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Convergence : Partially ordered soft topological space

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Abstract

This research paper introduce Symmetrical and sequentially dense soft set and isometry in the context of POS topological space. Inspired by Matthew's seminal work on dataflow network research, we extend the study to soft universe, bridging classical metrics with soft computing to represent imprecision in complex systems. We investigate sequence convergence in partial soft topological spaces, revealing essential insights and contributing to our understanding of partial soft metrics. Additionally, we introduce the idea of Convergence of partially ordered soft topological spaces along with we provide examples to support the obtained results and relations. In the future, these results can be further applied to analyze and understand the properties and convergence of soft topological spaces with the contraction condition for soft fixed point theory.

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1. Introduction

The concept of a metric space holds significant importance and serves as a versatile tool in various scientific fields, particularly in the domain of fixed point theory. In recent years, the notion of metric spaces has been extended and generalized in several directions, leading to the introduction of metric-like spaces such as Complex valued metric, Soft metric, b-metric, d-metric, generalized metric, quasi-metric, Cone metric and among others. Another development in the field came with the introduction of partial metric spaces by

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Matthew [11] in 1994, Matthew gives non zero self distance, gained considerable attention from researchers [5, 6, 7, 8,] recently Shinde S. R. [10] introduced non-linear second order Boundary value problem and multivalued mappings in fixed point theory subsequently Berrah et. Al. [3] given fixed point theorem for dynamic programming, Simultaneously the exploration of soft sets has studied by scholars, offering a new theoretical framework with applications in decision-making problems and information technology. In this study, we explore the field of soft topological set theory. To set the stage, we lay the foundation with fundamental definitions and profound insights, while the addition of topological spaces has been examined by Atay and Tutalar [2] in 2015, and some results of soft topological spaces has been studied by Al-shami et al [15-18] in 2018 and the uncertainty of fuzzy soft topological spaces have been successfully applied in soft set theory, for more [7, 8, 9, 12, 10, 11, 14]. In this study, we designate $\widetilde{\nabla}$ as the initial universe, $\Upsilon(\widetilde{\nabla})$ as the power set of $\widetilde{\nabla}$ and E as the set parameters with $\widetilde{\nabla}$, where A represents a specific element from E.

2. Preliminaries

In the following section, we provide essential preliminaries on soft sets, we refer readers to the [12, 1, 13, 4] for all the fundamental definitions and notations.

Definition 2.1: [13] Lets define ordering between soft real number $\hbar$ and ℓ on Ξ as:

1. For every $\widetilde{\omega} \in \Xi$, $\hbar(\widetilde{\omega}) \widetilde{\lesssim} \ell(\widetilde{\omega})$.
2. For every $\widetilde{\omega} \in \Xi$, $\hbar(\widetilde{\omega}) \widetilde{\gtrsim} \ell(\widetilde{\omega})$.
3. For every $\widetilde{\omega} \in \Xi$, $\hbar(\widetilde{\omega}) \widetilde{<} \ell(\widetilde{\omega})$.
4. For every $\widetilde{\omega} \in \Xi$, $\hbar(\widetilde{\omega}) \widetilde{>} \ell(\widetilde{\omega})$.

Definition 2.2: [13]

1. Let's Consider two soft point $\Upsilon_\hbar^\ell$ and Υ_ν^ω are equal if $\hbar = \nu$ & $\Upsilon(\hbar) = \Upsilon(\nu)$ which means $\ell = \omega$, hence $\Upsilon_\hbar^\ell \neq \Upsilon_\nu^\omega$ iff $\hbar \neq \nu$ or $\ell \neq \omega$.
2. Suppose soft set Υ over $\widetilde{\nabla}$ is called a soft point whenever their exist only $\hbar \in E$ has $\Upsilon(\hbar) = \{\ell\}$ and $\ell \in \widetilde{\nabla}$, for every $\nu \in E - \{\hbar\}$ and written as $\Upsilon_\hbar^\ell$.
3. Again Consider a soft point $\Upsilon_\hbar^\ell$ belongs to soft set Ξ whenever $\Upsilon(\widetilde{\omega}) = \{x\} \subset \Xi(\widetilde{\omega})$ then we write $\Upsilon_\hbar^\ell \in \Xi$.

Proposition 2.1: [13] The Collection of all soft points, $\Xi = \bigcup_{\Upsilon_\hbar^\ell \in \Xi} \Upsilon_\hbar^\ell$ that is the collection of all soft points can be supposed as a soft set & we write every soft set can be expressed as union of all soft points belongs to given set.

