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Common extension of two sigma-finite measures

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Abstract

A σ – finite measure is obtained as the unique common extension of two σ – finite measures defined on two disjoint rings. This is done by first considering an outer measure on a σ – ring containing both disjoint rings. This outer measure then gives rise to the unique extension in question.

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Keywords: *Hereditary sigma-ring, Sigma-finite measure, Common extension of measure.*

1. Introduction

Various people have studied extension of measures over a long period. For example, Kharizishvili [4] considers set-theoretical, topological and algebraic aspects of the measure extension problem and presents new methods for solving this problem. In particular, in [4], the measure extension problem is solved for invariant or quasi-invariant measures on solvable uncountable groups, and non-separable extensions of invariant measures are constructed. Also, Endou, Okazaki and Shidama [2] use a finite additive measure to induce a σ-additive measure. This is called an E. Hopf's extension of measure. In addition, Sahin, Olgun, Akyildiz and Karukus [5] provide a new proof of the Caratheodory extension theorem for a lattice valued fuzzy measure.

The problem of obtaining a common extension of more than one measure has also received attention. Rao and Shortt [1] investigated the common extension of finitely additive measures taking values in an abelian group. Sookoo [6] generalised a method (cf Halmos [3]) of extending a measure on a ring, which utilizes the concept of a measurable set and obtained a measure which is an extension of two measures defined on disjoint rings. In this paper, a method (cf Halmos [3]) is adapted to obtain a common extension of two σ-finite

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measures which are defined on non-intersecting rings. Some results on measurable covers are first generalized as a preliminary step.

2. Definitions

The definitions of ring, σ – ring, hereditary set, subadditive set function, finitely additive set function, countably subadditive set function, measure, finite measure, σ - finite measure, extension of a measure, outer measure, measurable set, measurable cover, symmetric difference of sets and hereditary σ – ring $\boldsymbol{H(E)}$ as used in this article are found in Halmos [3]. Also, Definitions 2.1, 2.5, 2.6, 2.7, and 2.8 closely follows definitions on pages 24, 31, 47, 47 and 47 respectively in Halmos [3].

Definition 2.1: $\boldsymbol{S(R)}$ is the smallest σ – ring containing the ring $\boldsymbol{R}$.

Definition 2.2: $\boldsymbol{S_{1,2}} = \{S_1 \cup S_2 \mid S_1 \in \boldsymbol{S(R_1)}, S_2 \in \boldsymbol{S(R_2)}\}$

Definition 2.3: $\boldsymbol{H(R_1, R_2)}$ represents the smallest σ – ring containing $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$, having the property of being hereditary.

Definition 2.4: Given 2 disjoint rings $\boldsymbol{R_1}, \boldsymbol{R_2}$ let $\Delta_2 = \{S \mid S = S_1 \cup S_2$, where $S_1 \in \boldsymbol{H(R_1)}, S_2 \in \boldsymbol{H(R_2)}\}$

Definition 2.5: Given an outer measure μ^* defined on $\boldsymbol{H(R_1, R_2)}$, **a s**et E in $\boldsymbol{H(R_1, R_2)}$ is said to be of σ- finite outer measure if $E \subseteq \bigcup_{n=1}^{\infty} E_i$ for sets $E_i \in \boldsymbol{H(R_1, R_2)}$; $i \in \{1, 2, \dots\}$ for which $\mu^*(E_j)$ is finite for all natural numbers j.

Definition 2.6: Let $\overline{\boldsymbol{S}}$ be the class of all sets in $\Delta_{\boldsymbol{2}}$ that are μ^* - measurable.

Definition 2.7: Let $\overline{\boldsymbol{S}}_{\boldsymbol{i}}$ be the class of all sets in $\boldsymbol{H(R_i)}$ that are μ_i^*- measurable, for $i \in \{1, 2\}$.

Definition 2.8: Given an outer measure μ^* on the hereditary ring σ – ring $\boldsymbol{\Delta_2}$**,** we define the function $\bar{\mu}$ on $\overline{\boldsymbol{S}}$ by $\bar{\mu}(E) = \mu^*(E), \forall\, E \in \overline{\boldsymbol{S}}$

3. A unique Extension of of $\sigma -$ finite Measures on non-intersecting Rings.

Theorem 3.1: *If* $E \in \boldsymbol{S_{1,2}}$, *then* E *is* $\mu^* -$ *measurable.*

Proof: From Theorem A, Page 49 of [3], every set in $\boldsymbol{S(R_1)}$ is $\mu_1^* -$ measurable and every set in $\boldsymbol{S(R_2)}$ is $\mu_2^* -$ measurable.

