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Fractional differential equation with movable boundary conditions

Babli Yadav *
Trilok Mathur †
Shivi Agarwal §
Ashish Yadav ‡
Department of Mathematics
Birla Institute of Technology and Science Pilani
Pilani Campus
Rajasthan 333031
India

Abstract

In this research paper, we discuss the complex-valued solutions for the nonlinear fractional boundary value problem (FBVP) of complex order ($\delta = \tau + \iota a; 1 < \tau \leq 2, a \in \mathbb{R}^+$) with movable boundary conditions. The fractional operators are taken in the sense of Riemann-Liouville (R-L) with complex order. By using the concept of Green's function, the existence and uniqueness of solutions are established in this article. Also, we prove that the FBVP of complex order with movable boundary conditions is Ulam-Hyers Stable. Using illustrative examples, the results for this nonlinear FBVP have been shown.

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Keywords: *Complex order R-L fractional integral, Complex order R-L fractional derivative, Gamma function, Fixed point theorem, Green's function, Contraction mapping, Stability.*

1. Introduction

Since the 17th century, pioneering works by Leibniz, Liouville, Euler, Riemann, Letnikov, Lagrange, Able, Lacroix, and others have contributed to the development of fractional calculus [1-4], which generalizes differentiation and integration to arbitrary order. The development of the study of fractional differential equations in recent research is evidence of the importance and application of fractional calculus in science and engineering. On the other hand,

* *E-mail:* `p20200046@pilani.bits-pilani.ac.in (Corresponding Author)`
† *E-mail:* `tmathur@pilani.bits-pilani.ac.in`
§ *E-mail:* `shivi@pilani.bits-pilani.ac.in`
‡ *E-mail:* `as20111995rao@gmail.com`

given the wide range of applications of fractional calculus to natural phenomena, including chemical physics [5], bioengineering [6], electrical networks [7], neural networks [8,9], porous media [10], and viscoelasticity [11] it attracted the attention of many scholars, see articles [12-14]. Xie et al. [15] studied the existence of extremal solutions for an FBVP

$$\begin{cases} \mathbf{D}_{0^+}^{\alpha} v(y) + h(y, v(y)) = 0; y \in [0,1] \\ v(0) = 0, g\big(v(1), v(\eta)\big) = 0; \eta \in (0,1) \end{cases} \tag{1}$$

with three-point nonlinear boundary conditions. Where $\mathbf{D}_{0^+}^{\alpha}$ is the $\alpha^{th}, (\alpha \in (1,2))$ order $\mathrm{R-L}$ fractional derivative, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $h \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Yadav et al. [14] studied the existence-uniqueness of the following FBVP

$$\begin{cases} \mathbf{D}_{u}^{\beta,\gamma} h(u) = g(u, h(u)), 0 \leq \beta < 1, \gamma \geq 0 \\ nh(0) + mh(1) = ph(\xi), u, \xi \in \Omega; m + n \neq p \end{cases} \tag{2}$$

of order β and m, n, p are real constants and $\Omega = \{u \in \mathbb{C}: |u| < 1\}$, with the analytic functions g and h defined in the unit disk Ω. The order of differentiation was real in each of the aforementioned work, we consider the complex-order $(\delta = \tau + \iota a)$ nonlinear FBVP with movable boundary conditions

$$\begin{cases} \mathbf{D}_{0^+}^{\delta} r(y) = q(y, r(y)); y \in [0,1] \\ r(0) = 0, f(r(1), r(\xi)) = 0; \xi \in (0,1) \end{cases} \tag{3}$$

where $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}), q \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and $\mathbf{D}_{0^+}^{\delta}$ is the standard R-L fractional derivative of order $\delta, \delta = \tau + \iota a; 1 < \tau \leq 2, a \in \mathbb{R}^+$. Our objective is to demonstrate the existence of a complex-valued solution to the nonlinear FBVP (3) of complex order.

The remaining portion of the article is divided as follows: In section 2, we provide some fundamental definitions and introductions to fractional derivatives and integrals. The corresponding Green's function and integral equation for the problem (3) were then obtained. In section 3, the existence and uniqueness of complex-valued solution will be determined by making some assumptions and applying fixed-point theorems [21-23]. The Ulam-Hyers stability of the nonlinear FBVP(3) is discussed in section 4. And in the final section, an illustration is provided.

2. Basic Definitions and Preliminaries

Some important definitions and lemmas of fractional calculus required in this manuscript, are provided here. Throughout the article, we assume that $\delta = \tau + \iota a, \tau \in (1,2], a \in \mathbb{R}^+, \xi \in (0,1)$.

