

**Advancing viscoelastic material modeling : Tackling time-dependent behavior with fractional calculus**

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## Abstract

A combination of inherent unique stress-strain response features, Viscoelastic Materials (VM) provide an integral part in many different fields of engineering. Although practical, these materials' highly complex time-dependent methods according to non-linear loading scenarios are impossible to model using conventional viscoelastic models such as Maxwell and Kelvin-Voigt. The present study introduces a framework that pushes over traditional Maxwell and Kelvin-Voigt approaches by employing fractional calculus in order to enhance the prediction of VM performance. The mathematical representation makes use of the Caputo fractional derivative for expressing an artificial viscoelastic polymer's non-linear and time-dependent responses. Dynamic Mechanical Analysis (DMA) and Stress Relaxation Tests (SRT) proved the polymer possessed 1500 MPa fractional modulus and 0.65 fractional order, respectively. The resulting model involved more significant computing resources, but contrasting testing indicated that it accurately depicted stress relaxation and dynamic responses. As mentioned, the technique integrates mathematical and actual viscoelasticity for industrial uses while offering a precise basis for advanced material analysis.

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**Subject Classification:** 74S40.

**Keywords:** Deep learning, Dynamic mechanical analysis, Viscoelastic models, Fractional calculus, Accuracy, Material modeling.

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## 1. Introduction

Viscoelastic Materials (VM) are essential to the field of material science because numerous industries use them [1]. Conventional models like the Maxwell and Kelvin-Voigt (MKV) systems explain viscoelastic behaviour by exhibiting materials such as springs and dashpots. Linear and time-invariant techniques are frequently unable to deal with all viscoelastic behaviour in non-linear and time-dependent scenarios [2]. The mechanical laws monitoring the stress-strain relationship have been incorporated with Fractional Calculus (FC) in the present investigation, which presents an entirely novel method of modelling VM. In order to simulate the complex, history-dependent behaviours seen in VM, FC enhances the range of possibilities of combination and distinction to non-integer orders, providing a versatile toolbox. A good instance is the Caputo Fractional Derivative (CFD), which presents a more physically accurate model, especially when dealing with first-value challenges.

With this approach, researchers can develop a more accurate representation of materials' dynamic behaviour prediction capability. The present investigation supplements the scientific literature through the implementation of an FC-based framework on a particular viscoelastic polymer, designated for its distinct combination of well-known viscoelastic properties and real uses in manufacturing. Dynamic mechanical analyses (DMA) and comprehensive stress relaxation measurements were employed to analyze the polymer's reaction under various loading conditions during the investigation. The findings of the demonstration emphasized the non-linear and time-dependent attributes of the material, showing that the stress decreased non-exponentially and that the moduli shifted frequency-dependently. The projected value of the fractional modulus  $E_\alpha$  was somewhere around 1500 MPa, and the fractional order of magnitude ' $\alpha$ ' has been determined to be close to 0.65, showing the material's deviation from conventional viscoelastic responses. The recommended viscoelastic model improves a conventional one concerning the effectiveness of computation, complexity collected, and accuracy of behaviour prediction. The comparison study shows that the FC-based model proves more predictive than the conventional methods, particularly if the other model fails to provide the predicted outcomes. The paper is organized as follows: the literature reviewed is presented in section 2, the problem is defined in section 3, section 4 presents the proposed work, section 5 presents the experiment analysis, and section 6 concludes the work.

## 2. Literature Review

Recent research in VM modeling has unveiled significant advancements across various materials and applications: [3] study focused on hydrogels for cartilage replacement, identifying hyper viscoelastic parameters for numerical simulations. Key findings include faster stress relaxation in hydrogels than natural cartilage, providing insights for simulating mechanical behavior in implants. [4] developed a shear-lag model for bio-composites in soft tissues. The study offered analytical solutions for deformation under dynamic loading, enhancing the understanding of soft tissue responses to impact and reciprocating loads. [5] investigated the mechanics of collagen fibrils through stress relaxation and creep tests. The adaptive quasilinear viscoelastic model provided a good fit for data, contributing to a quantitative understanding of collagen fibril mechanics. [6] formulated a non-linear viscoelastic model for PMMA bone types of cement, focusing on time-dependent behaviors under compressive creep loading. Mechanical attributes that are important for healthcare applications can be more clearly understood with the implementation of this model. Researchers in [7] explored the associations between ionomer temperature and strain rate as they applied it to laminated glass. In order to define viscoelastic behaviour [8], this research incorporated phenomenological models with DMA tests; this will be beneficial for the development of materials and uses.

## 3. Problem Definition

In VM, the complex correlation between stress ' $\sigma$ ' and strain ' $\varepsilon$ ' is manipulated by time and the history of functional stress/strain [11]. In non-linear and variable loading scenarios, conventional approaches that depend upon differential equations with linear solutions for describing this behaviour, such as the MKV models, are subject to drawbacks [12-15]. For example, the Maxwell model is described by  $\sigma + \lambda \frac{d\sigma}{dt} = E \frac{d\varepsilon}{dt}$ , where ' $\lambda$ ' is the relaxation time, and ' $E$ ' is the elastic modulus. However, the challenges with non-exponential relaxation are problematic for these mathematical models. By introducing differentiation and integration to non-integer orders, FC presents an increasingly refined methodology for solving these drawbacks. The stress-strain relationship describes VM, and the current investigation proposes an FC-based model to describe them:

$(t) = \int_0^t E_\alpha (t - \tau) \left( \frac{d^\alpha \varepsilon(\tau)}{d\tau^\alpha} \right) d\tau$ . As mentioned above, the model substantially enhances its ability to represent viscoelastic behaviours under numerous

circumstances by incorporating a time-dependent relaxation modulus, defined as  $E_a$ , where  $\alpha$  is a non-integer order.

