

A computational technique for studying the Heisenberg group

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Abstract

A Heisenberg group is a polycyclic, which allows elements to be handled as collected forms, leading to the polycyclic presentation style. In this study, we examine various types of Heisenberg groups from a polycyclic perspective, determine matrix representation and construct conjugation action orbits and stabilizers.

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1. Introduction

Heisenberg proposed a commutation relation for observable positions and momentum in quantum physics, namely, $[Q, P] = I$, where I is the identity, giving rise to a group that bears his name, [1]. Because of Hermann Weyl's early pioneering work in this sector, physicists preferred the term Weyl group instead of Heisenberg [2]. The Heisenberg group can be described in two ways: symplectic and upper triangular group matrices [3]. Traditional harmonic analysis approaches were used to investigate the algebraic structure of the Heisenberg group ([4], [5]). In 1938, K.A. Hirsch began leading research on polycyclic group: a solvable group whose subgroups are all finitely generated ([6], [7], [8]). The fact that a Heisenberg group is a polycyclic, allows for effective computations, which is the focus of this note.

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The Heisenberg group considered as an upper triangular $n \times n$ matrices over a commutative ring, which represent a subgroup of the general linear group. Let $\mathcal{PC}$ denote the class of all polycyclic groups $\mathcal{FG}$ and $\mathcal{FS}$ be the classes of all finitely generated nilpotent and finitely generated solvable groups respectively, we have, $\mathcal{FG} \subseteq \mathcal{PC} \subseteq \mathcal{FS}$. A polycyclic group, notably the Heisenberg group, have finite presentation, known as polycyclic presentation (abbreviated as pc-presentation). Many attempts have been undertaken to computationally analyze polycyclic groups, including Sinanan and Holt's [9], Bettina's [10], and Höfling's [11], to name a few.

Recently, P. Gunjan, and M. Pruthi, Heisenberg and other groups have been investigated via the notion of unit graphs [12]. B. Senoussi utilizing a 3-dimensional Heisenberg group viewpoint, analyze the surfaces of the helicoidal invariant [13].

From a polycyclic standpoint, we investigate several types of Heisenberg groups, identify matrix representation, and construct conjugation action orbits and stabilizers.

In our implementations, we used the computer algebra system GAP (version 4.10.1 as of 23-Feb-2019), [14], which ran on a PC with 4.00 GB RAM and an Intel(R) 2.50 GHz processor operating under Windows 10.

2. Preliminaries and notations

Let F be any field such that $\text{har}(F) \neq 2$, we may define m -dimensional Heisenberg group as upper triangular matrices as follows:

$$\mathcal{H}^m = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & I_m & \gamma \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \dots & \alpha_m & \beta \\ 0 & 1 & 0 & \dots & 0 & \gamma_1 \\ 0 & 0 & \ddots & \dots & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 & \gamma_{m-1} \\ 0 & 0 & \dots & 0 & 1 & \gamma_m \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}_{m+2 \times m+2} \left| \alpha = (\alpha_1 \dots \alpha_m), \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix} \in F^m, \beta \in F \right\},$$

The pair $(\mathcal{H}^m, \cdot)$ forms a group since multiplication and inversion of such matrices yields matrices of the same type.

If $m=1$, for example, we get: $H^1 = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha, \beta, \gamma \in F \right\}$ set of all

3×3 upper triangular matrices. Another method for creating a Heisenberg group is to adopt a symplectic form, such as, Assume that ω is a symplectic form and define the set $\mathcal{H}_\omega^m$ by;

$\mathcal{H}_\omega^m = \{(\alpha, \beta, \gamma) \in F^{2m+1} \mid \alpha, \gamma \in F^m, \beta \in F\}$, with multiplication, identity, and inverse are given as follows;

$$(\alpha, \beta, \gamma) \cdot (\dot{\alpha}, \dot{\beta}, \dot{\gamma}) = (\beta + \dot{\beta} + \frac{1}{2}\omega(\alpha, \gamma; \dot{\alpha}, \dot{\gamma}), \alpha + \dot{\alpha}, \gamma + \dot{\gamma}),$$

$$(0, 0, 0) \text{ and } (\beta, \alpha, \gamma)^{-1} = (-\beta, -\alpha, -\gamma).$$

It's worth noting that $\mathcal{H}^1$ is isomorphic to F^3 , implying that the homomorphism:

$\varphi: \mathcal{H}^1 \rightarrow M_3(F)$, $\varphi(\beta, \alpha, \gamma) = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}$ exists and can be used as a

bridge when investigating the Heisenberg group.