Definition 2.1: [1] The R-L fractional integral of order $\omega \in \mathbb{C}, (\mathrm{Re}(\omega) > 0)$ of a function $g: (0, \infty) \to \mathbb{R}$ is

$$\mathbf{I}_{0^+}^{\omega} g(y) = \frac{1}{\Gamma(\omega)} \int_0^y (y - \eta)^{\omega - 1} g(\eta) d\eta$$

Definition 2.2: [1] The R-L fractional derivative of order $\omega \in \mathbb{C}, (\mathrm{Re}(\omega) > 0)$ of a function $g: (0, \infty) \to \mathbb{R}$ is

$$\mathbf{D}_{0^+}^{\omega} g(y) = \frac{1}{\Gamma(m-\omega)} \frac{d^m}{dy^m} \int_0^y \frac{g(\eta)}{(y-\eta)^{\omega-m+1}} d\eta$$

where $m = [\mathrm{Re}(\omega)] + 1$.

Definition 2.3: [16] The Stirling asymptotic formula of the Gamma function for $u \in \mathbb{C}$ is following

$$\Gamma(u) = (2\pi)^{\frac{1}{2}} u^{u-\frac{1}{2}} e^{-u} \left[1 + O\left(\frac{1}{w}\right)\right];\ (|\arg(u)| < \pi; |u| \to \infty),$$

and its result for $|\Gamma(c + id)|, (c, d \in \mathbb{R})$ is

$$|\Gamma(c + id)| = (2\pi)^{\frac{1}{2}} |d|^{c-\frac{1}{2}} e^{-c-\frac{\pi|d|}{2}} \left[1 + O\left(\frac{1}{d}\right)\right]$$

Definition 2.4: [17] The mapping $\mathcal{J}: \mathfrak{B} \to \mathfrak{B}$ on a Banach space ($\mathfrak{B}$) is called contractive if there exists a constant $Q \in (0,1)$ such that

$$\|\mathcal{J}r - \mathcal{J}r'\| \le Q\|r - r'\| \tag{4}$$

Definition 2.5: [18] The complex order fractional differential equation

$$\mathbf{D}_{0^+}^{\delta} r(y) = q(y, r(y)); y \in [0,1] \tag{5}$$

is Ulam-Hyers stable if there exists a positive real number k_q such that for each $\epsilon > 0$ and for each solution $p(y) \in C([0,1], \mathbb{C})$ of the inequality

$$\left|\mathbf{D}_{0^+}^{\delta} p(y) - q(y, p(y))\right| \le \epsilon; y \in [0,1] \tag{6}$$

there exists a solution $r(y) \in C([0,1], \mathbb{C})$ of equation (5) with

$$|p(y) - r(y)| \le k_q \epsilon, y \in [0,1]$$

Lemma 2.1: [19] *Let $r \in L(0,1) \cap C(0,1)$ with a fractional derivative of order $\omega \in \mathbb{C}, (Re(\omega) > 0)$ that also belongs to $L(0,1) \cap C(0,1)$. Then*

$$\mathrm{I}_{0^+}^{\omega} \mathrm{D}_{0^+}^{\omega} r(y) = r(y) + \lambda_1 y^{\omega-1} + \lambda_2 y^{\omega-2} + \cdots + \lambda_m y^{\omega-m} \tag{7}$$

for some $\lambda_j \in \mathbb{R}, j = 1,2, \dots, m$, where $m = [\mathrm{Re}(\omega)] + 1$.

Lemma 2.2: *Let $g(y) \in C[0,1], \alpha \in (0,1], \theta \in \mathbb{R}, \xi \in (0,1)$. Then the FBVP*

$$\begin{cases} \mathbf{D}_{0^+}^{\delta} r(y) = g(y); y \in [0,1] \\ r(0) = 0, r(1) = \alpha r(\xi) + \theta \end{cases} \tag{8}$$

is equivalent to

$$\begin{aligned} r(y) = \int_0^y \frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta) d\eta - \frac{1}{(1-\alpha\xi^{\delta-1})} \int_0^1 \frac{[y(1-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta) d\eta \\ + \frac{\alpha}{(1-\alpha\xi^{\delta-1})} \int_0^{\xi} \frac{[y(\xi-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta) d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})} \end{aligned} \tag{9}$$

Proof: We can apply Lemma 2.1 to reduce FBVP (8) in an equivalent integral equation

$$r(y) = \mathbf{I}_{0^+}^{\delta} g(y) + k_1 y^{\delta-1} + k_2 y^{\delta-2}, \text{ for some } k_1, k_2 \in \mathbb{R}$$
$$\Rightarrow r(y) = \int_0^y \frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + k_1 y^{\delta-1} + k_2 y^{\delta-2} \tag{10}$$