Therefore $\boldsymbol{S(R_1)} \subseteq \overline{\boldsymbol{S_1}}$ and $\boldsymbol{S(R_2)} \subseteq \overline{\boldsymbol{S_2}}$

Therefore $\boldsymbol{S_{1,2}} \subseteq \{ S_1 \cup S_2 \mid S_1 \in \overline{\boldsymbol{S}}_{\boldsymbol{1}}, S_2 \in \overline{\boldsymbol{S}}_{\boldsymbol{2}} \} = \overline{\boldsymbol{S}}$

Therefore, if $E \in \boldsymbol{S}_{1,2}$, then E is $\mu^* -$ measurable.

Theorem 3.2: *The $\sigma -$ ring $\boldsymbol{S}_{1,2}$ contains $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$, and is the smallest such ring to do so.*

Proof: Let $E, F \in \boldsymbol{S}_{1,2}$.

Then $E = E_1 \cup E_2$, where $E_1 \in \boldsymbol{S}(\boldsymbol{R_1})$, and $E_2 \in \boldsymbol{S}(\boldsymbol{R_2})$.

Also, $F = F_1 \cup F_2$, where $F_1 \in \boldsymbol{S}(\boldsymbol{R_1})$, and $F_2 \in \boldsymbol{S}(\boldsymbol{R_2})$.

$$E - F = (E_1 - F_1) \cup (E_2 - F_2) \in \boldsymbol{S}_{1,2} \tag{3.1}$$

since $E_1 - F_1 \in \boldsymbol{S}(\boldsymbol{R_1})$ and $E_2 - F_2 \in \boldsymbol{S}(\boldsymbol{R_2})$.

Let $E_i \in \boldsymbol{S}_{1,2}$.

Then $E_i \in E_{i1} \cup E_{i2}$, $E_{i1} \in \boldsymbol{S}(\boldsymbol{R_1})$, and $E_{i2} \in \boldsymbol{S}(\boldsymbol{R_2})$.

Therefore

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} E_{n1} \cup E_{n2} = [\bigcup_{n=1}^{\infty} E_{n1}] \cup [\bigcup_{n=1}^{\infty} E_{n2}] \in \boldsymbol{S}_{1,2} \tag{3.2}$$

since $\bigcup_{n=1}^{\infty} E_{n1} \in \boldsymbol{S}(\boldsymbol{R_1})$ and $\bigcup_{n=1}^{\infty} E_{n2} \in \boldsymbol{S}(\boldsymbol{R_2})$.

From (3.1) and (3.2), $\boldsymbol{S}_{1,2}$ is a $\sigma -$ring.

Let $\boldsymbol{T}$ be the smallest $\sigma -$ ring containing $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$.

Then

$$\boldsymbol{T} \subseteq \boldsymbol{S}_{1,2} \tag{3.3}$$

Now, let E be an arbitrary element of $\boldsymbol{S}_{1,2}$, where $E = E_1 \cup E_2$; $E_1 \in \boldsymbol{S}(\boldsymbol{R_1})$ and $E_2 \in \boldsymbol{S}(\boldsymbol{R_2})$.

Then $E_1 \in$ any $\sigma -$ ring containing $\boldsymbol{R_1} \subseteq$ any $\sigma -$ ring containing $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$. Also $E_2 \in$ any $\sigma -$ ring containing $\boldsymbol{R_2} \subseteq$ any $\sigma -$ ring containing $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$. Therefore $E = E_1 \cup E_2 \in$ any $\sigma -$ ring containing $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$. Therefore $E \in \boldsymbol{T}$.

Therefore

$$\boldsymbol{S}_{1,2} \subseteq \boldsymbol{T} \tag{3.4}$$

From (3.3) and (3.4), $\boldsymbol{T} = \boldsymbol{S}_{1,2}$.

Hence the theorem is proved.

