Proof: According to Lemma 1,

$$\sum_{k|p^m} |Ch(\mathbb{Z}_k)| = \sum_{i=0}^{m} |Ch(\mathbb{Z}_{p^i})| = 1 + 2 + \cdots + 2^m = 2^{m+1} - 1.$$

So, the result is correct.

Corollary 2: *For a dihedral group* D_{2^m}, *the commutativity degree of cyclic chains of* D_{2^m} *is as follows:*

$$ccd(D_{2^m}) = \frac{7.2^{2m-1} - 2^{m+1} + 1}{8.2^{2m-1} - 2^{m+2} + 1}.$$

Proof: By replacing $n = 2^{m-1}$ in Theorem 2 (b),

$$ccd(D_{2^m}) = \frac{(2^m - 1)(2^{m+1} - 1) + (2^m(3 \cdot 2^{m-1} + 1))}{(2^{m+1} - 1)^2} = \frac{7.2^{2m-1} - 2^{m+1} + 1}{8.2^{2m-1} - 2^{m+2} + 1}.$$

Theorem 3: *The commutativity degree of cyclic chains of a Quaternion group* Q_{2^n} *is as follows:*

$$ccd(Q_{2^n}) = 1 - \frac{3(2^{2n-3} - 2^n)}{(2^{n+1} - 1)^2}.$$

Proof: The cyclic subgroups of this group are: $< a^{2^i} >$ and $< a^j b >$ where $0 \le i \le n-1$ and $0 \le j \le 2^{n-2} - 1$, and so the cyclic chains of this group are as follows:

(1) chains in form: $\cdots \subset < a^{2^k} > \subset Q_{2^n}$,
(2) chains in form: $< a^i b > \subset Q_{2^n}$ for $0 \le i \le 2^{n-2} - 1$,
(3) chains in form: $< a^{2^{n-2}} > \subset < a^i b > \subset Q_{2^n}$ for $0 \le i \le 2^{n-2} - 1$,
(4) chains in form: $1 \subset < a^i b > \subset Q_{2^n}$ for $0 \le i \le 2^{n-2} - 1$,
(5) chains in form: $1 \subset < a^{2^{n-2}} > \subset < a^i b > \subset Q_{2^n}$ for $0 \le i \le 2^{n-2} - 1$.

The number of chains in case (1) is $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$, therefore

$$|C^*(Q_{2^n})| = 2^n - 1 + 4(2^{n-2}) = 2^{n+1} - 1.$$

On the other hand, we know that cyclic subgroups $< a^{2^i} >$ where $0 \le i \le n-1$ commute with all cyclic subgroups of the group. Also, a subgroup $< a^j b >$ where $0 \le j \le 2^{n-2} - 1$ commutes with subgroups $< a^j b >$ and $< a^{j+2^{n-3}} b >$.

According to the above considerations, the following results are obtained:

- Chains in case (1) commute with all cyclic chains in the group.
- Each chain in case (2) is commute by two chains from (2) and all chains in (1), (3), (4) and (5).
- Each chain in case (3) is commute by two chains of (3) and two chains of (4), and all chains of (1), (2) and (5).
- Each chain in case (4) is commute by two chains of (3) and two chains of (4), and all chains of (1), (2) and (5).
- Each chain in case (5) is commute by two chains of group (5), and all chains of (1), (2), (3) and (4).

Therefore,

$$ccd(Q_{2^n}) = 1 - \frac{(2 \cdot 2^{n-2}(2^{n-2}-2) + 2 \cdot 2^{n-2}(2^{n-2}-2+2^{n-2}-2))}{(2^{n+1}-1)^2}$$

$$= 1 - \frac{(2^{n-1}(2^{n-2}-2) + 2^{n-1}(2^{n-1}-4)}{(2^{n+1}-1)^2} = 1 - \frac{3(2^{2n-3}-2^n)}{(2^{n+1}-1)^2}.$$

Theorem 4: *For a quasi-dihedral group QD_{2^n}, we have:*

$$ccd(QD_{2^n}) = \frac{115 \cdot 2^{2n-5} - 9 \cdot 2^{n-2} + 1}{128 \cdot 2^{2n-5} - 16 \cdot 2^{n-2} + 1}.$$

Proof: By Theorem 1, number of cyclic chains of the dihedral group $G = QD_{2^n}$ which ending in G is $|C^*(QD_{2^n})| = 2^{n+1} - 1$.