Since $r(0) = 0$, therefore k_2 must be zero as $(\mathrm{Re}(\delta) - 2) \le 0$ then

$$r(y) = \int_0^y \frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + k_1 y^{\delta-1}$$

Now, by using the condition $r(1) = \alpha r(\xi) + \theta$, we have

$$\int_0^1 \frac{(1-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + k_1 = \alpha \int_0^{\xi} \frac{(\xi-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \alpha k_1 \xi^{\delta-1} + \theta$$
$$\Rightarrow k_1\left(1 - \alpha\xi^{\delta-1}\right) = -\int_0^1 \frac{(1-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \alpha \int_0^{\xi} \frac{(\xi-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \theta$$
$$\Rightarrow k_1 = \frac{-1}{(1-\alpha\xi^{\delta-1})} \int_0^1 \frac{(1-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \frac{\alpha}{(1-\alpha\xi^{\delta-1})} \int_0^{\xi} \frac{(\xi-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \frac{\theta}{(1-\alpha\xi^{\delta-1})}$$

Therefore, the unique solution of FBVP (8) is given by (9).

Conversely, let $r(y)$ be given by formula (9). Since, $\mathbf{D}_{0^+}^{\delta} r(y) = \mathbf{D}^2 \mathbf{I}_{0^+}^{2-\delta} r(y)$. So, by operating $\mathbf{I}_{0^+}^{2-\delta}$ on both sides of (9), we get

$\mathrm{I}_{0^+}^{2-\delta}\mathrm{r(y)} = \mathrm{I}_{0^+}^{2}\mathrm{g(y)} + \kappa\mathrm{y} \Rightarrow \mathrm{D}^2\mathrm{I}_{0^+}^{2-\delta}\mathrm{r(y)} = \mathrm{D}^2\{\mathrm{I}_{0^+}^{2}\mathrm{g(y)}\} + \kappa\mathrm{D}^2\mathrm{(y)} \Rightarrow \mathrm{D}_{0^+}^{\delta}\mathrm{r(y)} = \mathrm{g(y)}$

Green's function of nonlinear FBVP (8) is presented in the following lemma.

Lemma 2.3: *Let* $g(y) \in C[0,1], \alpha \in (0,1], \theta \in \mathbb{R}$. *Then the* $FBVP(8)$ *is equivalent to*

$$r(y) = \int_0^1 G(y,t)g(\eta)d\eta + \frac{\theta y^{\delta-1}}{1-\alpha\xi^{\delta-1}} \tag{11}$$

$$G(y,\eta) = \begin{cases} \dfrac{\alpha[y(\xi-\eta)]^{\delta-1} + (1-\alpha\xi^{\delta-1})(y-\eta)^{\delta-1} - [y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)}, 0 \le \eta \le y \le 1, \eta \le \xi \\ \dfrac{(1-\alpha\xi^{\delta-1})(y-\eta)^{\delta-1} - [y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)}, 0 < \xi \le \eta \le y \le 1, \\ \dfrac{\alpha[y(\xi-\eta)]^{\delta-1} - [y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)}, 0 \le y \le \eta \le \xi < 1 \\ \dfrac{-[y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)}, 0 \le y \le \eta \le 1, \xi \le \eta \end{cases} \tag{12}$$

Proof: By Lemma 2.2 the unique solution of FBVP (8) is

$$r(y) = \int_0^y \frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta - \frac{1}{(1-\alpha\xi^{\delta-1})}\int_0^1 \frac{[y(1-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta$$
$$+\frac{\alpha}{(1-\alpha\xi^{\delta-1})}\int_0^\xi \frac{[y(\xi-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})}$$

Now, we only need to prove that (9) can be written as (11).