3. Convergence for $\mathcal{POS}$ topological space

Throughout this paper we denote $\mathfrak{S}\mathfrak{P}(\Xi)$ as a collection of all soft points from $\widetilde{\nabla}$.

Definition 3.1: In the concept of partial soft Topological spaces, let $(\widetilde{\nabla}, ð)$ represent $\mathcal{POS}$ topological space, Ξ denotes a soft set over $\widetilde{\nabla}$. We define the following notions:

1. *Sequential Dense:* Ξ is deemed sequentially dense in $\widetilde{\nabla}$ if, for any point $\Upsilon_{\hbar}^{\ell}$ belonging to the set of partial soft points $\Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\widetilde{\nabla})$, there exists a sequence of soft points in Ξ that converges to $\Upsilon_{\hbar}^{\ell}$ gives comprehensive coverage of Ξ in soft space.
2. *Symmetrical Dense:* Ξ is considered symmetrically dense in $\widetilde{\nabla}$ if, for any point in $\Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\Xi)$ and any nonnegative soft real number $\widetilde{\flat}$, there exists a point Υ_{ν}^{ω} in Ξ satisfying conditions $\Upsilon_{\hbar}^{\ell} \in \aleph(\Upsilon_{\nu}^{\omega}, \widetilde{\flat})$ and $\Upsilon_{\nu}^{\omega} \in \aleph(\Upsilon_{\hbar}^{\ell}, \widetilde{\flat})$. Gives ensures the all-encompassing nature of Ξ respect to balanced and symmetric soft neighborhoods.

Definition 3.2: [3] Consider a soft set (Ξ, A) over the universal set $\widetilde{\nabla}$ is defined by an collection of ordered pairs, $(\Xi, A) = \{(e, \Xi(e)) \text{such that} e \in E, \Xi(e) \in \Upsilon(\widetilde{\nabla})\}$.

Where we have, $\Xi: E \to \Upsilon(\widetilde{\nabla})$ having $\Xi(e) \neq \emptyset$ whenever $e \in A \subseteq E$ and $\Xi(e) = \emptyset$ if $e \notin A$. $(\widetilde{\nabla}, \tau)$ be a topological space such that Ξ be a family of which contains all neighborhoods of a points from $x \in \Xi$ $\Xi(x) = \{V \in \tau | x \in V\}$ which represent ordered pair $(\Xi, \widetilde{\nabla})$ is a soft set over $(\widetilde{\nabla})$.

Definition 3.3: Lets assume $(\widetilde{\nabla}, ð)$ be a $\mathcal{POS}$ topological space and a complete $\mathcal{POS}$ topological space $(\widetilde{\nabla}^*, ð^*)$ said to be a completion of $(\widetilde{\nabla}, ð)$ whenever it has an isometry $\mho_{\tau}: \widetilde{\nabla} \to \widetilde{\nabla}^*$ such that $\mho_{\tau}(\widetilde{\nabla})$ is dense in $(\widetilde{\nabla}^*, ð^*)$.

Definition 3.4: Let's consider two $\mathcal{POS}$ topological space $(\widetilde{\nabla}_1, ð)$ and $(\widetilde{\nabla}_2, ð)$, which has soft mappings $\mho_{\tau}: (\widetilde{\nabla}_1, ð) \to (\widetilde{\nabla}_2, ð)$ called as an isometry if for all $\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega} \in \mathfrak{S}\mathfrak{P}(\Xi)$ such that $p(\mho_{\tau}(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega})) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega})$.

Definition 3.5: Consider $\mathcal{POS}$ topological space $(\widetilde{\nabla}, ð)$, having following properties

1. Consider a sequence of a soft points $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is a Cauchy sequence whenever their exist a non-negative real number $\widetilde{\lambda}$ such that $\lim_{m,n\to\infty} ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_m}^{\ell_m}) = \widetilde{\lambda}$.
2. $(\widetilde{\nabla}, ð)$ is said to satisfies the property of completeness if all Cauchy sequence in $\widetilde{\nabla}$ are convergent in $(\widetilde{\nabla}, ð)$.
3. Finally, a sequence of all soft points $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is called as Convergent in $(\widetilde{\nabla}, ð)$, if their exist $\Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\Xi)$ satisfies $\lim_{n\to\infty} ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) = \lim_{n\to\infty} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell})$.