Theorem 3.3: $\boldsymbol{S}_{1,2} \subseteq \overline{\boldsymbol{S}}$

Proof: $\boldsymbol{S}_{1,2} = \{S_1 \cup S_2 \mid S_1 \in \boldsymbol{S}(\boldsymbol{R_1}), S_2 \in \boldsymbol{S}(\boldsymbol{R_2})\}$

$$\subseteq \{S_1 \cup S_2 \mid S_1 \in \overline{\boldsymbol{S}}_1, S_2 \in \overline{\boldsymbol{S}}_2\} = \overline{\boldsymbol{S}}$$

since $\boldsymbol{S}(\boldsymbol{R_i}) \subseteq \overline{\boldsymbol{S}}_i$; $i \in \{1, 2\}$.

The next theorem is based on Theorem B, Section 12, Chapter III of [3].

Theorem 3.4: $\mu^*(E) = \inf\{\overline{\boldsymbol{\mu}}(\boldsymbol{F}) \mid E \subseteq F \in \overline{\boldsymbol{S}}\} = \inf\{\overline{\boldsymbol{\mu}}(\boldsymbol{F}) \mid E \subseteq F \in \boldsymbol{S}_{1,2}\}$

Proof: For every E in Δ_2,

$$\mu^*(E) = \inf\{\textstyle\sum_{n=1}^{\infty}[\mu_1(E_{n1}) + \mu_2(E_{n2})] \mid E_{ni} \in \boldsymbol{R_i}; i \in \{1,2\};$$
$$n = 1, 2, \dots; E \subseteq \textstyle\bigcup_{n=1}^{\infty}(E_{n1} \cup E_{n2})$$
$$\geq \inf\{\textstyle\sum_{n=1}^{\infty}\left[\overline{\mu^1}(E_{n1}) + \overline{\mu_2}(E_{n2})\right] \mid E_{ni} \in \boldsymbol{R_i}; i \in \{1,2\}; n = 1, 2, \dots;$$
$$E \subseteq \textstyle\bigcup_{n=1}^{\infty}(E_{n1} \cup E_{n2})\}$$
$$\geq \inf\{\textstyle\sum_{n=1}^{\infty}[\overline{\mu_1}(E_{n1}) + \overline{\mu_2}(E_{n2})] \mid E_{ni} \in \boldsymbol{S}(\boldsymbol{R_i}); i \in \{1, 2\}; n = 1, 2, \dots;$$
$$E \subseteq \textstyle\bigcup_{n=1}^{\infty}(E_{n1} \cup E_{n2})\}$$

Since every sequence $\{E_{ni}\}$ of sets in $\boldsymbol{S}(\boldsymbol{R_i})$ such that $E_i \subseteq \bigcup_{n=1}^{\infty} E_{ni} = F_i$, for $i = 1, 2$, is replaceable by a disjoint sequence satisfying the same condition, with no resulting increase in the sum $\sum_{n=1}^{\infty} \overline{\mu_i}(E_{ni})$,

$$\inf\{\textstyle\sum_{n=1}^{\infty}[\overline{\mu_1}(E_{n1}) + \overline{\mu_2}(E_{n2})] \mid E_{ni} \in \boldsymbol{S}(\boldsymbol{R_i}); i \in \{1,2\}; n = 1, 2, \dots;$$
$$E \subseteq \textstyle\bigcup_{n=1}^{\infty}(E_{n1} \cup E_{n2})\}$$
$$\geq \inf\left\{[\overline{\mu_1}\,(F_1) + \overline{\mu_2}\,(F_2)] \middle| E \subseteq (F_1 \cup F_2) \in \boldsymbol{S_{1,2}}\right\}$$
$$\geq \inf\{[\overline{\mu_1}\,(F_1) + \overline{\mu_2}\,(F_2)] \mid E \subseteq (F_1 \cup F_2) \in \overline{\boldsymbol{S}}\}, \text{[since } \boldsymbol{S_{1,2}} \subseteq \overline{\boldsymbol{S}}]$$
$$= \inf\{\bar{\mu}\,(F) \mid E \subseteq F \subseteq \overline{\boldsymbol{S}}; F = F_1 \cup F_2\}$$
$$= \inf\{\mu^*(F) \mid E \subseteq F \in \overline{\boldsymbol{S}}\}$$
$$\geq \mu^*(E)$$

since $\bar{\mu}\,(E) = \mu^*(E)$, $\forall\, E \in \overline{\boldsymbol{S}}$.