Case (i) $y \le \xi$,

$$r(y) = \int_0^y \frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta - \frac{1}{(1-\alpha\xi^{\delta-1})}\left(\int_0^y + \int_y^\xi + \int_\xi^1\right)\frac{[y(1-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta +$$
$$\frac{\alpha}{(1-\alpha\xi^{\delta-1})}\left(\int_0^y + \int_y^\xi\right)\frac{[y(\xi-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \frac{\theta}{(1-\alpha\xi^{\delta-1})} y^{\delta-1}$$

$\Rightarrow r(y)$

$$= \int_0^y \frac{(1-\alpha\xi^{\delta-1})(y-\eta)^{\delta-1} + \alpha[y(\xi-\eta)]^{\delta-1} - [y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)} g(\eta)d\eta$$
$$+ \int_y^\xi \frac{\alpha[y(\xi-\eta)]^{\delta-1} - [y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)} g(\eta)d\eta + \int_\xi^1 \frac{-[y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)} g(\eta)d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})}$$
$$= \int_0^1 G(y,\eta)g(\eta)d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})}$$

Case (ii) $y \ge \xi$

$$r(y) = \left(\int_0^\xi + \int_\xi^y\right)\frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta - \frac{1}{(1-\alpha\xi^{\delta-1})}\left(\int_0^\xi + \int_\xi^y + \int_y^1\right)\frac{[y(1-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta$$
$$+ \frac{\alpha}{(1-\alpha\xi^{\delta-1})}\int_0^\xi \frac{[y(\xi-\eta)]^{\delta-1}}{\Gamma(\delta)} g(\eta)d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})}$$
$$= \int_0^\xi \frac{(1-\alpha\xi^{\delta-1})(y-\eta)^{\delta-1} + \alpha[y(\xi-\eta)]^{\delta-1} - [y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)} g(\eta)d\eta$$
$$+ \int_\xi^y \frac{(1-\alpha\xi^{\delta-1})(y-\eta)^{\delta-1} - [y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)} g(\eta)d\eta$$
$$+ \int_y^1 \frac{-[y(1-\eta)]^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)} g(\eta)d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})} = \int_0^1 G(y,\eta)g(\eta)d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})}$$

From case(i) & case(ii), we have $G(y,\eta)$ defined in (12) .

3. Existence and Uniqueness Results

Suppose that $\mathfrak{B} = C[0,1]$ is a Banach space of continuous functions with the norm

$\| r(y) \| = \max_{0\le y\le 1}|r(y)|$. Define the operator $\mathcal{J}: \mathfrak{B} \to \mathfrak{B}$ as follows:

$$(\mathcal{J}r)(y) = \frac{1}{\Gamma(\delta)}\int_0^y (y-\eta)^{\delta-1}q(\eta,r(\eta))d\eta - \frac{y^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)}\int_0^1 (1-\eta)^{\delta-1}q(\eta,r(\eta))d\eta + \frac{\alpha y^{\delta-1}}{(1-\alpha\xi^{\delta-1})\Gamma(\delta)}\int_0^\xi (\xi-\eta)^{\delta-1}\, q(\eta,r(\eta))d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})} \tag{13}$$

Lemma 2.2 states that, the solutions of the FBVP (3) and the fixed points of the operator $\mathcal{J}$ are equivalent.

Theorem 3.1: *Under the assumptions:*

$(\mathbf{Q_1})$ $q \in C([0,1] \times [0,\infty), [0,\infty))$

$(\mathbf{Q_2})$ $\forall y \in [0,1], r, r' \in \mathfrak{B}$, there exists a constant $Q > 0$ such that

$$|q(y, r(y)) - q(y, r'(y))| \leq Q|r(y) - r'(y)|$$

($\mathbf{Q}_3$) $\beta_1 = \frac{Q}{\tau|\Gamma(\delta)|}\left[1 + \frac{2}{|(1-\alpha\xi^{\delta-1})|}\right] < 1,$

Then the FBVP (3) has a solution on $[0,1]$.

Proof: Let $B_s = \{r \in \mathfrak{B} : \parallel r \parallel \leq s\}$ be convex, bounded, and closed subset of the Banach space $\mathfrak{B}$. Also assume that $M = \max_{0\leq y\leq 1}|q(y, r(y))| + 1$ for $r \in B_s$.

(i) $\mathcal{J}$ is uniformly bounded

$$|\mathcal{J}r(y)| \leq \frac{M}{|\Gamma(\delta)|}\int_0^y (y-\eta)^{\tau-1}d\eta + \frac{M}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^1 (1-\eta)^{\tau-1}d\eta$$
$$+ \frac{M}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^\xi (\xi-\eta)^{\tau-1}d\eta + \frac{|\theta|}{|(1-\alpha\xi^{\delta-1})|}$$
$$\leq \frac{M}{\tau|\Gamma(\delta)|}\left[1 + \frac{2}{|(1-\alpha\xi^{\delta-1})|}\right] + \frac{|\theta|}{|(1-\alpha\xi^{\delta-1})|}$$

Since $\beta_1 = \frac{Q}{\tau|\Gamma(\delta)|}\left[1 + \frac{2}{|(1-\alpha\xi^{\delta-1})|}\right] <, \Rightarrow \frac{1}{|(1-\alpha\xi^{\delta-1})|} < \frac{\tau|\Gamma(\delta)|}{2Q} - \frac{1}{2}$ (14)

From equations (11) and (12), we have

$$|\mathcal{J}r(y)| \leq \frac{M}{Q} + \frac{|\theta|}{2}\left[\frac{\tau|\Gamma(\delta)|}{Q} - 1\right]$$

So, $\mathcal{J}$ is uniformly bounded.