Definition 3.6: The mappings $ð: \mathfrak{S}\mathfrak{P}(\Xi) \times \mathfrak{S}\mathfrak{P}(\Xi) \to \mathbb{R}(\mathbb{A})^*$ be a $\mathcal{POS}$ metric space on $\widetilde{\nabla}$ if following axioms are satisfies:

1. $\Upsilon_{\hbar}^{\ell} = \Upsilon_{\nu}^{\omega}$ iff $\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega}) = \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) = \eth(\Upsilon_{\nu}^{\omega}, \Upsilon_{\nu}^{\omega})$, for all $\Upsilon_{\nu}^{\omega}, \Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\Xi)$.
2. $\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega}) = \eth(\Upsilon_{\nu}^{\omega}, \Upsilon_{\hbar}^{\ell})$, for all $\Upsilon_{\nu}^{\omega}, \Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\Xi)$.
3. $\eth(\Upsilon_{\nu}^{\omega}, \Upsilon_{\nu}^{\omega}) \lessapprox \eth(\Upsilon_{\nu}^{\omega}, \Upsilon_{\hbar}^{\ell})$, for all $\Upsilon_{\nu}^{\omega}, \Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\Xi)$.
4. $\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{p}^{o}) \lessapprox \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega}) + \eth(\Upsilon_{\nu}^{\omega}, \Upsilon_{p}^{o}) - \eth(\Upsilon_{\nu}^{\omega}, \Upsilon_{\nu}^{\omega})$, for all $\Upsilon_{\nu}^{\omega}, \Upsilon_{\hbar}^{\ell}, \Upsilon_{p}^{o} \in \mathfrak{S}\mathfrak{P}(\Xi)$.

Here $\widetilde{\nabla} \neq \emptyset$ and $\eth$ be a partial soft metric on $\widetilde{\nabla}$ then $(\widetilde{\nabla}, \eth)$ is a $\mathcal{POS}$ metric space.

Lets consider the mapping $\eth: \mathfrak{S}\mathfrak{P}(\Xi) \times \mathfrak{S}\mathfrak{P}(\Xi) \to \mathbb{R}(\mathbb{A})^*$ and $\widetilde{\nabla} = \mathbb{A} = \mathbb{R}$, then we define $\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega}) = \max\{\ell, \omega\} + \max\{\hbar, \nu\}$ is said to be a $\mathcal{POS}$ metric space.

Remark 3.1: Suppose $\mathcal{POS}$ topological space $(\widetilde{\nabla}, \eth)$ having map $\eth^*: \mathfrak{S}\mathfrak{P}(\widetilde{\nabla}) \times \mathfrak{S}\mathfrak{P}(\widetilde{\nabla}) \to \mathbb{R}(\mathbb{A})$ defined as follows:

$$\eth^*(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega}) = 2\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu}^{\omega}) - \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) - \eth(\Upsilon_{\nu}^{\omega}, \Upsilon_{\nu}^{\omega}), for all \Upsilon_{\nu}^{\omega}, \Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\widetilde{\nabla}).$$

Theorem 3.7: *Let $(\tilde{\nabla}, \eth)$ be a $\mathcal{POS}$ topological space. If sequence of soft point $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is a Convergent in $\tilde{\nabla}$ then we say sequence of soft point $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is a Cauchy sequence.*

Theorem 3.8: *Consider $\mathcal{POS}$ topological space $(\tilde{\nabla}, \eth)$ having soft set $\tilde{\nabla}$. Ξ is symmetrically dense if and only if Ξ is sequentially dense in $\tilde{\nabla}$.*

Proof: We prove above result in two folds **Firstly,** suppose $\Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\widetilde{\nabla})$ and Ξ is sequentially dense soft set in $\mathcal{POS}$ topological space $\widetilde{\nabla}$, then $\forall n \in \mathbb{N}$, $\Upsilon_{\hbar_n}^{\ell_n} \in \Xi$ gives,

$$\Upsilon_{\hbar}^{\ell} \in \chi(\Upsilon_{\hbar_n}^{\ell_n}, \frac{1}{n}) \ \& \ \Upsilon_{\hbar_n}^{\ell_n} \in \chi(\Upsilon_{\hbar}^{\ell}, \frac{1}{n})$$

$$\eth(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) + \frac{1}{n} \gtrsim \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) \ \& \ \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) + \frac{1}{n} \gtrsim \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n})$$