Since

$$\inf\left\{[\overline{\mu_1}\,(F_1) + \overline{\mu_2}\,(F_2)] \middle| E \subseteq (F_1 \cup F_2) \in \boldsymbol{S_{1,2}}\right\} = \inf\left\{\bar{\mu}\,(F) \middle| E \subseteq F \in \boldsymbol{S_{1,2}}\right\}$$

we have established that

$$\mu^*(E) \geq \inf\left\{\bar{\mu}\,(F) \middle| E \subseteq F \in \boldsymbol{S_{1,2}}\right\} \geq \inf\{\bar{\mu}\,(F) \mid E \subseteq F \subseteq \overline{\boldsymbol{S}}\} \geq \mu^*(E)$$

and so the result follows.

The theorem below is based on Theorem C, Section 12, Chapter III of [3].

Theorem 3.5: *If $E \in \Delta_2$ is of σ – finite outer measure, then $\exists$ a measurable cover F of E in $\boldsymbol{S_{1,2}} \ni \bar{\mu}(F) = \mu^*(E)$.*

Proof: We commence by proving the theorem when $\bar{\mu}(E) < \infty$. From Theorem 3.4, given $n \in \{1, 2, \dots\}$, $\exists$

$$F_n \in \boldsymbol{S_{1,2}} \ni E \subseteq F_n, \bar{\mu}(F_n) \leq \mu^*(E) + \frac{1}{n}.$$

Let $F = \bigcap_{n=1}^{\infty} F_n$

Then $E \subseteq F \in \boldsymbol{S_{1,2}}$

and $\mu^*(E) \leq \mu^*(F) = \bar{\mu}(F) \leq \bar{\mu}(F_n) \leq \mu^*(E) + \frac{1}{n}$

We must have $\bar{\mu}(F) = \mu^*(E)$, as n is arbitrary.

Now, if $G \in \boldsymbol{S_{1,2}} \ni G \subseteq F - E$, then $E \subseteq F - G$.

Therefore $\bar{\mu}(F) = \mu^*(E) \leq \mu^*(F - G)$

$$= \bar{\mu}(F - G) \leq \bar{\mu}(F) - \bar{\mu}(G)$$
$$\leq \bar{\mu}(F)$$

Therefore $\bar{\mu}(F) - \bar{\mu}(G) = \bar{\mu}(F)$

Therefore $\bar{\mu}(G) = 0$

Therefore F is a measurable cover of E.

We now establish the theorem when $\bar{\mu}(E) = \infty$.

If $E \in \Delta_2$ and E is an element of an element T of $\boldsymbol{S_{1,2}}$, then $\bar{\mu}(T) = \infty$.

Clearly, $\Delta_2 = \boldsymbol{H}(\boldsymbol{R_1}, \boldsymbol{R_2})$.

Because E is an element of Δ_2 having σ −finite outer measure, it can be expressed as a disjoint union of countably many sets each having finite outer measure.

That is, $\exists$ sets $E_1, E_2, \ldots \in \boldsymbol{H}(\boldsymbol{R_1}, \boldsymbol{R_2}) \ni$

$$E_i \cup E_j = \phi; i, j \in \{1, 2, \ldots\}; i \neq j$$
$$E = \bigcup_{i=1}^{\infty} E_i$$

and $\mu^*(E) < \infty$.

Therefore $\exists\ K_i \in \boldsymbol{S_{1,2}} \ni K_i$ is a measurable cover of E_i.

Therefore $E_i \subseteq K_i$

Let $K = \bigcup_{i=1}^{\infty} K_i$.

Then $E \subseteq K \in \boldsymbol{S_{1,2}}$.

Therefore $\bar{\mu}(K) = \infty$.

Let $Q \in \boldsymbol{S_{1,2}} \ni Q \subseteq K - E$.

Then $\exists\ Q_1, Q_2, \ldots \in \boldsymbol{S_{1,2}} \ni Q = Q_1 \cup Q_2 \cup Q_3 \cdots$ and $Q_i \subseteq K_i - E_i$; for each i in $\{1, 2, \ldots\}$.

Since K_i is a measurable cover of E_i,

$$\bar{\mu}\,(Q_i) = 0$$

Therefore $\bar{\mu}\,(Q) = 0$

Therefore K is a measurable cover of E.

The next theorem is based on Theorem D, Section 12, Chapter III of [3].

Theorem 3.6: *Let $E \in \boldsymbol{H}(\boldsymbol{R_1}, \boldsymbol{R_2})$. If E has a measurable cover F which is an element of $\boldsymbol{S_{1,2}}$, then $\mu^*(E) = \bar{\mu}\,(F)$; if E has two measurable covers F_1 and F_2, then $\bar{\mu}(F_1 \Delta F_2) = 0$*

Proof: Assume that E has two measurable covers F_1 and F_2.