(ii) $\mathcal{J}(B_s)$ is equicontinuous.

For $r \in B_s$ and $y_1, y_2 \in [0,1]$ let $y_1 < y_2$ we have,

$$|\mathcal{J}r(y_2) - \mathcal{J}r(y_1)|$$
$$\leq \left|\frac{1}{\Gamma(\delta)}\left[\int_0^{y_2} (y_2-\eta)^{\delta-1} - \int_0^{y_1} (y_1-\eta)^{\delta-1}\right]\right| |q(\eta, r(\eta))|d\eta$$
$$+ \frac{|y_2^{\delta-1} - y_1^{\delta-1}|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^1 |(1-\eta)^{\delta-1}||q(\eta, r(\eta))|d\eta$$
$$+ \frac{|\alpha(y_2^{\delta-1} - y_1^{\delta-1})|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^\xi |(\xi-\eta)^{\delta-1}||q(\eta, r(\eta))|d\eta$$
$$+ \frac{|\theta(y_2^{\delta-1} - y_1^{\delta\delta-1})|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}$$

$$\leq M\left|\frac{1}{\Gamma(\delta)}\left[\frac{y_2^\delta-y_1^\delta}{\delta}\right]\right|+\frac{M\left|y_2^{\delta-1}-y_1^{\delta-1}\right|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^1 \left|(1-\eta)^{\delta-1}\right|d\eta$$

$$+\frac{M\left|\alpha\left(y_2^{\delta-1}-y_1^{\delta-1}\right)\right|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^\xi \left|(\xi-\eta)^{\delta-1}\right|d\eta+\frac{\left|\theta\left(y_2^{\delta-1}-y_1^{\delta-1}\right)\right|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}$$

$$=\frac{M\left|\left(y_2^\delta-y_1^\delta\right)\right|}{|\Gamma(\delta+1)|}+\frac{\left|y_2^{\delta-1}-y_1^{\delta-1}\right|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta+1)|}\left\{M+M\alpha\left|(\xi)^\delta\right|+|\delta\theta|\right\}$$

$\mathcal{J}r(y)$ is equicontinuous since the functions $\left(y^\delta, y^{\delta-1}\right)$ and ξ^δ are uniformly continuous on $[0,1]$. So by the Arzela-Ascoli theorem, $\overline{\mathcal{J}r(y)}$ is compact and so $\mathcal{J}: B_s \to B_s$ is completely continuous. Thus, using Schauder's fixed point theorem [20], the FBVP (3) has a solution.

Theorem 3.2: *Under the assumptions of Theorem 3.1, the FBVP (3) has unique solution.*

Proof: $\mathcal{J}$ is continuous since $q, G(y,\eta)$ are continuous. Regarding the condition $(\mathbf{Q_1})$, let $K=\max_{0\leq y\leq 1}|q(y,0)|$.

Consider the ball $B_s=\{r\in\mathfrak{B}: \| r \|\leq s\}$, where $\beta_2=\frac{K}{\tau|\Gamma(\delta)|}\left\{1+\frac{2}{|(1-\alpha\xi^{\delta-1})|}\right\}+\frac{|\theta|}{|(1-\alpha\xi^{\delta-1})|}$ and $\frac{\beta_2}{1-\beta_1}\leq s$.

Firstly, we prove that $\mathcal{J}(B_s)\subset B_s$ for $r\in B_s$.

$$|\mathcal{J}r(y)|\leq\frac{1}{|\Gamma(\delta)|}\left[\int_0^y \left|(y-t)^{\delta-1}\right|\left(\left|q\left(\eta,r(\eta)\right)-q(\eta,0)\right|+|q(\eta,0)|\right)d\eta\right.$$

$$+\frac{1}{|(1-\alpha\xi^{\delta-1})|}\int_0^1 \left|(1-\eta)^{\delta-1}\right|\left(\left|q\left(\eta,r(\eta)\right)-q(\eta,0)\right|+|q(\eta,0)|\right)d\eta$$

