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Restricted size Ramsey numbers and restricted size Ramsey minimal graphs for matching versus wheel upto order six

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Abstract

The notion of the restricted size Ramsey number arises from the amalgamation of the theory of Ramsey numbers and the size Ramsey number for graphs. The concept of the (restricted) size Ramsey number of small graphs was first introduced by Harary and Miller in 1983. They derived specific values for certain pairs of small graphs, each having a maximum count of four vertices. In that same year, R. J. Faudree and J. Sheehan furthered the research and expanded the findings to encompass all the pairs of small graphs with a maximum order of four. The restricted size Ramsey number and size Ramsey number for all pairs of small forests of maximum order five was given by Lortz and Mengersen in 1998. In this study, our research endeavours to advance the understanding of the restricted size Ramsey number by ascertaining the restricted size Ramsey numbers and restricted size Ramsey minimal graphs involving $2K_2$ and wheel graph upto order six. More specifically, we find $r^*(2K_2, W_3) = r^*(2K_2, W_4) = 15$ and $r^*(2K_2, W_5) = 19$.

Subject Classification: *Primary 05C55, Secondary 05C15.*

Keywords: *Ramsey number, Size Ramsey number, Restricted size Ramsey number, Restricted size Ramsey minimal graph, $2K_2$, Wheel- graph.*

1. Introduction and Preliminaries

All the graphs used in this study are finite, undirected, and free of multiple edges and loops. We represent the set of all the vertices for the given graph F as V(F) and the set of edges by E(F). The cardinality of the set of vertices, V(F) (or order of the graph F), is represented as $\nu(F)$ and the cardinality of the set of

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edges, E(F) (or size of the graph F), is represented as $\varepsilon(F)$. The degree of a vertex u, represented as $d(u)$, refers to the count of edges that are incident to a vertex u in the graph and the least degree of vertices in a graph F is represented as $\delta(F)$. A complete graph, represented as K_n, is a graph with n vertices such that each pair of vertices is adjacent. A graph with n vertices which are connected, wherein all its vertices have degree two, is referred to as a cycle, C_n. A bipartite graph is a type of graph with the property that the set of vertices is partitioned into two disjoint and non-overlapping subsets such that all edges connect vertices from one subset to the other. A complete bipartite graph is a specific kind of bipartite graph where each vertex in one subset is connected to all vertices in the other subset. Let $\phi \neq V_1 \subseteq V(F)$ is the subgraph of F generated by the vertex set V_1 and the edge set contains those edges of F whose both ends are contained in V_1, denoted by $F(V_1)$. If G is a subgraph of F, it is denoted as $G \subseteq F$. Let H and G are graphs then, $H - G$ is a graph obtained by removing all the edges of graph G from the edge set of graph H if $G \subseteq H$. Let F is a graph and v is one of its vertex then, $F - v$ is a graph obtained by eliminating a vertex v along with all the edges incident to this vertex. For more terminologies in graphs, refer [10, 13, 14, 16].

Let P, Q and R are any finite and simple graphs. A graph R arrowing (P, Q), is represented as $R \rightarrow (P, Q)$, if in any given two-colouring (e.g., red-blue) of the edges in R, there exists either a red-coloured subgraph isomorphic to P or a blue-coloured subgraph isomorphic to Q. Otherwise, if the graph R does not satisfy the property of arrowing (P, Q), it is denoted as $R \nrightarrow (P, Q)$. If a graph P does not contain isolates, it implies that there are no vertices in the graph with a degree of zero. Given graph P and Q without isolates. If a complete graph of order m, K_m satisfies $K_m \longrightarrow (P, Q)$, then such smallest positive integer m is the Ramsey number (RN) of P and Q, denoted as $r(P, Q)$. If a graph R having size $\widehat{m}$ and satisfies $R \longrightarrow (P, Q)$, then such smallest positive integer $\widehat{m}$ is referred to as the size Ramsey number (SRN) of P and Q, represented as $\hat{r}(P, Q)$. If a graph R having order $r(P, Q)$ (i.e., m) and having size m^* satisfies $R \longrightarrow (P, Q)$, then such smallest positive integer m^* is referred to as the restricted size Ramsey number (RSRN) of P and Q, denoted as $r^*(P, Q)$. Restricted size Ramsey minimal graphs are those graphs that have the smallest possible number of vertices and size while still satisfying a certain RSRN property. Specifically, it refers to a graph that cannot be further reduced in size while preserving the desired property.