Let G be a group, a polycyclic series of length n for G is a sequence of subgroups;

$G_1 \subseteq G_2 \supseteq \dots \supseteq G_n \supseteq G_{n+1} = \{e\}$, such that: $\forall i \in [1 \dots n]$ I) $G_{i+1} \trianglelefteq G_i$

II) G_i/G_{i+1} is cyclic, a group which has polycyclic series is called polycyclic group (pc-group).

A generating sequence for a polycyclic group G is a sequence $u_1, \dots, u_n$, $u_i \in G_i$, $1 \leq i \leq n$, where G_i are the polycyclic factors of the polycyclic series of G of length n such that $\langle u_i G_{i+1} \rangle = G_i/G_{i+1}$ and $G_i = \langle u_i, \dots, u_n \rangle$. Let $\{e\} \neq u \in G$, then $u = u_1^{\alpha_1} \dots u_n^{\alpha_n}$ is a word, where the exponents α_i 's, $i \in [1 \dots n]$, are integers. Such word called a collected form, the language of collected form words enables us to get unique relations for product of elements of the generating sequence and their inverses, and these relations comprise the standard polycyclic presentation. See ([7], [8], [13]) for further details.

3. Presentations of Heisenberg groups

In GAP, the function HeisenbergPcpGroup(n) can be used to convert a Heisenberg group to a polycyclic group. For example, if $n=7$, the resulting group has Hirsch length 15 and contains 15 polycyclic generators, $u_1, u_2, \dots, u_{15}$, which satisfy the conjugation relations $u_{7+i}^{a_i} = u_{7+i} u_{15}$ for $1 \leq i \leq 7$.

Table 1 shows several types of Heisenberg groups, together with their finitely presented (FP-presentation) and polycyclic presentations (PC-presentation).

Table 1
FP- presentation and PC- presentation for some Heisenberg groups

H_q^n	Order	FP- presentation	Orders of pcp-Series	PC- presentation
H_2^1	8	$\langle u_1, u_2, u_3 $ $u_1^2, u_2^2, u_3^2,$ $(u_1 u_3)^2, (u_2 u_3)^2,$ $u_3 (u_1 u_2)^2 \rangle$	[8,4,2,1]	$u_1^2 = 1$ $u_2^2 = 1$ $u_3^2 = 1$ $u_2^{a_1} = u_2 u_3$
H_3^1	27	$\langle u_1, u_2, u_3 $ $u_1^3, u_2^3, u_3^3,$ $u_1 u_2 u_1^{-1} u_3^{-1},$ $u_2 u_3 u_2^{-1} u_3^{-1},$ $u_3^{-1} u_1 u_2 u_1^{-1} u_2^{-1} \rangle$	[27,9,3,1]	$u_1^3 = 1$ $u_2^3 = 1$ $u_3^3 = 1$ $u_2^{a_1} = u_2 u_3^2$

