

On the combinatorial identities using the n^{th} power of square matrix

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Abstract

In this paper, we provide an independent proof for calculating the power of a matrix, where our proof complements the theorem established by McLaughlin in 2004 [10, p-3]. Additionally, we obtain some novel combinatorial identities from the above result.

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1. Introduction

The concept of the power of matrix holds the significant role in linear algebra. For finding applications in diverse fields such as mathematics, engineering, physics, neural processing, and machine learning. It plays a crucial role in solving systems of linear equations and computing eigenvalues and eigenvectors. In the century, Cayley developed the foundation for this discipline, and many mathematicians have thoroughly investigated it over the years. In 1998, Kwapisz [4] published a note detailing an elementary approach to calculating the powers of a matrix. During the same year, Saber and William [3] introduced alternative methods to compute for a nonsingular matrix which depends on the Cayley-Hamilton Theorem.

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Their paper presented the following theorem: If A is a $k \times k$ nonsingular matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_i$ and if $\phi_j(n)$ satisfies the system of linear differential equation

$$\begin{aligned}\phi_1(n+1) &= \lambda \phi_1(n), \phi_1(0) = 1, \phi_{j+1}(n+1) = \lambda_{j+1} \phi_{j+1}(n) + \phi_j(n), \\ \phi_{j+1}(0) &= 0, 0 \leq j \leq i-1.\end{aligned}$$

For $k \leq j$, $A^n = \sum_{j=0}^{i-1} \phi_{j+1}(n) M(j)$.

In 2004, Kwapisz [5] addressed the scenario of calculating the power of a matrix A when its determinant is zero. Around the same time, in 2004, McLaughlin [10] derived an explicit formula for the n^{th} power of a 2×2 matrix, which is dependent on its trace and determinant. McLaughlin also explored the use of the n^{th} power of specific matrices to derive various combinatorial identities. Furthermore, in 1993, Sury [12] noted a crucial observation concerning the subsequent polynomial identity:

$$\sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} \binom{t-k}{k} (rs)^k (r+s)^{t-2k} = \sum_{k=0}^t r^k s^{t-k}.$$

McLaughlin and Sury [6] derived a general polynomial identity involving k variables for $k \geq 2$. Using this identity, they conducted additional investigations into the expressions for the elements within the matrix raised to different powers. On a related note, Ma et al. [8] conducted a study focusing on the properties of matrix powers. They specifically characterized complex square matrices whose powers become and remain diagonal, Toeplitz and normal respectively.

In 2019, Karia et al. [4] derived a formula for $su(A^n)$ concerning a 2×2 matrix, where $n \in \mathbb{N}$ and $su(A^n)$ represent the sum of elements of matrix A . The obtained formula relied solely on the trace and determinant of A . Additionally, they applied their findings to derive the Marcus Newman inequality for a positive symmetric matrix $A \in M_2$ as following:

$$2su(A^3) \geq su(A)su(A^2).$$

N. Annamalai and C. Durairajan [13] investigated the codes generated through the analysis of the incidence matrix of zero-divisor graphs in various commutative rings with unity, demonstrating its practical application.

Moreover, In 2023, Mohammed Al-Dolat and Khaldoun Al-Zoubi [14] have established new upper limits for the numerical radius of off-diagonal operator matrices and extended the primary finding initially presented by Abu-Omar and Kittaneh. [15].

In 2021, Patil [11] obtained the formula for $Tr(A^n)$ by using the Cayley Hamilton Theorem. They studied the trace of negative powers of an invertible matrix. The trace of a matrix finds application in various fields, including network analysis, number theory, matrix theory, differential equations, as studied by Brezinski et al. [2]. The power of the matrix can also be used to compute the exponential of the matrix, which has crucial applications in differential equations. In 2022, Lee et al. [7] studied different methods for computing the matrix exponential using the different power of a matrix and developed a novel algorithm that reduces computational complexity compared to traditional methods. In 2022, Barai and Garge [1] investigated The trace and discriminant criteria are used to investigate whether a matrix can be expressed as the sum of sixth and eighth powers of other matrices.