$2K_2$ refers to a matching consisting of two edges, where each edge connects two distinct vertices. A wheel graph, denoted as W_k, is the graph on $k + 1$ vertices, is constructed by adding a hub vertex, labelled as u, to a cycle C_k on k vertices.

The determination of the SRN of graphs was initially found by Erdos et al. [6]. The notion of the RSRN emerges from the combination of the theory of RN and the SRN for graphs. The survey conducted by Faudree and Schelp [19], along with the survey by Burr [22], presents intriguing findings regarding (restricted) SRN. Harary and Miller [8] were the first to introduce the (restricted) SRN for small graphs in their study on generalised Ramsey theory. Faudree and Sheehan [17, 18] carry on with the study on the (restricted) SRN for small graphs and they give comprehensive findings regarding the (restricted) SRN for all pair of small graphs with a maximum count of vertices (i.e., order) is four. The results on the SRN and the RSRN for all pairs of paths versus star-graphs of maximum order 5 were given by Lortz and Mengersen [20] in 1998. For more results on the RSRN of graphs, please refer to [2-4, 6, 11, 12, 21,24,25,26].

Chvatal and Harary's Theorem 1.1 [23] presents the value of the RN involving $2\,K_2$ and a graph Q that does not contain any isolated vertices. The following is the statement of Theorem 1.1. The result of this theorem provides the count of vertices (i.e., order) of the arrowing graph R which satisfies $R \longrightarrow (2\,K_2, W_n)$, where W_n represents a wheel graph of order upto six, in determining the RSRN $r^*(2\,K_2, W_n)$.

Theorem 1.1: [23] *For any graph Q without isolated vertices,*

$$r(2\,K_2, Q) = \begin{cases} \nu(Q) + 2 & Q \text{ is complete} \\ \nu(Q) + 1 & \text{otherwise,} \end{cases}$$

In order to determine the precise value of $r^*(2\,K_2, W_n)$, where $n = 3, 4$ and 5. To support our findings, we will make use of theorem 1.2 to prove some of our results.

Theorem 1.2: [5] *For $n \geq 3$,*

$$r^*(2\,K_2, K_n) = \begin{cases} \binom{n}{2} - 1, & n = 3 \\ \binom{n+2}{2}, & n \geq 4 \end{cases},$$

where $\binom{m}{n}$ is a combination of m objects taken n at a time.

We have the monotonicity property for a pair of graphs through the statement of the RSRN as follows. If $G_1 \subseteq H_1$ and $G_2 \subseteq H_2$, then

$$r^*(G_1, G_2) \leq r^*(H_1, H_2). \tag{1.1}$$

Recall that these inequalities for the RN of graphs is provided by Chvatal and Harary [23].

Silaban et al. [1] have given the RSRN for the path of order three versus the graph of order five. The RSRN for matching versus tree and triangle unicyclic graph with order 6 is given by E. Safitri et al. [7].

In this study, we aim to further investigate RSRN by focusing on the involvement of $2\,K_2$ and wheel graphs upto order six. Our objective is to determine the specific RSRN for these structures.

2. Main Results

The RSRN and restricted size Ramsey minimal graphs involving $2K_2$ and wheel upto order six are presented in this section. The aim of this study is to find $r^*(2\,K_2, W_n)$, where $n = 3, 4$ and 5. It is important to note that for any pair of graphs F and G, a known inequality states that $\varepsilon(F) + \varepsilon(G) - 1 \leq r^*(F, G) \leq$

$\binom{r(G,H)}{2}$, where $\binom{m}{n}$ is a combination of m objects taken n at a time. Lemma 2.1 and Lemma 2.2 provide the properties of graphs $F \longrightarrow (2\,K_2, H)$ for any graph H that does not contain any isolated vertices. Lemma 2.1 is an expansion of a lemma presented in the previous study by Mengenser and Oeckermann on matching-star Ramsey number [9] that specifically applies to the case where H is a complete bipartite graph $K_{1,n}$. It is important to note that Lemma 2.1 holds for any graph Q, and the proof can be conducted in a similar manner as the proof provided in the aforementioned study. We will utilise these lemmas as a foundation for proving our findings. By employing the properties outlined in these lemmas, we can establish the validity of our results.