H_5^1	125	$\langle u_1, u_2, u_3 $ $u_1^5, u_2^5,$ $u_1 u_3 u_1^{-1} u_3^{-1},$ $u_2 u_3 u_2^{-1} u_3^{-1},$ $u_3^{-1} u_1 u_2 u_1^{-1} u_2^{-1} \rangle$	[125, 25, 5, 1]	$u_1^5 = 1$ $u_2^5 = 1$ $u_3^5 = 1$ $u_2^{u_1} = u_2 u_3^4$
H_2^2	64	$\langle u_1, u_2, u_3, u_4, u_5, u_6 $ $u_1^2, u_2^2, u_3^2, u_4^2, u_5^2, u_6^2,$ $(u_1 u_3)^2, (u_1 u_4)^2,$ $(u_2 u_4)^2, (u_2 u_5)^2,$ $(u_3 u_5)^2, (u_4 u_5)^2,$ $(u_1 u_6)^2, (u_2 u_6)^2,$ $(u_3 u_6)^2, (u_4 u_6)^2,$ $(u_5 u_6)^2, u_4 (u_1 u_2)^2,$ $u_6 (u_1 u_5)^2,$ $u_5 (u_2 u_3)^2 \rangle$	[64, 32, 16, 8, 4, 2, 1]	$u_1^2 = 1$ $u_2^2 = 1$ $u_3^2 = 1$ $u_4^2 = 1$ $u_5^2 = 1$ $u_6^2 = 1$ $u_2^{u_1} = u_2 u_4$ $u_3^{u_2} = u_3 u_5$ $u_4^{u_3} = u_4 u_6$ $u_5^{u_1} = u_5 u_6$
H_2^3	1024	$\langle u_1, u_2, u_3, u_4, u_5, u_6 $ $u_1^2, u_2^2, u_3^2, u_4^2, u_5^2,$ $u_6^2, u_7^2, u_8^2,$ $u_9^2, u_{10}^2,$ $(u_1 u_3)^2, (u_1 u_4)^2, (u_2 u_4)^2$ $(u_1 u_5)^2, (u_2 u_5)^2, (u_4 u_5)^2,$ $(u_2 u_6)^2, (u_3 u_6)^2, (u_5 u_6)^2,$ $(u_1 u_7)^2, (u_3 u_7)^2, (u_4 u_7)^2,$ $(u_6 u_7)^2, (u_1 u_8)^2, (u_2 u_8)^2$ $(u_3 u_8)^2, (u_5 u_8)^2, (u_6 u_8)^2,$ $(u_7 u_8)^2, (u_2 u_9)^2, (u_3 u_9)^2,$ $(u_4 u_9)^2, (u_5 u_9)^2, (u_6 u_9)^2,$ $(u_7 u_9)^2, (u_8 u_9)^2, (u_1 u_{10})^2,$ $(u_2 u_{10})^2, (u_3 u_{10})^2, (u_4 u_{10})^2,$ $(u_5 u_{10})^2, (u_6 u_{10})^2, (u_7 u_{10})^2,$ $(u_8 u_{10})^2, (u_9 u_{10})^2$ $u_5 (u_1 u_2)^2, u_8 (u_1 u_6)^2,$ $u_{10} (u_1 u_9)^2, u_6 (u_2 u_3)^2,$ $u_2 u_7 u_2 u_9 u_7, u_7 (u_3 u_4)^2 \rangle$	[1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1]	$u_1^2 = 1$ $u_2^2 = 1$ $u_3^2 = 1$ $u_4^2 = 1$ $u_5^2 = 1$ $u_6^2 = 1$ $u_7^2 = 1$ $u_8^2 = 1$ $u_9^2 = 1$ $u_{10}^2 = 1$ $u_2^{u_1} = u_2 u_5$ $u_3^{u_2} = u_3 u_6$ $u_4^{u_3} = u_4 u_7$ $u_5^{u_3} = u_5 u_8$ $u_6^{u_1} = u_6 u_8$ $u_6^{u_4} = u_6 u_9$ $u_7^{u_2} = u_7 u_9$ $u_7^{u_5} = u_7 u_{10}$ $u_8^{u_4} = u_8 u_{10}$ $u_9^{u_1} = u_9 u_{10}$

Now, we restate some facts with their references, which provides a strategy for analyzing polycyclic groups.

Theorem 3.1 (Mal'cev): *A normal subgroup with a finite index exists for every polycyclic group. [6, Th. 15.1.6].*

Proposition 3.2: The subgroups and quotient groups inherit the polycyclic attribute from the parent group.

[7, Proposition 3.7] and [7, Proposition 3.6] respectively.

Table 2 shows the computation of the induced generating sequence (IGS) and polycyclic presentation (Pcp) for both a normal subgroup selected to the indicated Heisenberg groups and its associative factor group. Recall that D_m stands for the order $2m$ dihedral group and that C_n stands for the order n cyclic group.

Table 2
Pcp-presentation for normal and factor subgroups

H	N	H/N	H: N	IGS (N)	IGS(H/N)	Pcp (N)	Pcp(H/N)
H_2^1	C2	$C2 \times C2$	4	$\langle u_3 \rangle$	$\langle u_1, u_2 \rangle$	$u_1^2 = 1$	$u_1^2 = 1$ $u_2^2 = 1$
H_3^1	C3	$C3 \times C3$	9	$\langle u_3 \rangle$	$\langle u_1, u_2 \rangle$	$u_1^3 = 1$	$u_1^3 = 1$ $u_2^3 = 1$
H_5^1	C5	$C5 \times C5$	25	$\langle u_3 \rangle$	$\langle u_1, u_2 \rangle$	$u_1^5 = 1$	$u_1^5 = 1$ $u_2^5 = 1$
H_2^2	$C2 \times C2$	$C2 \times D8$	16	$\langle u_4, u_6 \rangle$	$\langle u_1, u_2, u_3, u_4 \rangle$	$u_1^2 = 1$ $u_2^2 = 1$	$u_1^2 = 1$ $u_2^2 = 1$ $u_3^2 = 1$ $u_4^2 = 1$ $u_3 u_2 = u_3 u_4$
H_2^3	$C2 \times C2 \times C2 \times C2$	$D8 \times D8$	64	$\langle u_6, u_7, u_8, u_9, u_{10} \rangle$	$\langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$	$u_1^2 = 1$ $u_2^2 = 1$ $u_3^2 = 1$ $u_4^2 = 1$	$u_1^2 = 1$ $u_2^2 = 1$ $u_3^2 = 1$ $u_4^2 = 1$ $u_5^2 = 1$ $u_6^2 = 1$ $u_2 u_1 = u_2 u_5$ $u_3 u_2 = u_3 u_6$ $u_4 u_3 = u_4 u_6$

4. Matrix representation

Assume that G is a group and that V is a vector space over a field F of dimension n . A homomorphism $\rho: G \rightarrow GL(V) \cong GL(n, F)$ is a matrix representation of G of degree n .