These and several other applications motivate us to study the powers of a matrix. This paper aims to give independent proof of the theorem developed by McLaughlin [10] explored the derivation of a formula for the powers of a matrix, exploring combinatorial identities that incorporate trace and determinant is a significant area of study.

2. n^{th} Power of Matrix A

In this section, we present an autonomous proof for calculating the n^{th} power of a matrix $A \in M_2(\mathbb{K})$, where $(\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})$. Our proof stands apart from the theorem proposed by McLaughlin in 2004 [10, p-3]. Furthermore, we unveil a set of new combinatorial identities resulting from our findings.

Theorem 2.1: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T_A = Tr(A) = a + d$ and $D_A = \det(A) = ad - bc$. Then, for $n \geq 1$,

$$A^n = \begin{bmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_{n-1} & y_n - ay_{n-1} \end{bmatrix} \quad (1)$$

where

$$y_n = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} (T_A)^{n-2l} (D_A)^l. \quad (2)$$

Proof: We prove (2.1) by the principle of mathematical induction on n . Note that $y_1 = \text{Tr}(A) = a + d$ and $y_0 = 1$. Hence, the theorem holds true for $n = 1$. Additionally, for $n = 2$, we have

$$A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}.$$

And

$$\begin{aligned} y_2 &= \sum_{l=0}^1 (-1)^l \binom{2-l}{l} (T_A)^{2-2l} (D_A)^l \\ &= (T_A)^2 - D_A \\ &= a^2 + 2ad + d^2 - ad + bc. \end{aligned}$$

The right-hand side of equation (1) can be observed as,

$$\begin{bmatrix} y_2 - dy_1 & by_1 \\ cy_1 & y_2 - ay_1 \end{bmatrix}.$$

Putting the values of y_2 and y_1 and after simplifying we get,

$$\begin{bmatrix} a^2+bc & ab+bd \\ ca+cd & d^2+bc \end{bmatrix}.$$

Thus, the theorem holds for $n = 2$. Now we are assuming that the theorem holds true for $n = r$. Using the induction hypothesis and $A^{r+1} = A^r A$, we get

$$A^{r+1} = \begin{bmatrix} ay_r - ady_{r-1} + bcy_{r-1} & by_r - bdy_{r-1} + bdy_{r-1} \\ acy_{r-1} + cy_r - acy_{r-1} & bcy_{r-1} + dy_r - ady_{r-1} \end{bmatrix}.$$

The result will follow, if it can be shown that

$$y_{r+1} - dy_r = ay_r - D_A y_{r-1},$$

$$\text{i.e., } y_{r+1} = T_A y_r - D_A y_{r-1}.$$

Using (2), we have,

$$\begin{aligned} T_A y_m - D_A y_{m-1} &= \sum_{l=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^l \binom{r-l}{l} T_A^{r-2l+1} D_A^l + \sum_{l=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^{l+1} \binom{r-1-l}{l} T_A^{r-2l-1} D_A^{l+1} \\ &= \sum_{l=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^l \binom{r-l}{l} T_A^{r-2l+1} D_A^l + \sum_{l=1}^{\lfloor \frac{r-1}{2} \rfloor + 1} (-1)^l \binom{r-l}{l-1} T_A^{r-2l+1} D_A^l \\ &= T_A^{r+1} + \sum_{l=1}^{\lfloor \frac{r}{2} \rfloor} (-1)^l \binom{r-l}{l} T_A^{r-2l+1} D_A^l + \sum_{l=1}^{\lfloor \frac{r-1}{2} \rfloor + 1} (-1)^l \binom{r-l}{l-1} T_A^{r-2l+1} D_A^l. \end{aligned}$$

Using the Pascal Identity, $\binom{p-q}{q} + \binom{p-q}{q-1} = \binom{p-q+1}{q}$, we have,

$$\begin{aligned} T_A y_r - D_A y_{r-1} &= T_A^{m+1} + \sum_{l=1}^{\lfloor \frac{r+1}{2} \rfloor} (-1)^l \binom{r+1-l}{l} T_A^{r-2l+1} D_A^l \\ &= \sum_{l=0}^{\lfloor \frac{r+1}{2} \rfloor} (-1)^l \binom{r+1-l}{l} T_A^{r-2l+1} D_A^l \\ &= y_{r+1}. \end{aligned}$$

This completes the proof.