Then $E \subseteq F_1$ and $E \subseteq F_2$.

Therefore $\bar{\mu}[F_i - (F_1 \cap F_2)] = 0$ for $i \in \{1, 2\}$

Therefore $\bar{\mu}(F_1 \Delta F_2) = \bar{\mu}\,\{[F_1 - (F_1 \cap F_2)] \cup [F_2 - (F_1 \cap F_2)]\} = 0$

If $\mu^*(E)$ is infinite, then it must be the case that $\mu^*(E)$ and $\bar{\mu}(F)$ are equal.

If $\mu^*(E)$ is not infinite, then from Theorem 3.5, $\exists$ a measurable cover F_0 of $E \ni \bar{\mu}(F_0)$ and $\mu^*(E)$ are equal.

Now $\bar{\mu}(F_0) = \bar{\mu}\{[F_0 - (F_0 \cap F)] \cup [F_0 \cap F]\}$

$$= \bar{\mu}\{F_0 - (F_0 \cap F)\} + \bar{\mu}\{F_0 \cap F\}$$
$$= \bar{\mu}\{F - (F_0 \cap F)\} + \bar{\mu}\{F_0 \cap F\}$$
$$= \bar{\mu}\{[F - (F_0 \cap F)] \cup [F_0 \cap F]\}$$
$$= \bar{\mu}(F)$$

Therefore $\bar{\mu}(F) = \mu^*(E)$

The next theorem is based on Theorem E, Section 12, Chapter III of [3].

Theorem 3.7: *If μ_i on $\boldsymbol{R_i}$ is σ –finite; $i \in \{1, 2, \ldots\}$, then $\bar{\mu}$ is σ –finite on both $\boldsymbol{S_{1,2}}$ and $\overline{\boldsymbol{S}}$.*

Proof: Assume that μ_i on $\boldsymbol{R_i}$ is σ –finite. From Theorem 3.4 of [6], μ^* is also σ –finite. Hence for any set $E \in \overline{\boldsymbol{S}}$, $\exists$ a sequence $\{E_i\}$ of sets in $\boldsymbol{H}(\boldsymbol{R_1}, \boldsymbol{R_2}) \ni E$ is a subset of $\bigcup_{i=1}^{\infty} E_i$ and $\mu^*(E_i)$ is finite for i in $\{1, 2, \ldots\}$.

From Theorem 3.5, $\exists$ a subset F_i of $\boldsymbol{S_{1,2}} \ni \mu^*(E_i) = \bar{\mu}(F_i)$ and F_i is a measurable cover of E_i.

Therefore E is a subset of $\bigcup_{i=1}^{\infty} F_i$, $\bar{\mu}(F_i)$ is finite, and F_i is an element of $\boldsymbol{S_{1,2}}$ which is a subset of $\overline{\boldsymbol{S}}$. Hence $\bar{\mu}$ is $\sigma -$ finite on $\overline{\boldsymbol{S}}$. From the foregoing, obviously μ is a $\sigma -$ finite measure with respect to $\boldsymbol{S_{1,2}}$.

The theorem below is based on Theorem A, Section 13, Chapter III of [3].

Theorem 3.8: *Let μ_i be a measure on $\boldsymbol{R_i}$; for $i \in \{1, 2\}$, where $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ are disjoint rings. If μ_1 and μ_2 are both $\sigma -$ finite, then there exists a unique measure $\bar{\mu}$ defined on $\boldsymbol{S_{1,2}}$ satisfying the condition that $\bar{\mu}(E_i) = \mu_i(E_i)$, for all $E_i \in \boldsymbol{R_i}$, for $i \in \{1, 2\}$.*

Proof: From Theorem 5.1 of [6], we see that $\exists$ a measure $\bar{\mu}$ on $\overline{\boldsymbol{S}} \ni \bar{\mu}$ is an extension of of $\mu_1$and μ_2; that is $\bar{\mu}(E_i) = \mu_i(E_i)$ for $E_i \in \boldsymbol{R_i}$ $i \in \{1, 2\}$. From Theorem 3.1, every set in $\boldsymbol{S_{1,2}}$ is in $\overline{\boldsymbol{S}}$.

Hence the measure $\bar{\mu}$ exists as claimed in this theorem (present theorem).