now we claim Sequence of soft point $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is convergent to $\Upsilon_{\hbar}^{\ell}$. for that we have,

$$\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) + \frac{1}{n} \gtrsim \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) \ \& \ \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) \gtrapprox \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell})$$

$$\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) = \lim_{n\to\infty} \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}), \forall n \in \mathbb{N}.$$

similarly by other side,

$$\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) + \frac{1}{n} \gtrapprox \eth(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) + \frac{1}{n} \ \& \ \eth(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) + \frac{1}{n} \gtrsim \eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}).$$

letting $\lim_{n\to\infty}$ then we write $\eth(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) = \lim_{n\to\infty} \eth(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n})$

Secondly let's suppose $\Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\widetilde{\nabla})$ and Ξ is sequentially dense soft set in a $\mathcal{POS}$ topological space $\widetilde{\nabla}$, then we get

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) = \lim_{n\to\infty} ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) \text{and} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) = \lim_{n\to\infty} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n})$$

that is $\exists$ a sequence, $\Upsilon_{\hbar_n}^{\ell_n} \in \Xi$ converges to $\Upsilon_{\hbar}^{\ell}$ and for all $n_0 < n$ we write, $\forall 0 \precsim \widetilde{b}\ \exists\ n_0 \in \mathbb{N}$ such that

$$\frac{\widetilde{b}}{2} \gtrsim |ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) - ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell})| \text{and} \frac{\widetilde{b}}{2} \gtrsim |ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) - ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell})|$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) + \widetilde{b} \gtrsim ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) + \frac{\widetilde{b}}{2} \text{and} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) + \frac{\widetilde{b}}{2} \gtrsim ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n})$$

$$ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) + \frac{\widetilde{b}}{2} + \frac{\widetilde{b}}{2} \gtrsim ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) + \frac{\widetilde{b}}{2} \text{and} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) + \frac{\widetilde{b}}{2} \gtrsim ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}).$$

let's take $n_0 \precsim n$ and putting in to $\Upsilon_{\hbar_{n_1}}^{\ell_m} = \Upsilon_{\nu}^{\omega}$ then finally we get,

$$\Upsilon_{\hbar}^{\ell} \in \aleph(\Upsilon_{\nu}^{\omega}, \widetilde{b}) \text{and} \Upsilon_{\nu}^{\omega} \in \aleph(\Upsilon_{\hbar}^{\ell}, \widetilde{b})$$

hence finally it gives, In $\mathcal{POS}$ topological space $\widetilde{\nabla}$, Ξ is a sequentially dense soft set.

Theorem 3.9: *Consider the two $\mathcal{POS}$ topological space $(\tilde{\nabla}_1, ð)$, $(\tilde{\nabla}_2, ð^*)$ and map $\mho_\tau: (\tilde{\nabla}_1, ð) \to (\tilde{\nabla}_2, ð^*)$. If sequence of soft point $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ converging to $\{\Upsilon_{\hbar}^{\ell}\}$ then $\{\mho_\tau(\Upsilon_{\hbar_n}^{\ell_n})\} \in \mathfrak{S}\mathfrak{P}(\tilde{\nabla}_2)$ converges to $\{\mho_\tau(\Upsilon_{\hbar}^{\ell})\}$ and $\Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\tilde{\nabla}_1)$ & $\mho_\tau$ is an Isometry.*

Proof: Suppose $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ converges $\{\Upsilon_{\hbar}^{\ell}\}$ in a partial order soft topological space $(\widetilde{\nabla}_1, ð)$,

$$\lim_{n\to\infty} ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) \ \& \ \lim_{n\to\infty} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell})$$

by using Isometry of $\mho_\tau$,

$$\lim_{n\to\infty} ð^*(\mho_\tau(\Upsilon_{\hbar_n}^{\ell_n}), \mho_\tau(\Upsilon_{\hbar_n}^{\ell_n})) = ð^*(\mho_\tau(\Upsilon_{\hbar}^{\ell}), \mho_\tau(\Upsilon_{\hbar}^{\ell})) \ \&$$
$$\lim_{n\to\infty} ð^*(\mho_\tau(\Upsilon_{\hbar}^{\ell}), \mho_\tau(\Upsilon_{\hbar_n}^{\ell_n})) = ð^*(\mho_\tau(\Upsilon_{\hbar}^{\ell}), \mho_\tau(\Upsilon_{\hbar}^{\ell})).$$

then, sequence $\{\mho_\tau(\Upsilon_{\hbar_n}^{\ell_n})\}$ converges $\mho_\tau(\Upsilon_{\hbar}^{\ell})$ under $\mathcal{POS}$ topological space $(\widetilde{\nabla}_2, ð^*)$.