The Heisenberg group over rationales and integers are torsion free and Nilpotent groups, therefore the function UnitriangularMatrixRepresentation can be used to get the following results for Heisenberg groups over Z or rationales as subgroups, "upper triangular matrix group of $GL(3, Z)$ and $GL(4, Z)$ ", respectively;

We start with the Heisenberg group H with pc-presentation given by;

$$H = \langle u_1, u_2, u_3 \mid u_2^{u_1} = u_2 u_3^{-1}, u_2^{u_1^{-1}} = u_2 u_3 \rangle$$

$$\rho: H \rightarrow GL(3, Z),$$

$$\rho: (u_1) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rho: (u_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \rho: (u_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

The matrix representation for the Heisenberg H with the following pc -presentation;

$$H = \left\langle u_1, u_2, u_3, u_4, u_5, u_6 \left| \begin{array}{l} u_2^{u_1} = u_2 u_4^{-1}, u_2^{u_1^{-1}} = u_2 u_4, u_3^{u_2} = u_3 u_5^{-1}, u_3^{u_2^{-1}} = u_3 u_5, u_4^{u_3} = u_4 u_6, \\ u_4^{u_3^{-1}} = u_4 u_6^{-1}, u_5^{u_1} = u_5 u_6^{-1}, u_5^{u_1^{-1}} = u_5 u_6 \end{array} \right. \right\rangle$$

given by; $\rho: H \rightarrow GL(7, Z)$,

$$\begin{aligned} \rho: (u_1) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \rho: (u_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix} \\ \rho: (u_3) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} \rho: (u_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \\ \rho: (u_5) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \rho: (u_6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Note that Heisenberg over rationales or integers fields are infinite so the irreducibility of their matrix representations cannot be checked. Also, over finite fields they are not torsion free which is an essential condition to find their matrix representation by this method. GAP used lower triangular matrices instead of upper, which are isomorphic.

5. Orbit and stabilizer for actions of Heisenberg group

Let G be a polycyclic group acting on a set X . The question of determining the orbits and stabilizers of elements of X under the action of G is a crucial problem in computational group theory. Two possibilities arising according to the orbit size, finite or infinite. Recall that an elementary free abelian series is a normal series of a polycyclic group G with elementary or free abelian factors. The next sections compute orbits and stabilizers for a few Heisenberg groups

with finite orbits; the infinite case is significantly more difficult to deal with. We will concentrate on two cases.

Case I: Consider the Heisenberg group over the Integers,

$$H^1 = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha, \gamma, \beta \in Z \right\}, \text{ which has the } pc\text{-presentation;}$$

$H^1 = \langle u_1, u_2, u_3 \mid u_2^{u_1} = u_2 u_3^{-1}, u_2^{u_1^{-1}} = u_2 u_3 \rangle$, it has the following elementary free abelian series, $H^1 = \langle u_1, u_2, u_3 \rangle > \langle u_2, u_3 \rangle > \langle u_3 \rangle > \langle 1 \rangle$

Choose $N = \langle u_2, u_3 \rangle$ a free abelian subgroup of H^1 and define conjugation action σ of H^1 on N , we have;

$$\sigma(u_1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \sigma(u_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma(u_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$Stab_{H^1}([2,3]) = N$, the orbit $Orb_{H^1}([1,1], [1,2]) = \sigma(u_1^{-1})$. The orbit of a subgroup $\langle [1,0], [0,1] \rangle$ of Z^2 is H^1 itself.