We now establish that $\bar{\mu}$ is unique.

Assume that $\exists$ two measures $\overline{\mu_a}$ and $\overline{\mu_b}$ on $\boldsymbol{S_{1,2}} \ni$ for E_i in $\boldsymbol{R_i}$
$\overline{\mu_a}(E_i) = \mu_i(E_i)$ and $\overline{\mu_b}(E_i) = \mu_i(E_i)$; $i \in \{1, 2\}$. Also, let

$$\boldsymbol{M} = \{E \mid E \in \boldsymbol{S_{1,2}} \text{ and } \overline{\mu_a}(E) = \overline{\mu_b}(E)\}.$$

We first consider the case where one of the measures $\overline{\mu_a}$ and $\overline{\mu_b}$ is finite. If $\{G_n\}$ is a monotone sequence in $\boldsymbol{M} \ni G_n = G_{n1} \cup G_{n2}$ for each $n \in$

$\{1, 2, \dots\}$, where $G_{n1} \in \boldsymbol{S}(\boldsymbol{R_1})$ and $G_{n2} \in \boldsymbol{S}(\boldsymbol{R_2})$, then $\{G_{ni}\}$ is a monotone sequence in $\boldsymbol{S}(\boldsymbol{R_i})$, for $i \in \{1, 2\}$.

$$\begin{aligned}\therefore \overline{\mu_a}[\lim_n(G_n)] &= \lim_n\,[\overline{\mu_a}(G_n)]\\ &= \lim_n[\overline{\mu_a}(G_{n1} \cup G_{n2}\,)]\\ &= \overline{\mu_a}[\lim_n(G_{n1} \cup G_{n2})]\\ &= \overline{\mu_b}\,[\lim_n(G_{n1} \cup G_{n2})]\\ &= \overline{\mu_b}[\lim_n(G_n)]\end{aligned}$$

Hence $\lim_n(G_n) \in \boldsymbol{M}$ and so $\boldsymbol{M}$ is a monotone class.

We next show that $\boldsymbol{M} = \boldsymbol{S_{1,2}}$.

$\boldsymbol{M}$ contains $\boldsymbol{R_1}$. Hence from Theorem B, Section 6, Chapter I of [3], it is clear that $\boldsymbol{M}$ contains $\boldsymbol{S}(\boldsymbol{R_1})$.

Let

$$\boldsymbol{H} \in \boldsymbol{S_{1,2}} \tag{3.5}$$

Then $\boldsymbol{H} = \boldsymbol{H_1} \cup \boldsymbol{H_2}$ for some $\boldsymbol{H_i} \in \boldsymbol{S}(\boldsymbol{R_i})$, for $i \in \{1, 2\}$.

Hence $\overline{\mu_a}(\boldsymbol{H_i}) = \overline{\mu_b}(\boldsymbol{H_i})$, for $i \in \{1, 2\}$.

Therefore $\overline{\mu_a}(\boldsymbol{H}) = \sum_{i=1}^{2} \overline{\mu_a}(\boldsymbol{H_i}) = \sum_{i=1}^{2} \overline{\mu_b}(\boldsymbol{H_i}) = \overline{\mu_b}(\boldsymbol{H_1} \cup \boldsymbol{H_2}\,) = \overline{\mu_b}(\boldsymbol{H}\,)$

Therefore

$$\boldsymbol{H} \in \boldsymbol{M} \tag{3.6}$$

From (3.5) and (3.6), $\boldsymbol{S_{1,2}} \subseteq \boldsymbol{M}$.

Since clearly $\boldsymbol{M} \subseteq \boldsymbol{S_{1,2}}$, $\boldsymbol{M} = \boldsymbol{S_{1,2}}$.

Hence for every set $E \in \boldsymbol{S_{1,2}}$, $\mu_a(E) = \mu_b(E)$.

Hence $\bar{\mu}$ is unique, as claimed in this theorem.

We now consider the general case (i. e. not necessarily finite).

Let $\Phi = \{P_1 \cup P_2 \mid P_1 \in \boldsymbol{R_1}, P_2 \in \boldsymbol{R_2}\}$

and let A be a fixed set in $\Phi \ni A = A_1 \cup A_2$, where $A_1 \in \boldsymbol{R_1}, A_2 \in \boldsymbol{R_2}$, and either $\mu_1(A_1)$ and $\mu_1(A_2)$ are finite or $\mu_2(A_1)$ and $\mu_2(A_2)$ are finite.