Theorem 3.10: *Assume $(\tilde{\nabla}, ð)$ be a $\mathcal{POS}$ topological space. If sequence of soft point $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is a Cauchy sequence in $\tilde{\nabla}$ has a convergent subsequence $\{\Upsilon_{n_k}^{\ell_{n_k}}\}$, then sequence of soft point $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is convergent.*

Proof: We have $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ be a Cauchy sequence of soft points. Inside $(\tilde{\nabla}, ð)$ we have $\{\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}\}$ is convergent subsequence of a sequence $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ then, $\lim_{m,n\to\infty} ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_m}^{\ell_m}) = \tilde{\lambda}$.

hence we write, $ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) = \tilde{\lambda}$. we note that

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) \lesssim ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_n}^{\ell_n}) + ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar}^{\ell}) - ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \ \&$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \lesssim ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_n}^{\ell_n}) + ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar}^{\ell}) - ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n})$$

$$(1)\ ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) - ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_n}^{\ell_n}) + ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) \lesssim ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar}^{\ell})$$

$$(2)\ ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) \lesssim ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_n}^{\ell_n}) + ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar}^{\ell}) - ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}})$$

now, letting $\lim_{k,n\to\infty}$ (1) and (2) then we get,

$$\tilde{\lambda} + ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) - ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) \geqq \lim_{n\to\infty} ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar}^{\ell}) \geqq ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) - \tilde{\lambda} + \tilde{\lambda}$$

$$\lim_{n\to\infty} ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\hbar_n}^{\ell_n}) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) \ \& \ \lim_{n\to\infty} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_n}^{\ell_n}) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell})$$

sequence $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is convergent in $(\tilde{\nabla}, ð)$. Hence, proof is complete.

4. Completeness property for $\mathcal{POS}$ Topological space.

Theorem 4.1: *$\mathcal{POS}$ topological space $(\tilde{\nabla}, ð)$ is complete whenever every Cauchy sequence in Ξ is convergent in $\tilde{\nabla}$. Where $(\tilde{\nabla}, ð)$ be $\mathcal{POS}$ metric space and Ξ be a soft set over ∇ and sequentially dense in $\tilde{\nabla}$.*

Proof: Let's consider a Cauchy sequence $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ in the $\mathcal{POS}$ topological space $(\tilde{\nabla}, ð)$ then their exist $\tilde{\lambda} \geq 0$ such that,

$$\lim_{m,n\to\infty} ð(\Upsilon_{\hbar_m}^{\ell_m}, \Upsilon_{\hbar_n}^{\ell_n}) = \tilde{\lambda} \tag{1}$$

without loss of generality, we take $\ldots \succ n_k \succ \ldots \succ n_3 \succ n_2 \succ n_1$
and for every $k \in \mathbb{N}$ their exist n_k we have,

$$\tilde{\lambda} + \frac{1}{k} \gtrsim ð(\Upsilon_{\hbar_m}^{\ell_m}, \Upsilon_{\hbar_n}^{\ell_n}) \ \& \ ð(\Upsilon_{\hbar_m}^{\ell_m}, \Upsilon_{\hbar_n}^{\ell_n}) \gtrsim \tilde{\lambda} - \frac{1}{k}, \qquad \forall n, m \geq n_k$$

for every k we have symmetrically density of Ξ and $\exists\ \Upsilon_{\nu_k}^{\omega_k} \in \Xi$ with,

$$\Upsilon_{\nu_k}^{\omega_k} \in \aleph(\frac{1}{k}, \Upsilon_{\hbar_{nk}}^{\ell_{n_k}}) \ \textit{and} \ \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}} \in \aleph(\frac{1}{k}, \Upsilon_{\nu_k}^{\omega_k})$$

$$\frac{1}{k} + ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_k}^{\omega_k}) \gtrsim ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \ \textit{and} \ \frac{1}{k} + ð(\Upsilon_{\hbar_{nk}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \geqq ð(\Upsilon_{\hbar_{nk}}^{\ell_{n_k}}, \Upsilon_{\nu_k}^{\omega_k})$$