The cyclic subgroups of this group are:

- $< a^{2^i} >$ where $0 \le i \le n-1$,
- $< a^{2i}b >$ where $0 \le i \le 2^{n-2} - 1$,
- $< a^{2i+1}b >$ where $0 \le i \le 2^{n-3} - 1$.

Therefore, the cyclic chains of this group are as follows:

(1) Chains in the form: $\cdots \subset < a^{2^i} > \subset QD_{2^n}$,

(2) Chains in the form: $< a^{2i}b > \subset QD_{2^n}$ for $0 \le i \le 2^{n-2} - 1$,

(3) Chains in the form: $< a^{2i+1}b > \subset QD_{2^n}$ for $0 \le i \le 2^{n-3} - 1$,

(4) Chains in the form: $1 \subset < a^{2i}b > \subset QD_{2^n}$ for $0 \le i \le 2^{n-2} - 1$,

(5) Chains in the form: $1 \subset < a^{2i+1}b > \subset QD_{2^n}$ for $0 \le i \le 2^{n-3} - 1$,

(6) Chains in the form: $<a^{2^{n-2}}>\subset<a^{2i+1}b>\subset QD_{2^n}$ for $0\le i\le 2^{n-3}-1$,

(7) Chains in the form: $1\subset<a^{2^{n-2}}>\subset<a^{2i+1}b>\subset QD_{2^n}$ for $0\le i\le 2^{n-3}-1$.

The number of chains in case (1) is $1+2+2^2+\cdots+2^{n-1}=2^n-1$.

Regarding the commutativity of the cyclic subgroups, the following results are obtained with a simple study:

- Cyclic chains $<a^{2^i}>$ for $0\le i\le n-1$, commute with all cyclic subgroups of the group.
- Subgroups $<a^{2i+1}b>$ and $<a^{2j+1}b>$ where $0\le i,j\le 2^{n-3}-1$, commute with together if and only if $i=j$.
- Subgroups $<a^{2i}b>$ and $<a^{2j}b>$ where $0\le i,j\le 2^{n-2}-1$, commute together if and only if $2^{n-3}\mid i-j$.
- Subgroups $<a^{2i}b>$ and $<a^{2j+1}b>$ where $0\le i\le 2^{n-2}-1$ and $0\le j\le 2^{n-3}-1$ do not commute together.

According to the above considerations, the following results are obtained:

- Chains in case (1) commute with all chains in this group.
- Each chain in case (2) commutes with two chains in case (2) and all chains in cases (1), (4), (5), (6) and (7).
- Each chain in case (3) commutes with a chain in case (3) and all chains in cases (1), (4), (5), (6) and (7).
- Each chain in case (4) commutes with two chains in case (4) and all chains in cases (1), (2), (3) and (7).
- Each chain in case (5) commutes with a chain in case (5), a chain in case (6) and all chains in cases (1), (2), (3) and (7).
- Each chain in case (6) commutes with a chain in case (5), a chain in case (6) and all chains in cases (1), (2), (3) and (7).
- Each chain in case (7) commutes with a chain in case (7) and all chains in cases (1), (2), (3), (4), (5) and (6).

So, we have:

$$ccd(QD_{2^n})=\frac{(2^n-1)(2^{n+1}-1)+2^{n-2}(2^n-1+2+2^{n-2}+2^{n-3}+2^{n-3}+2^{n-3})}{(2^{n+1}-1)^2}$$

$$+\frac{2^{n-3}(2^n-1+1+2^{n-2}+2^{n-3}+2^{n-3}+2^{n-3})}{(2^{n+1}-1)^2}$$
$$+\frac{2^{n-2}(2^n-1+2^{n-2}+2^{n-3}+2+2^{n-3})}{(2^{n+1}-1)^2}$$
$$+\frac{2^{n-3}(2^n-1+2^{n-2}+2^{n-3}+1+1+2^{n-3})}{(2^{n+1}-1)^2}$$
$$+\frac{2^{n-3}(2^n-1+2^{n-2}+2^{n-3}+1+1+2^{n-3})}{(2^{n+1}-1)^2}$$
$$+\frac{2^{n-3}(2^n-1+2^{n-2}+2^{n-3}+2^{n-2}+2^{n-3}+2^{n-3}+1)}{(2^{n+1}-1)^2}$$
$$=\frac{115\cdot 2^{2n-5}-9\cdot 2^{n-2}+1}{128\cdot 2^{2n-5}-16\cdot 2^{n-2}+1}.$$

Therefore, the above relation is resulted.