3. Combinatorial Identities

Throughout this section we use the following conventions:

$$\binom{0}{0} = 1 \quad \text{and} \quad \binom{n}{k} = \begin{cases} 0, & \text{if } n < k \\ 0, & \text{if } k \text{ is negative} \\ \frac{n!}{k!(n-k)!}, & \text{otherwise.} \end{cases}$$

The multinomial formula is defined as follows for any positive integer n nonnegative integer m ,

$$(y_1 + y_2 + y_3 + y_4 + \cdots + y_r)^n = \sum_{\eta_1 + \eta_2 + \eta_3 + \eta_4 + \cdots + \eta_r = n} \binom{n}{\eta_1, \eta_2, \eta_3, \eta_4, \dots, \eta_r} \prod_{1 \leq t \leq r} y_t^{\eta_t}$$

$$\text{Where } \binom{n}{\eta_1, \eta_2, \eta_3, \eta_4, \dots, \eta_r} = \frac{n!}{\eta_1! \eta_2! \eta_3! \eta_4! \dots \eta_r!}.$$

Theorem 3.1: Let $n \in \mathbb{N}$ and $1 \leq t \leq n$. Then

$$\binom{n}{t} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{t-1} (-1)^k \binom{k}{j} \binom{n-k-1}{k} \binom{n-1-2k}{t-j-1} 2^{n-2k-t+j}.$$

Proof: Let $A = \begin{bmatrix} u+2v & 1 \\ 0 & u \end{bmatrix}$. Here $\text{Tr}(A) = 2(y+x)$ and $\det(A) = x(x+2y)$. Observed that

$$A^n = \begin{bmatrix} (u+2v)^n & \frac{(2v+u)^n - u^n}{2y} \\ 0 & u^n \end{bmatrix}. \quad (3)$$

Equating the $(1, 2)$ matrix entry from the equations (3) and (1), we get

$$(2v + u)^n - u^n = 2v \sum_{w=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^w \binom{n-1-w}{w} T_A^{n-2w-1} D_A^w.$$

Putting the values of $T_A = 2(v + u)$ and $D_A = u(u + 2v)$ we get,

$$(2v + u)^n = u^n + 2v \sum_{w=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^w \binom{n-1-w}{w} 2^{n-2w-1} (v + u)^{n-2w-1} u^w (2v + u)^w.$$

Now, by replacing v by $\frac{v}{2}$ and setting $u = 1$ we get

$$(v + 1)^n - 1 = v \sum_{w=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^w \binom{n-1-w}{w} (v + 2)^{n-1-2w} (v + 1)^w. \quad (4)$$

Observe that

$$\begin{aligned} & (2 + v)^{n-2w-1} (1 + v)^w \\ &= \left[2^{n-1-2w} + \binom{n-2w-1}{1} 2^{n-2-2w} v^1 + \binom{n-1-2w}{2} 2^{n-2w-3} v^2 + \dots + v^{n-2w-1} \right] \\ & \quad \left[1 + \binom{w}{1} v + \binom{w}{2} v^2 + \dots + v^w \right] \\ &= \sum_{r=0}^{n-w-1} \sum_{j=0}^r \binom{w}{j} \binom{n-2w-1}{r-j} 2^{n-2w-r+j-1} v^r. \end{aligned}$$

Thus, (4) becomes,

$$(v + 1)^n - 1 = \sum_{w=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{r=0}^{n-w-1} \sum_{j=0}^r (-1)^w \binom{n-1-w}{w} \binom{w}{j} \binom{n-2w-1}{r-j} v^{r+1} 2^{n-2w-r+j-1}.$$

Using the Binomial expansion on the left hand side of the above equation and equating the coefficient of v^t from both the sides, we get

$$\binom{n}{t} = \sum_{w=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{t-1} (-1)^w \binom{n-1-w}{w} \binom{w}{j} \binom{n-2w-1}{t-j-1} 2^{n-2w-t+j}.$$

This completes the proof.