$$\left.+\frac{1}{|(1-\alpha\xi^{\delta-1})|}\int_0^\xi \left|(\xi-\eta)^{\delta-1}\right|\left(\left|q\left(\eta,r(\eta)\right)-q(\eta,0)\right|+|q(\eta,0)|\right)d\eta\right]+\frac{|\theta|}{|(1-\alpha\xi^{\delta-1})|}$$

$$\leq\frac{(Q\|r\|+K)}{|\Gamma(\delta)|}\left[\int_0^y (y-\eta)^{\tau-1}d\eta\right.$$

$$\left.+\frac{1}{|(1-\alpha\xi^{\delta-1})|}\left\{\int_0^1 (1-\eta)^{\tau-1}d\eta+\int_0^\xi (\xi-\eta)^{\tau-1}d\eta\right\}\right]+\frac{|\theta|}{|(1-\alpha\xi^{\delta-1})|}$$

$$\leq\frac{Qs}{\tau|\Gamma(\delta)|}\left[1+\frac{2}{|(1-\alpha\xi^{\delta-1})|}\right]+\frac{K}{\tau|\Gamma(\delta)|}\left\{1+\frac{2}{|(1-\alpha\xi^{\delta-1})|}\right\}+\frac{|\theta|}{|(1-\alpha\xi^{\delta-1})|}=\beta_1 s+\beta_2\leq s$$

For $r, r'\in\mathfrak{B}, y\in[0,1]$,

$$|\mathcal{J}r(y)-\mathcal{J}r'(y)|\leq\frac{1}{|\Gamma(\delta)|}\int_0^y \left|(y-\eta)^{\delta-1}\right||q(\eta,r(\eta))-q(\eta,r'(\eta))|d\eta$$

$$+\frac{|y^{\delta-1}|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^1 \left|(1-\eta)^{\delta-1}\right|\left|q\left(\eta,r(\eta)\right)-q\left(\eta,r'(\eta)\right)\right|d\eta$$

$$+\frac{|\alpha y^{\delta-1}|}{|(1-\alpha\xi^{\delta-1})||\Gamma(\delta)|}\int_0^\xi \left|(\xi-\eta)^{\delta-1}\right||q(\eta,r(\eta))-q(\eta,r'(\eta))|d\eta$$

$$\leq \frac{Q\||r-r'\||}{\tau|\Gamma(\delta)|}\left\{1+\frac{2}{|(1-\alpha\xi^{\delta-1})|}\right\} = \beta_1||r-r'\|$$

So, $\mathcal{J}$ is a contraction. By Banach fixed point theorem, FBVP (3) has unique solution.

4. Stability of FBVP (3)

Theorem 4.1: *Under the assumptions* $(\boldsymbol{Q}_2)$ *and* $(\boldsymbol{Q}_3)$ *of Theorem 3.1. and* $\| r(y)-p(y) \| = max_{0\leq y\leq 1}|r(y)-p(y)|$, *the FBVP (3) is Ulam-Hyers stable, where* $p \in C([0,1],\mathbb{C})$ *be a function satisfying the inequality (6) with boundary condition* $r(1)=p(1)$.

Proof: Let $\epsilon > 0$ and $r \in C([0,1],\mathbb{C})$ be the unique solution of the FBVP (3), then from Lemma 2.2 the function $r(y)$ and constant k_1 is defined as

$$r(y) = \int_0^y \frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} q\big(\eta, r(\eta)\big)d\eta + k_1 y^{\delta-1}$$

$$k_1 = \frac{-1}{(1-\alpha\xi^{\delta-1})}\int_0^1 \frac{(1-\eta)^{\delta-1}}{\Gamma(\delta)} q(\eta, r(\eta))d\eta + \frac{\alpha}{(1-\alpha\xi^{\delta-1})}$$
$$\int_0^\xi \frac{(\xi-\eta)^{\delta-1}}{\Gamma(\delta)} q(\eta, r(\eta))d\eta + \frac{\theta}{(1-\alpha\xi^{\delta-1})}$$