Lemma 2.1. [5] *Let Q be any graph, then $R \longrightarrow (2\,K_2, Q)$ iff the following conditions satisfies:*

(i) $Q \subseteq R - v$ for all $v \in V(R)$ and

(ii) $Q \subseteq R - C_3$ for all C_3 in R.

Lemma 2.2: [5] *Consider a graph Q without isolated vertices. If $R \longrightarrow (2K_2, Q)$ and $v(R) = r(2K_2, Q)$, then it follows that the least degree of R, represented as $\delta(R)$, is at least 2.*

Theorem 2.3: *RSRN,* $r^*(2\,K_2, W_3) = 15$.

Proof: In order to find the RSRN, the initial step is to calculate the RN. Given that W_3 represents a complete graph of order 4 as shown in Figure 1. We can conclude, based on Theorem 1.1, that $r(2K_2, W_3) = 6$. Applying Theorem 1.2 for W_3 which is complete graph of order 4. We get $r^*(2\,K_2, W_3) = \binom{6}{2} = 15$. Thus, the RSRN for $2K_2$ and W_3 is determined as 15.

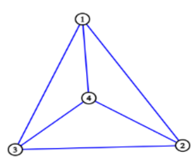


Figure 1
Wheel graph of order 4, W_3.

Corollary 2.4: *There is one restricted size Ramsey minimal graph for* $r^*(2\,K_2, W_3) = 15$*, as shown in Figure 2.*

Proof: As observed that there exists only one graph of order six and size fifteen and that graph arrows to $(2\,K_2, W_3)$, implies that there is one restricted size Ramsey minimal graph for $r^*(2\,K_2, W_3) = 15$, as shown in Figure 2.

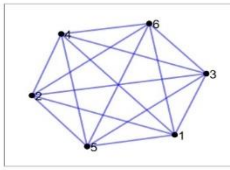


Figure 2
Minimal graph, M_1 of order 6 and size 15.

Theorem 2.5: *RSRN,* $r^*(2\,K_2, W_4) = 15$.

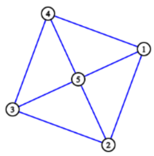


Figure 3
Wheel graph of order 5, W_4.

Proof: In order to find the RSRN, the initial step is to calculate the RN. Given that W_4 is a graph of order 5 as shown in Figure 3, based on Theorem 1.1, it is concluded that $r(2K_2, W_4) = 6$.

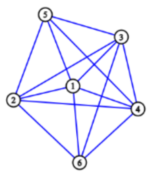


Figure 4
Graph of order six and size fourteen, G_1.

As we know that $\varepsilon(2\,K_2) + \varepsilon(W_4) - 1 \leq r^*(2\,K_2, W_4) \leq \binom{r(2\,K_2, W_4)}{2}$, which implies that $9 \leq r^*(2\,K_2, W_4) \leq 15$, beacause size of $2\,K_2$ is 2 and of W_4 is 8. Implies that $r^*(2\,K_2, W_4) \leq 15$.

It remains to prove that $r^*(2\,K_2, W_4) \geq 15$. It will be enough to show that there does not exist any graph of order 6 and size 14 which follows both the Lemmas 2.1 and 2.2 together.

As we know that there is only one possible non-isolated graph with an order of 6 and a size of 14, G_1 as depicted in the accompanying Figure 4. If we remove a cycle of order three, C_3, i.e. $C_3(1231)$ then we observe that, $W_4 \nsubseteq G_1 - C_3$ which implies that $G_1 \nrightarrow (2K_2, W_4)$, by Lemma 2.1. Therefore, we conclude that $r^*(2\,K_2, W_4) \geq 15$. Hence, $r^*(2\,K_2, W_4) = 15$.

Corollary 2.6: *There is one restricted size Ramsey minimal graph for* $r^*(2\,K_2, W_4) = 15$*, as shown in Figure 2.*

Proof: As observed that there exists only one graph of order six and size fifteen and that graph arrows to $(2\,K_2, W_4)$, implies that there is one restricted size Ramsey minimal graph for $r^*(2\,K_2, W_4) = 15$, as shown in Figure 2.