since we know every Cauchy sequence is convergent so, we need to prove inside Ξ $\{\Upsilon_{\nu_k}^{\omega_k}\}$ is a Cauchy sequence. We select for every $\widetilde{\flat} \gtrsim 0$ and $i \in \mathbb{N}$ such that, $\frac{\widetilde{\flat}}{3} \gtrsim \frac{1}{i}$ and let $i < k, \eta$ then, $ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_\eta}^{\omega_\eta}) \lessapprox$

$$ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) + ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}}) + ð(\Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}}, \Upsilon_{\nu_k}^{\omega_k}) - ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) - ð(\Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}}, \Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}})$$

$$ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_\eta}^{\omega_\eta}) \lessapprox \widetilde{\lambda} + \frac{3}{i} \lesssim \widetilde{\lambda} + \widetilde{\flat}$$

$$ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}}) \gtrsim \widetilde{\lambda} - \tfrac{1}{i} \text{ and } ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}}) \lessapprox$$

$$ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) + ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_\eta}^{\omega_\eta}) + ð(\Upsilon_{\nu_\eta}^{\omega_\eta}, \Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}}) - ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) - ð(\Upsilon_{\nu_\eta}^{\omega_\eta}, \Upsilon_{\nu_\eta}^{\omega_\eta})$$

$$ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_\eta}}^{\ell_{n_\eta}}) \lessapprox \frac{2}{i} + ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_\eta}^{\omega_\eta})$$

$ð\left(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_\eta}^{\omega_\eta}\right) \gtrsim \widetilde{\lambda} - \frac{3}{i} = \widetilde{\lambda} - \widetilde{\flat}$ *and* $\lim_{k,\eta \to \infty} ð\left(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_\eta}^{\omega_\eta}\right) = \widetilde{\lambda}$. Hence sequence of soft point $\{\Upsilon_{\nu_k}^{\omega_k}\}$ Cauchy inside Ξ and in $\widetilde{\nabla}$ all Cauchy sequence is convergent, which gives sequence of soft point $\{\Upsilon_{\nu_k}^{\omega_k}\}$ is convergent in $\widetilde{\nabla}$. now it is clear that, $\exists \Upsilon_{\hbar}^{\ell} \in \mathfrak{S}\mathfrak{P}(\widetilde{\nabla})$ such that,

$$\lim_{k\to\infty} ð\left(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\hbar}^{\ell}\right) = ð\left(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}\right) = \widetilde{\lambda}, \qquad \lim_{k\to\infty} ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_k}^{\omega_k}) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}) = \widetilde{\lambda}.$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \lessapprox ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu_k}^{\omega_k}) + ð(\Upsilon_{\nu_n}^{\omega_n}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) - ð(\Upsilon_{\nu_k}^{\omega_k}, \Upsilon_{\nu_k}^{\omega_k})$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \lessapprox \frac{1}{k} + ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu_k}^{\omega_k}), \qquad \forall\ k \in \mathbb{N}$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \gtrsim ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu_k}^{\omega_k}) - \frac{1}{k}, \qquad \forall\ k \in \mathbb{N}$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu_k}^{\omega_k}) \lessapprox ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) + ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\nu_k}^{\omega_k}) - ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}})$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu_k}^{\omega_k}) \lessapprox \frac{1}{k} + ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}), \qquad \forall\ k \in \mathbb{N}$$

$$ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\nu_k}^{\omega_k}) + \frac{1}{k} \gtrsim ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}), \qquad \forall\ k \in \mathbb{N}$$

letting $k \to \infty$, $\lim_{k\to\infty} ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell})$ and similarly we write,

$$\lim_{k\to\infty} ð(\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}, \Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}) \widetilde{\lambda} = ð(\Upsilon_{\hbar}^{\ell}, \Upsilon_{\hbar}^{\ell}).$$

gives $\{\Upsilon_{\hbar_{n_k}}^{\ell_{n_k}}\}$ be convergent subsequence with the help of (3.9)(2) we get, sequence of soft point $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is convergent. Hence $(\widetilde{\nabla}, ð)$ be a complete $\mathcal{POS}$ topological space.

Theorem 4.1: *In $\mathcal{POS}$ topological space Completeness remains preserved under the concept of soft isometry.*

Proof: Let's consider inside $\mathcal{POS}$ topological space we have $(\widetilde{\nabla}_1, ð)$ and $(\widetilde{\nabla}_2, ð)$ be two soft isometric spaces has soft mappings $\mho_\tau: (\widetilde{\nabla}_1, ð) \to (\widetilde{\nabla}_2, ð)$ be a isometry.