Let, $T_r = \frac{-1}{(1-\alpha\xi^{\delta-1})}\int_0^1 \frac{[y(1-\eta)]^{\delta-1}}{\Gamma(\delta)} q_r(\eta, r(\eta))d\eta \frac{\alpha}{(1-\alpha\xi^{\delta-1})}$
$$\int_0^\xi \frac{[y(\xi-\eta)]^{\delta-1}}{\Gamma(\delta)} q_r(\eta, r(\eta))d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})}$$

and $T_p = \frac{-1}{(1-\alpha\xi^{\delta-1})}\int_0^1 \frac{[y(1-\eta)]^{\delta-1}}{\Gamma(\delta)} q_p\big(\eta, p(\eta)\big)d\eta +$
$$\frac{\alpha}{(1-\alpha\xi^{\delta-1})}\int_0^\xi \frac{[y(\xi-\eta)]^{\delta-1}}{\Gamma(\delta)} q_p(\eta, p(\eta))d\eta + \frac{\theta y^{\delta-1}}{(1-\alpha\xi^{\delta-1})}$$

$$\left|T_r - T_p\right| \leq \frac{1}{|(1-\alpha\xi^{\delta-1})|}\int_0^1 \frac{|(1-\eta)^{\delta-1}|}{|\Gamma(\delta)|}\left|q_r(\eta, r(\eta)) - q_p(\eta, p(\eta))\right|d\eta$$
$$+\frac{|\alpha|}{|(1-\alpha\xi^{\delta-1})|}\int_0^\xi \frac{|(\xi-\eta)^{\delta-1}|}{|\Gamma(\delta)|}\left|q_r\big(\eta, r(\eta)\big) - q_p\big(\eta, p(\eta)\big)\right|d\eta$$
$$\leq \frac{Q}{|(1-\alpha\xi^{\delta-1})|}\mathrm{I}_{0+}^{\tau}|r(1)-p(1)| + \frac{Q|\alpha|}{|(1-\alpha\xi^{\delta-1})|}\mathrm{I}_{0+}^{\tau}|r(\xi)-p(\xi)| \leq 0$$
$$|p(y)-r(y)| = \left|p(y) - \frac{1}{\Gamma(\delta)}\int_0^y (y-\eta)^{\delta-1} q_r(\eta, r(\eta))d\eta - T_r\right|$$
$$\leq \left|p(y) - \frac{1}{\Gamma(\delta)}\int_0^y (y-\eta)^{\delta-1} q_p(\eta, p(\eta))d\eta - T_p\right| + \frac{Q}{|\Gamma(\delta)|}\int_0^y (y-\eta)^{\tau-1}|p(\eta)-r(\eta)|d\eta$$
$$\Rightarrow \| p-r \| \leq \frac{\epsilon}{\tau|\Gamma(\delta)|} + \frac{Q\|p-r\||}{\tau|\Gamma(\delta)|} y^\tau \int_0^y \big(y-\eta\big)^{\delta-1} q_p\big(\eta, p(\eta)\big)d\eta - T_p\Big|$$
$$\leq \frac{\epsilon}{|\tau\Gamma(\delta)|} \leq \frac{\epsilon}{\tau|\Gamma(\delta)|} + \frac{Q\|p-r\|}{\tau|\Gamma(\delta)|} = \frac{\epsilon}{\tau|\Gamma(\delta)|-Q} = k_q\epsilon; \text{ where } k_q = \frac{1}{\tau|\Gamma(\delta)|-Q}$$

which completes the proof.

5. Example

Consider the complex order FBVP:

$$\begin{cases} \mathbf{D}_{0^+}^{\frac{8}{5}+\iota} r(y) = \frac{\tan^{-1}(r)}{y+3}; y \in [0,1] \\ r(0) = 0, r(1) = \alpha r(\xi) + \theta; \xi \in (0,1) \end{cases} \tag{15}$$

Here $\tau = \frac{8}{5}, \eta = 1$ and $q(y,r) = \frac{\tan^{-1}(r)}{y+3}, (y,r) \in [0,1] \times [0,\infty)$.

For $(y,r),(y,r') \in [0,1] \times [0,\infty)$, we have

$$|q(y,r) - q(y,r')| = \frac{1}{y+3}|\tan^{-1} r - \tan^{-1} r'| \leq \frac{1}{3}|r - r'|$$

$\Rightarrow Q = \frac{1}{3}$. Now our aim is to show that $\beta_1 < 1$. Since $\left|\Gamma\left(\frac{8}{5} + \iota\right)\right| > 1$, then

$$\beta_1 = \frac{3Q}{n|\Gamma(\delta)|} = \frac{3\frac{1}{3}}{\frac{8}{5}\left|\Gamma\left(\frac{8}{5}+\iota\right)\right|} < \frac{5}{8} < 1$$

By Theorem 3.2, FBVP (15) has unique solution.

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References

[1] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations. Wiley (1993).

[2] R.Alchikh, and S. Khuri, On the solutions of the fractional bratu's problem," *Journal of Interdisciplinary Mathematics* vol. 25,no.6, pp. 1093-1107 (2020).

[3] Z. Baitiche, and C. Derbazi, "Implicit fractional differential equations with φ–caputo fractional derivative in banach spaces," *Journal of Interdisciplinary Mathematics,* vol.25, no. 5, pp. 1237–1252 (2022).