Theorem 2.7: *RSRN,* $r^*(2\,K_2, W_5) = 19$.

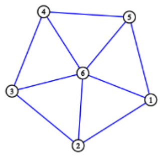
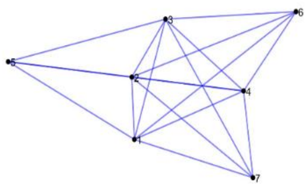


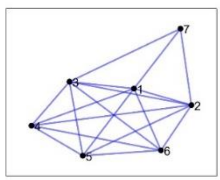
Figure 5
Wheel graph of order 6, W_5.

Proof: In order to find the RSRN, the first step is to calculate the RN. Given that W_5 is a graph of order 6, as shown in Figure 5. Concluded, based on Theorem 1.1, that $r(2K_2, W_5) = 7$.

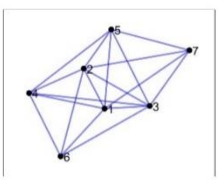
As we know that $\varepsilon(2K_2) + \varepsilon(W_5) - 1 \le r^*(2\,K_2, W_5) \le \binom{r(2\,K_2,W_5)}{2}$, which implies that $11 \le r^*\,(2\,K_2, W_5) \le 21$, beacause size of $2\,K_2$ is 2 and of W_5 is 10, as shown in Figure 5 and $\binom{r(2\,K_2,W_5)}{2} = 21$.



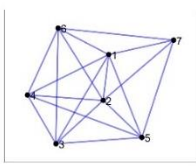
(A) F_1



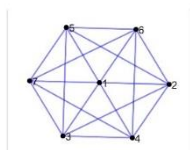
(B) F_2



(C) F_3



(D) F_4



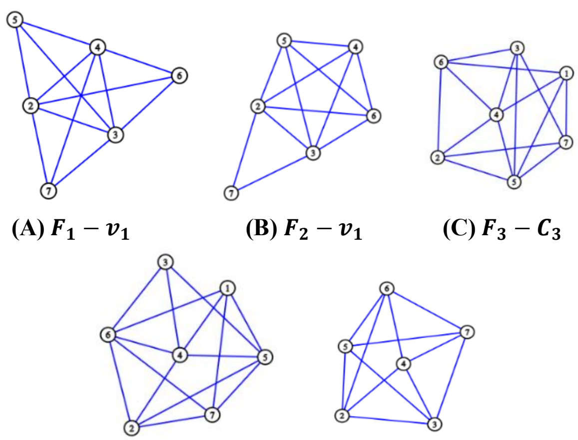
(E) F_5

Figure 6
List of all graphs of order 7 and size 18.

Firstly, we will show that $r^*(2\,K_2, W_5) \ge 19$. Given that there are five possible without isolates graphs with an order of 7 and a size 18, as depicted in the accompanying Figure 6. By removing vertex v_1 from graph F_1, we obtain $F_1 - v_1$ as shown in the Figure 7(A). In this case, it can be observed that

$W_5 \nsubseteq F_1 - v_1$. This implies that $F_1 \nrightarrow (2K_2, W_5)$ based on the application of Lemma 2.1. Similarly, by removing vertex v_1 from graphs F_2 and F_5, we obtain $F_2 - v_1$ and $F_5 - v_1$, respectively as shown in the Figure 7(B) and 7(E). In this case, it can be observed that $W_5 \nsubseteq F_2 - v_1$ and $W_5 \nsubseteq F_5 - v_1$. This implies that $F_2 \nrightarrow (2K_2, W_5)$ and $F_5 \nrightarrow (2K_2, W_5)$ based on the application of Lemma 2.1.

By removing a cycle of order three, C_3 from graph F_3, i.e., $C_3(1231)$, we obtain $F_3 - C_3(1231)$ as shown in the Figure 7(C). In this case, it can be observed that $W_5 \nsubseteq F_3 - C_3$. This implies that, $F_3 \nrightarrow (2K_2, W_5)$ based on the application of Lemma 2.1. Similarly, By removing a cycle of order three, C_3 from graph F_4, i.e. $C_3(1231)$, we obtain $F_4 - C_3(1231)$ as shown in Figure 7(D). In this case, it can be observed that $W_5 \nsubseteq F_4 - C_3$. This implies that $F_4 \nrightarrow (2K_2, W_5)$ based on the application of Lemma 2.1. Hence, all without isolates graphs of order 7 and size 18, i.e., $F_i, i = 1,2,\ldots,5$ does not arrow $(2K_2, W_5)$. Therefore, $r^*(2\,K_2, W_5) \geq 19$.