We assume $\widetilde{\nabla}_1$ is a complete, the sequence $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is a cauchy and $\{\Upsilon_{\hbar_n}^{\ell_n}\} \in (\widetilde{\nabla}_2, ð)$ then for all $\widetilde{\lambda} \gtrsim 0$ which has $\mathbb{N}$ such that ,

$$p(\mho_\tau^{-1}(\Upsilon_{\hbar_m}^{\ell_m}, \Upsilon_{\hbar_n}^{\ell_n})) = ð(\Upsilon_{\hbar_m}^{\ell_m}, \Upsilon_{\hbar_n}^{\ell_n}) \precapprox \widetilde{\lambda}, forall m, n \gtrapprox \mathbb{N}.$$

Then we say $\{\mho_\tau^{-1}\{\Upsilon_{\hbar_n}^{\ell_n}\}\}$ is a cauchy inside $\widetilde{\nabla}_1$ and therefor we have limit of $\Upsilon_\nu^\omega \in \mathfrak{S}\mathfrak{P}(\widetilde{\nabla}_1)$, which gives $\widetilde{\nabla}_1$ is a Complete. Our next claim is $(\widetilde{\nabla}_2, ð)$ also Complete, thus we have $\mho_\tau(\Upsilon_\nu^\omega) = \lim_{n\to\infty}\{\Upsilon_{\hbar_n}^{\ell_n}\}$ and hence we say is $(\widetilde{\nabla}_2, ð)$ also Complete. Similarly if $(\widetilde{\nabla}_2, ð)$ Complete then we say $(\widetilde{\nabla}_1, ð)$ is also a Complete.

The Completion of discrete soft topological space $(\widetilde{\nabla}_1, ð)$ is a $(\widetilde{\nabla}_1, ð)$ itself. Lets assume $\Upsilon_\hbar^\ell, \Upsilon_\nu^\omega \in \mathfrak{S}\mathfrak{P}(\widetilde{\nabla})$ be a two soft point in discrete soft topological space $(\widetilde{\nabla}_1, ð)$.

$$ð(\Upsilon_\hbar^\ell, \Upsilon_\nu^\omega) = \begin{cases} \tilde{0} & whenever \Upsilon_\hbar^\ell = \Upsilon_\nu^\omega \\ \tilde{1} & whenever \Upsilon_\hbar^\ell \neq \Upsilon_\nu^\omega \end{cases}$$

lets condiser $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ be any Cauchy sequence in $(\widetilde{\nabla}_1, ð)$ then we write $\widetilde{\lambda} \gtrsim 0$ which has a positive integer k^* such that $ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\nu_j}^{\omega_j}) \precapprox \widetilde{\lambda}$, for every $n, j \gtrsim k^*$.

Suppose $\widetilde{\lambda} = \frac{1}{2}$ then there exists k^* is a Natural number such that $ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\nu_j}^{\omega_j}) \precapprox \frac{1}{2}$ for every $n, j \gtrsim k^*$. Which gives

$$ð(\Upsilon_{\hbar_n}^{\ell_n}, \Upsilon_{\nu_j}^{\omega_j}) = 0 \Longrightarrow \Upsilon_{\hbar_n}^{\ell_n} = \Upsilon_{\nu_j}^{\omega_j} forall\ n, j \gtrsim k^*.$$

thus $\{\Upsilon_{\hbar_n}^{\ell_n}\}$ is a constant sequence and we know every constant sequence is a convergent sequence. $\{\Upsilon_{\hbar_n}^{\ell_n}\} \to \Upsilon_\hbar^\ell \in (\widetilde{\nabla}_1, ð)$. Then we write $(\widetilde{\nabla}_1, ð)$ is a complete. Hence Completion of discrete soft topological space $(\widetilde{\nabla}_1, ð)$ is a $(\widetilde{\nabla}_1, ð)$ itself.

5. Conclusion

In this paper Symmetrical dense soft set, sequentially dense soft set, isometry in the context of $\mathcal{POS}$ topological space and Convergence of soft fixed point have been studied in the domain of topological spaces, from literature Matthew [11] established the notion of partial metrics by drawing influence from the field of computer science. Subsequently, with the goal of contributing to the progress of soft structures. Our future attempts will go into analyzing fixed point outcomes inside partial soft topological spaces.

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