Theorem 3.2: Let α be any natural number ($\alpha \in \mathbb{N}$). Also for $1 \leq t \leq \alpha$. Then

$$\begin{aligned} \binom{\alpha}{t} &= \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{\alpha-r-1}{\alpha-r-t, \alpha-2r-1, 2r-\alpha+t} \\ & \quad + 2 \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{\alpha-r-1}{\alpha-r-1-t, \alpha-1-2r, 2r+t+1-\alpha}. \end{aligned}$$

Proof: Let $A = \begin{bmatrix} p+kq & 1 \\ 0 & (k-1)q \end{bmatrix}$. Here, $T_A = k(q+p)$ and $D_A = p(p+kq)(k-1)$. Observe that

$$A^\alpha = \begin{bmatrix} (kq+p)^\alpha & \frac{(p+kq)^\alpha - p^\alpha (k-1)^\alpha}{2p+kq-kp} \\ 0 & p^\alpha (k-1)^\alpha \end{bmatrix}. \quad (5)$$

Now equate the (1, 2) matrix entry from the equation (5) and (1) we get,

$$\frac{(p+kq)^\alpha - p^\alpha (k-1)^\alpha}{2p+kq-kp} = \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} (-1)^r \binom{\alpha-1-r}{r} T_A^{\alpha-2r-1} D_A^r.$$

Putting the values of T_A and D_A and simplifying we get.

$$(p+kq)^\alpha = p^\alpha (k-1)^\alpha + (2p+kq-kp) \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} (-1)^r \binom{\alpha-1-r}{r} k^{\alpha-2r-1} \times (p+q)^{\alpha-2r-1} (k-1)^r p^r (p+kq)^r.$$

Now, by replacing q by $\frac{q}{k}$ and setting $p=1$ we get,

$$(1+q)^\alpha = (k-1)^\alpha + (2-k+q) \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} (-1)^r \binom{\alpha-1-r}{r} (k+q)^{\alpha-2r-1} (k-1)^r (1+q)^r. \quad (6)$$

Comparing the constant term from both the sides of equation (6), we get

$$\begin{aligned} (k-1)^\alpha - (k-2) \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} (-1)^r \binom{\alpha-1-r}{r} k^{\alpha-1-2r} (k-1)^r &= 1 \\ \Rightarrow (k-1)^\alpha &= 1 + (k-2) \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} (-1)^r \binom{\alpha-1-r}{r} k^{\alpha-2r-1} (k-1)^r \\ \Rightarrow (k-1)^\alpha &= 1 + \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=0}^r (-1)^r \binom{\alpha-1-r}{r} k^{\alpha-2r} (-1)^j \binom{r}{j} k^{r-j} \\ &\quad - 2 \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=0}^r (-1)^r \binom{\alpha-1-r}{r} k^{\alpha-2r-1} (-1)^j \binom{r}{j} k^{r-j} \\ \Rightarrow (k-1)^\alpha &= 1 + \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=0}^r (-1)^{r+j} \binom{\alpha-1-r}{r} \binom{r}{j} k^{\alpha-r-j} \\ &\quad - 2 \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=0}^r (-1)^{j+r} \binom{\alpha-1-r}{r} \binom{r}{j} k^{\alpha-1-r-j}. \end{aligned}$$

Now, equating the coefficient of k^t from both the sides, we get

$$(-1)^{t+\alpha} \binom{\alpha}{t} = \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} (-1)^{\alpha-t} \binom{n-1-r}{r} \binom{r}{\alpha-t-r} - 2 \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} (-1)^{\alpha-t-1} \binom{\alpha-1-r}{r} \binom{r}{\alpha-r-t-1}.$$

This implies that

$$\begin{aligned} \binom{\alpha}{t} &= \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{\alpha-r-1}{\alpha-r-t, \alpha-2r-1, 2r-\alpha+t} \\ &\quad + 2 \sum_{r=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{\alpha-r-1}{\alpha-r-1-t, \alpha-1-2r, 2r+t+1-\alpha}. \end{aligned}$$

The proof is now finished.

4. Conclusion

This paper explores the application of the n^{th} power of a matrix to derive diverse combinatorial identities..

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