(A) $F_1 - v_1$ **(B) $F_2 - v_1$** **(C) $F_3 - C_3$**

(D) $F_4 - C_3$ **(E) $F_5 - v_1$**

Figure 7
Resultant graphs of $F_i, i = 1, 2..., 5$ after removing either of a vertex or any cycle of order three

It remains to show that $r^*(2\,K_2, W_5) \leq 19$. Given a graph M_2 with an order of 7 and a size 19 as depicted in the accompanying Figure 8. It is observed that $W_5 \subseteq M_2 - v_i$ for all $v_i \in V(M_2)$ and $W_5 \subseteq M_2 - C_3$ for all C_3 in M_2. This implies that $M_2 \longrightarrow (2K_2, W_5)$ by Lemma 2.1. Also, $\delta(M_2) \geq 2$, which follows Lemma 2.2. Therefore, $r^*(2\,K_2, W_5) \leq 19$. Hence, $r^*(2\,K_2, W_5) = 19$.

Corollary 2.8: *There is one restricted size Ramsey minimal graph for* $r^{*}(2\,K_2, W_5) = 19$ *i.e.* M_2, *as shown in Figure 8.*

Proof: As observed that there exist two graphs of order 7 and size 19, i.e. M_2 and M_3, as shown in Figure 8 and Figure 9, respectively. After applying Lemma 2.1 and Lemma 2.2 on both the graphs M_2 and M_3. Observe that $M_2 \rightarrow (2\,K_2, W_5)$, implies that M_1 is restricted size Ramsey minimal graph for $r^{*}(2\,K_2, W_3) = 15$. But after removing a cycle of order three from M_3, i.e. $C_3(1231)$, it is noted that $W_5 \nsubseteq F_2 - C_3(1231)$. It implies that $M_3 \nrightarrow (2\,K_2, W_5)$. So, M_3 is not restricted size Ramsey minimal graph for $r^{*}(2\,K_2, W_5) = 19$.

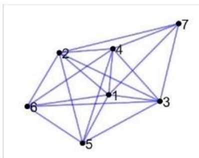


Figure 8
Minimal graph, M_2 of order 7 and size 19.

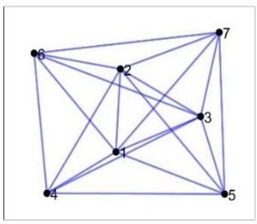


Figure 9
Graph, M_3 of order 7 and size 19.

3. Conclusion

This study presents the RSRN between matching and wheel graphs upto order six and their respective restricted size Ramsey minimal graphs. We conclude $r^{*}(2\,K_2, W_3) = r^{*}(2\,K_2, W_4) = 15$ and $r^{*}(2\,K_2, W_5) = 19$, where W_i is wheel- graph. In future, the RSRN and their minimal graphs involving $2\,K_2$, P_3 and unexplored non-isomorphic connected graphs can be investigated.

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References

[1] D.R. Silaban, E.T. Baskoro and S. Uttunggadewa, Restricted size Ramsey number for P_3 versus dense connected graphs of order six, AIP Conf. Proc. 1862, p. 030136 (2017), AIP Publishing LLC.

[2] D.R. Silaban, E.T. Baskoro and S. Uttunggadewa, Restricted size Ramsey number for path of order three versus graph of order five, *Elec. J. Graph Th. Appl.* 5, p. 55843 (2017).

[3] D.R. Silaban, E.T. Baskoro and S. Uttunggadewa, Restricted size Ramsey number for $2K_2$ versus dense connected graph of order six, *J. Phy.: Conf. Series* 1008, p. 012034 (2018).

[4] D.R. Silaban, E.T. Baskoro and S. Uttunggadewa, On the restricted size Ramsey number, Proc. Comp. Sci. 74, 21-26 (2015).