Theorem 5: *The commutativity degree of cyclic chains of a modular group* $M_{p^n}, (n\geq 3, p^n\neq 8)$ *is one, where p is a prime number.*

Proof: The cyclic subgroups of this group are: $<a^{p^i}>$ where $0\leq i\leq n-1$, and $<a^{jp^i}b>$ where $0\leq i\leq n-2$ and $0\leq j\leq p-1$. The derived subgroup of M_{p^n} satisfies $|G'|=p$, and any subgroup of order at least p^2 contains G' and so is normal. Therefore, any two subgroups G_1 and G_2 (whether or not they are cyclic) will commute if at least one of them has order greater than *p*. If both have order less than or equal to *p*, then (since $p^n\neq 8$) G_1 and G_2 are contained in the unique elementary abelian subgroup of order p^2 of *G*, and hence commute. Therefore $ccd(M_{p^n})=1$, and this completes the proof.

According to Theorems 2, 3 and 4, we can have.

Corollary 3: *For the dihedral groups* D_{2^n}, *quaternion groups* Q_{2^n}, *and quasi-dihedral groups* QD_{2^n}, *we have:*

- $\lim_{n\to+\infty} ccd(D_{2^n})=\frac{7}{8}$,
- $\lim_{n\to+\infty} ccd(Q_{2^n})=\frac{29}{32}$,
- $\lim_{n\to+\infty} ccd(QD_{2^n})=\frac{115}{128}$.

References

[1] W. H. Gustafson, What is the probability that two group elements commute? *Amer. Math. Monthly*, 80, 1031-1034 (1973).

[2] D. J. Rusin, What is the probability that two elements of a finite group commute? *Pacific J. Math.*, 82, 237-247 (1979).

[3] P. Lescot, Sur certains groupes finis. *Rev. Math. Spciales*, 8, 276-277 (1987).

[4] P. Lescot, Degr de commutativit et structure dun groupe fini (1). *Rev. Math. Spciales*, 8, 276-279 (1988).

[5] P. Lescot, Central extensions and commutativity degree. *Comm. Algebra*, 29, 4451-4460 (2001).

[6] A. Erfanian, P. Lescot, R. Rezaei, On the relative commutativity degree of a subgroup of a finite group. *Comm. Algebra*, 35, 4183-4197 (2007).

[7] M. Tarnauceanu, Subgroup commutativity degrees of finite groups. *J. Algebra*, 321, 2508-2520 (2009).

[8] M. Tarnauceanu, M. S. Lazorec, Cyclic subgroup commutativity degrees of finite groups. *Rend. Semin. Mat. Uni. Pad.*, 139, 225-240 (2018).

[9] X. Chen, H. Wang, Notes on conjugacy classes of finite groups, *J. Dis. Math. Sci. Cryptogr.*, 21 : 7-8, 1519-1522 (2018).

[10] H. Darabi, M. Imanparast, Counting number of fuzzy subgroups of some of dihedral groups. *Inter. J. of Pure and Applied Math.*, 85 (3), 563-575 (2013).

[11] M. Imanparast, H. Darabi, A recursive formula for the number of fuzzy subgroups of finite cyclic groups, *J. Adv. Computer Research*, 85(4), 55-63 (2013).

[12] H. Darabi, F. Saeedi, D. Farrokhi, The number of fuzzy subgroups of some non-abelian groups. *Iran J. Fuzzy Syst.*, 10 (6), 101-107 (2013).

[13] H. Darabi, M. Imanparast, Commutativity degree of chains of finite groups. *J. Dis. Math. Sci. Cryptogr.*, , 26 (8), 2153-2166, (2023).

[14] M. Sajedi, Mohammad Reza Rajabzadeh Moghaddam, Probability of cyclicity of chains in finite groups. *Southeast Asian Bulletin of Mathematics*, 47, 563–571 (2023).

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