[4] B. Ross, Fractional calculus and its applications: Proceedings of the international conference held at the University of New Haven, June 1974, vol. 457. Springer (2006).

[5] R. Metzler, W. Schick, H.-G. Kilian, and T. F. Nonnenmacher, "Relaxation in filled polymers: A fractional calculus approach," *The Journal of Chemical Physics*, vol. 103, no. 16, pp. 7180-7186 (1995).

[6] R. Magin, "Fractional calculus in bioengineering, part 1," *Critical Reviews in Biomedical Engineering*, vol. 32, no. 1 (2004).

[7] Y. Shen, J. Hua, W. Fan, Y. Liu, X. Yang, and L. Chen, "Optimal design and dynamic performance analysis of a fractional-order electrical network-based vehicle mechatronic ISD suspension," *Mechanical Systems and Signal Processing*, vol. 184, p. 109718 (2023).

[8] S. Arora, T. Mathur, S. Agarwal, K. Tiwari, and P. Gupta, "Applications of fractional calculus in computer vision: A survey," *Neurocomputing,* vol. 489, pp. 407-428 (2022).

[9] N. Singh, S. Arora, T. Mathur, S. Agarwal, and K. Tiwari, "Stock price prediction using fractional gradient-based long short term memory," in *Journal of Physics: Conference series*, vol. 1969, p. 012038 (2021).

[10] J. He, "Approximate analytical solution for seepage flow with fractional derivatives in porous media," *Computer Methods in Applied Mechanics and Engineering*, vol. 167, no. 1-2, pp. 57-68 (1998).

[11] T. M. Atanacković, S. Konjik, S. Pilipović, and D. Zorica, "Complex order fractional derivatives in viscoelasticity," *Mechanics of Time-Dependent Materials*, vol. 20, pp. 175-195 (2016).

[12] S.P. Goyal, and T. Mathur, "On generalized fractional diffusion equations," *South East Asian Journal of Mathematics and mathematical Sciences*, vol. 4, pp. 53-61 (2006).

[13] T. Mathur, "On generalized fractional diffusion equation-II," *Journal of Rajasthan Academy of Physical Sciences*, vol. 3, pp. 183-190 (2004).

[14] A. Yadav, T. Mathur, S. Agarwal, and B. Yadav, "Fractional boundary value problem in complex domain," *Journal of Mathematical Analysis and Applications*, vol. 526, no. 1 (2023).

[15] W. Xie, J. Xiao, and Z. Luo, "Existence of extremal solutions for nonlinear fractional differential equation with nonlinear boundary conditions," *Applied Mathematics Letters*, vol. 41, pp. 46-51 (2015).

[16] A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and applications of fractional differential equations, vol. 204. Elsevier (2006).

[17] K. Hussain, A. Hamoud, and N. Mohammed, "Some new uniqueness results for fractional integro-differential equations," *Nonlinear Functional Analysis and Applications*, vol. 24, no. 4, pp. 827-836 (2019).

[18] M. Benchohra and S. Bouriah, "Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order," *Moroccan Journal of Pure and Applied Analysis*, vol. 1, pp. 1-16 (2015).

[19] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications,* vol. 72, no. 2, pp. 916-924 (2010).

[20] K. Ma, X. Li, and S. Sun, "Boundary value problems of fractional q-difference equations on the half-line," *Boundary Value Problems,* vol. 2019, no. 1, pp. 1-16 (2019).

[21] Kumar, Anil , Gupta, Amit Kumar, Panwar, Deepak , Chaurasia, Sandeep & Goyal, Dinesh. Operating system security with discrete mathematical structure for secure round robin scheduling method with intelligent time quantum, *Journal of Discrete Mathematical Sciences and Cryptography*, 26:5, 1519–1533 (2023).

[22] Kushwaha, Satpal Singh, Joshi, Sandeep & Gupta, Amit Kumar. An efficient approach to secure smart contract of Ethereum blockchain using hybrid security analysis approach, *Journal of Discrete Mathematical Sciences and Cryptography*, 26:5, 1499–1517 (2023), DOI: https://doi.org/10.47974/JDMSC-1815.

[23] Joshi, Ruchi, Mathur, Priya, Gupta, Amit Kumar, Singh, Suyesha, Paliwal, Vismita & Nayar, Sejal. Mathematical modeling of intelligent system for predicting effectiveness of premenstrual syndrome, *Journal of Interdisciplinary Mathematics*, 26:3, 551-562 (2023).