[5] D.R. Silaban, E.T. Baskoro and S. Uttunggadewa, Restricted size Ramsey number involving matching and graph of order five, *J. Fund. Sci.* 52, 133- 142 (2020).

[6] E. Safitri, P. John and D. R. Silaban, Restricted size Ramsey number for 2K_2 versus disconnected graph of order six, *J. Phy.: Conf. series* 1722, 012048 (2021).

[7] E. Safitri, P. John and D. R. Silaban, Restricted size Ramsey number for matching versus tree and triangle unicyclic graphs of Order six, (IJCSAM) *Int. J. Comp. Sci. Appl. Math*. 8, 17-21 (2022).

[8] F. Harary and Z. Miller, Generalized Ramsey theory VIII. the size Ramsey number of small graphs, *Stud. Pure Math.*, 271-283 (1983).

[9] I. Mengenser and J. Oeckermann, Matching- star Ramsey sets, *Disc. App. Math*. 95, 417-424 (1999).

[10] J. A. Bondy and U. S. R. Murty, Graph theory with applications (1982).

[11] J. Balogh, F. C. Clemen, E. Heath and M. Lavrov, Ordered size Ramsey number of paths, *Disc. App. Math.* 276, 13-18 (2020).

[12] J. Cyman and T. Dzido, Restricted size Ramsey number for P_3 versus cycle, *Elec. J. Graph Th. Appl.* 8, 365-372 (2020).

[13] M. H. Kadhim and R. K. K. Ajeena, Triple vertex of cycle graph for polyalphabetic encryption scheme, *J. Disc. Math. Sci. & Crypto.*, 26, 1183–1188 (2023).

[14] N. J. Khalel and N. E. Arif, Associate graph of a commutative ring, *J. Disc. Math. Sci. & Crypto.*, 26, 1883–1887 (2023).

[15] P. Erdos, R. J. Faudree, C. C. Rousseau, R. H. Schelp, The size Ramsey number, Period. Hung. 9, 145-61 (1978).

[16] R. Diestel, Graph theory, Springer- Verlag Heidelberg, New York, 4 edition, 2005.

[17] R. J. Faudree and J. Sheehan, Size Ramsey numbers involving stars, *Disc. Math.* 46, 151-157 (1983).

[18] R. J. Faudree and J. Sheehan, Size Ramsey numbers for small-order graphs, *J. Graph Th.* 7, 53-55 (1983).

[19] R. J. Faudree and R. H. Schelp, A survey of results on the size Ramsey number, Paul Erdos and his math. II (Budapest, 1999) 11, 291-309 (2002).

[20] R. Lortz and I. Mengersen, Size Ramsey results for paths versus stars, *Aus. J. Comb.* 18, 3-12 (1998).

[21] R. Lortz, I. Mengersen, Size Ramsey results for the path of order three, *Graphs and Comb*. 37, 2315-2331(2021).

[22] S. A. Burr, A survey of noncomplete Ramsey theory for graphs, in: Topics in graph theory, *Annals New York Acad. Sd.* 328, 58-75 (1979).

[23] V. Chvatal and F. Harary, Generalized Ramsey theory for graphs III: small off-diagonal numbers, *Pac. J. Math*. 41, 335-345 (1972).

[24] Kumar, Anil , Gupta, Amit Kumar , Panwar, Deepak , Chaurasia, Sandeep & Goyal, Dinesh. Operating system security with discrete mathematical structure for secure round robin scheduling method with intelligent time quantum, *Journal of Discrete Mathematical Sciences and Cryptography*, 26:5, 1519–1533 (2023).

[25] Kushwaha, Satpal Singh, Joshi, Sandeep & Gupta, Amit Kumar. An efficient approach to secure smart contract of Ethereum blockchain using hybrid security analysis approach, *Journal of Discrete Mathematical Sciences and Cryptography,* 26:5, 1499–1517 (2023), DOI: https://doi.org/10.47974/JDMSC-1815.

[26] Joshi, Ruchi, Mathur, Priya, Gupta, Amit Kumar, Singh, Suyesha, Paliwal, Vismita & Nayar, Sejal. Mathematical modeling of intelligent system for predicting effectiveness of premenstrual syndrome, *Journal of Interdisciplinary Mathematics*, 26:3, 551-562 (2023).