Case II: The Heisenberg group H^3 over integers

$$H^3 = \left\{ \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & t \\ 0 & 1 & 0 & 0 & \gamma_1 \\ 0 & 0 & 1 & 0 & \gamma_2 \\ 0 & 0 & 0 & 1 & \gamma_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \alpha_i, \gamma_i, \beta \in Z, 1 \leq i \leq 3 \right\}$$

with pc -presentation;

$$H^3 = \left\langle u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10} \middle| \begin{array}{l} u_2^{u_1} = u_2 u_5^{-1}, u_2^{u_1^{-1}} = u_2 u_5 \\ u_3^{u_2} = u_3 u_6^{-1}, u_3^{u_2^{-1}} = u_3 u_6 \\ u_4^{u_3} = u_4 u_7^{-1}, u_4^{u_3^{-1}} = u_4 u_7 \\ u_5^{u_3} = u_5 u_8, u_5^{u_3^{-1}} = u_5 u_8^{-1} \\ u_6^{u_1} = u_6 u_8^{-1}, u_6^{u_1^{-1}} = u_6 u_8 \\ u_6^{u_4} = u_6 u_9, u_6^{u_4^{-1}} = u_6 u_9^{-1} \\ u_7^{u_2} = u_7 u_9^{-1}, u_7^{u_2^{-1}} = u_7 u_9 \\ u_7^{u_5} = u_7 u_{10}^{-1}, u_7^{u_5^{-1}} = u_7 u_{10} \\ u_8^{u_4} = u_8 u_{10}, u_8^{u_4^{-1}} = u_8 u_{10}^{-1} \\ u_9^{u_1} = u_9 u_{10}^{-1}, u_9^{u_1^{-1}} = u_9 u_{10} \end{array} \right\rangle$$

For which the elementary free abelian series is of length 11, pick a free abelian subgroup $N = \langle u_6, u_7, u_8, u_9, u_{10} \rangle$ of H^3 the conjugation action δ of H^3 on N given by:

$$\begin{aligned}
\delta(u_1) &= \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \delta(u_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
\delta(u_3) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \delta(u_4) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
\delta(u_5) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \delta(u_6) = \delta(u_7) = \delta(u_8) = \delta(u_9) = \delta(u_{10}) = I_5
\end{aligned}$$

Moreover, the stabilizer of the vector $[2, 3, 4, 0, -1]$ under the action δ is:
 $Stab_{H^3}([2, 3, 4, 0, -1]) = \langle u_2^2 u_3^7 u_4^3 u_5^4 u_6^{76} u_7^{-120} u_8^{81} u_9^{-855} u_{10}^{-735}, u_3, u_6, u_7, u_8, u_9, u_{10} \rangle$
 The orbit of the subgroup $K = \langle [1, 0, -1, 0, 0], [0, 1, 0, 0, 0] \rangle$ of Z^5 is:

$$Orb_{H^3}(K) = \left\langle u_3, u_6, u_7, u_8, u_9, u_{10} \begin{array}{l} u_2^{u_1} = u_2, u_2^{u_1^{-1}} = u_2, u_3^{u_1} = u_3, u_3^{u_1^{-1}} = u_3, \\ u_3^{u_2} = u_3, u_3^{u_2^{-1}} = u_3, u_4^{u_1} = u_4, u_4^{u_1^{-1}} = u_4, \\ u_4^{u_2} = u_4, u_4^{u_2^{-1}} = u_4, u_4^{u_3} = u_4, u_4^{u_3^{-1}} = u_4, \\ u_5^{u_1} = u_5, u_5^{u_1^{-1}} = u_5, u_5^{u_2} = u_5, u_5^{u_2^{-1}} = u_5, \\ u_5^{u_3} = u_5, u_5^{u_3^{-1}} = u_5, u_5^{u_4} = u_5, u_5^{u_4^{-1}} = u_5, \\ u_6^{u_1} = u_6, u_6^{u_1^{-1}} = u_6, u_6^{u_2} = u_6, u_6^{u_2^{-1}} = u_6, \\ u_6^{u_3} = u_6, u_6^{u_3^{-1}} = u_6, u_6^{u_4} = u_6, u_6^{u_4^{-1}} = u_6, \\ u_6^{u_5} = u_6, u_6^{u_5^{-1}} = u_6 \end{array} \right\rangle.$$

Conclusion

Heisenberg groups are significant in many scientific disciplines. For instance, it demonstrates how momentum in quantum physics is connected to observable positions. The finitely presented groups, which include polycyclic groups, are a major topic in computational group theory.

The computer algebra system "GAP" was used to computationally study Heisenberg groups since they are polycyclic and so finitely presented. We compute pc -presentations, determine the matrix representation, establish conjugacy actions on normal subgroups, and identify orbits and stabilizers as results.

We limit ourselves to finite cases in many situations. Working on infinite applications seems to continue to be a struggle. Our findings may prove helpful for those academics who are interested in computationally analyzing and

deciphering Heisenberg groups or, in a more general sense, the polycyclic group.

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