$\boldsymbol{R_1} \cap A_1$ is a ring and $\boldsymbol{S}(\boldsymbol{R_1}) \cap \boldsymbol{A_1}$ is the $\sigma -$ ring it generates (from Theorem E, Section 5, Chapter 1 of [3]. Similarly, $\boldsymbol{R_2} \cap A_2$ is a ring and $\boldsymbol{S}(\boldsymbol{R_2}) \cap \boldsymbol{A_2}$ is the $\sigma -$ ring it generates.

Now, let $\Psi = \{Q_1 \cup Q_2 \mid Q_i \in \boldsymbol{S}(\boldsymbol{R_i}) \cap \boldsymbol{A_i}; i = 1, 2\}$

Also, for $i \in \{1, 2\}$, let $\boldsymbol{R_i} \cap \boldsymbol{A_i}$ play the role of $\boldsymbol{R_i}$, $\boldsymbol{S}(\boldsymbol{R_i}) \cap \boldsymbol{A_i}$ play the role of $\boldsymbol{S}(\boldsymbol{R_i})$, and Ψ play the role of $\boldsymbol{S}(\boldsymbol{R_1}, \boldsymbol{R_2})$ in the earlier part of this proof. If we proceed as we did in this earlier part of the proof, we can conclude that for every set E in Ψ, $\mu_a(E) = \mu_b(E)$.

Let F be an arbitrary set in $\boldsymbol{S_{1,2}}$. Then $F = F_1 \cup F_2$, for some F_i in $\boldsymbol{S}(\boldsymbol{R_i})$, for $i \in \{1, 2\}$.

Also, $\exists$ some set $\dot{E}$ in $\Psi \ni \dot{E} = \dot{E}_1 \cup \dot{E}_2$, where $\dot{E}_i \in \boldsymbol{S}(\boldsymbol{R}_i) \cap \boldsymbol{A}_i$ and $\dot{E}_i = F_i \cap A_i$ for $i \in \{1,2\}$.

Now $F_1 \cap A_1 = C_1 \cup C_2 \cup \cdots$

[where $\{C_i \in \boldsymbol{R}_1 \mid i = 1,2,\ldots\}$ consists of mutually disjoint sets, each of which has finite measure with respect to μ_a or μ_b]

Also, $F_1 - A_1 = D_1 \cup D_2 \cup \cdots$

[where $\{D_i \in \boldsymbol{R}_1 \mid i = 1,2,\ldots\}$ consists of mutually disjoint sets, each of which has finite measure with respect to μ_a or μ_b].

Similarly, $F_2 - A_2 = V_1 \cup V_2 \cup \cdots$

[where $\{V_i \in \boldsymbol{R}_2 \mid i = 1,2,\ldots\}$ consists of mutually disjoint sets, each of which has finite measure with respect to μ_a or μ_b]

and $F_2 - A_2 = W_1 \cup W_2 \cup \cdots$

[where $\{W_i \in \boldsymbol{R}_2 \mid i = 1,2,\ldots\}$ consists of mutually disjoint sets, each of which has finite measure with respect to μ_a or μ_b].

Therefore $\mu_a(F) = \mu_a(F_1 \cup F_2) = \sum_{i=1}^{2} \mu_a(F_i)$

$$= \sum_{i=1}^{2} \mu_a[(F_i \cap A_i) \cup (F_i - A_i)]$$
$$= \sum_{i=1}^{\infty} \mu_a(D_i) + \sum_{i=1}^{\infty} \mu_a(C_i) + \sum_{i=1}^{\infty} \mu_a(W_i) + \sum_{i=1}^{\infty} \mu_a(V_i)$$
$$= \sum_{i=1}^{\infty} \mu_b(D_i) + \sum_{i=1}^{\infty} \mu_b(C_i) + \sum_{i=1}^{\infty} \mu_b(W_i) + \sum_{i=1}^{\infty} \mu_b(V_i)$$
$$= \sum_{i=1}^{2} \mu_b[(F_i \cap A_i) \cup (F_i - A_i)]$$
$$= \sum_{i=1}^{2} \mu_b(F_i)$$
$$= \mu_b(F)$$

Hence, since F is arbitrary in $\boldsymbol{S}_{1,2}$, μ_a and μ_b are equal for any set in $\boldsymbol{S}_{1,2}$.

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