#### 4. Integrating Fractional Calculus in VM Modeling

When subjected to stress and strain, VM responds in a time-dependent manner. While foundational, traditional models like MKV fail to capture their non-linear, time-dependent behaviors fully. Addressing this model integrates FC into the constitutive equations, enhancing accuracy in depicting these complex, time-dependent responses.

$$\sigma(t) = E_a \frac{d^\alpha \varepsilon(t)}{dt^\alpha} \quad (1)$$

Here,  $\sigma(t)$  denotes the stress,  $\varepsilon(t)$  the strain,  $E_a$  the fractional modulus, and  $\alpha$  the order of the fractional derivative. The constitutive equation for VM is expressed as a fractional differential equation using the Caputo Fractional Derivative (CFD). This modification of the standard derivative for non-integer orders is ideal for initial value problems, providing a physically interpretable and preferred approach for modeling dynamic material behavior. For a function  $f(t)$ , the CFD of order  $\alpha$  is defined as EQU (2)

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad (2)$$

' $n$ ' represents the smallest integer greater than or equal to ' $\alpha$ ', and ' $\Gamma$ ' denotes the gamma function. Applying the Caputo derivative to the strain function  $\varepsilon(t)$ , this EQU (3) directly relates current stress to the entire strain history, highlighting the memory effect inherent in VM. This study expresses the fractional derivative of strain as:

$$\frac{d^\alpha \varepsilon(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\varepsilon^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad (3)$$

Substituting this into the constitutive equation, this work gets EQU (4)

$$\sigma(t) = E_a \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\varepsilon^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad (4)$$

Real-world materials often exhibit non-linear and time-dependent responses. To accommodate this, the model can include non-linear terms and Time-Dependent Fractional Order.

- i) *Non-Linear Extension (NLE)*: To account for NLE behaviors often observed in materials, such as strain hardening or softening, the model includes a non-linear term  $g(\sigma(t), \varepsilon(t))$ . In addition to linear elasticity, this expression allows the mathematical model to represent precise material-specific responses accurately. To incorporate NLE behavior, the constitutive EQU (5) is modified as follows:

$${}^C D^\alpha \sigma(t) + g(\sigma(t), \varepsilon(t)) = E_a^C D^\alpha \varepsilon(t) \quad (5)$$

- ii) *Time-Dependent Fractional Order (TDFO)*: Applying the previously mentioned modification, material characteristics could develop, making the fractional sequence of events ' $\alpha$ ' dependent on time. By allowing ' $\alpha$ ' to vary with time ' $\alpha(t)$ ', the model becomes more adaptable, accurately reflecting time-varying material behaviors, EQU (6)

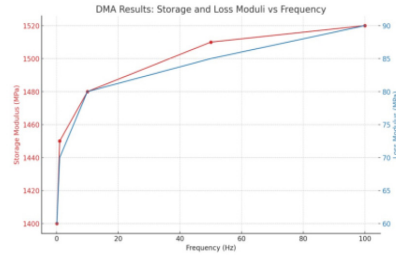
$${}^C D^{\alpha(t)} \sigma(t) + g(\sigma(t), \varepsilon(t)) = E_a^C D^{\alpha(t)} \varepsilon(t) \quad (6)$$

Based on this mathematical behaviour, a VM framework will more accurately depict the material's natural environments' dynamic behaviours.

## 5. Experimental Determination of Fractional Parameters in VM Modeling

The authors explored both the fractional order ' $\alpha$ ' and fractional modulus ' $E_a$ ' of this viscoelastic polymer. After performing multiple SRTs, the investigation team deployed DMA.

- A. Stress Relaxation Test (SRT):** In this study, SRT first assessed the polymer with a conventional mechanical testing machine. A predefined strain was rapidly introduced to the polymer sample and then developed at the specified level. The stress-relieving measures were reported in this article for the remainder of a new trend. The study's results demonstrated that, as is true of complicated VM, the stress disintegrates non-exponentially. The fractional modulus ' $E_a$ ' has been determined to be approximately  $1500 \text{ MPa}$ , which is more stimulating, after the study evaluated the behaviour using a fractional Zener model. The material's deviation from conventional VM responses was demonstrated by the fractional order, which was approximately 0.65. The final results of the relaxation test for stress are presented in Figure 1. It exhibited viscoelastic behaviour, which

**Figure 1****Stress relaxation test****Figure 2****Frequency-dependent viscoelastic behavior**

has been defined by a slow reduction in stress over a period of 24 hours.

- B. DMA:** DMA was conducted at frequencies ranging from 0.1 to 100 Hz to validate these results. The DMA outcomes provided insights into the material's frequency-dependent stiffness and energy dissipation characteristics. Notably, the DMA data correlated well with the stress relaxation test findings, affirming the previously established values of  $E_\alpha$  and  $\alpha$ . This consistency across various testing methods bolstered the credibility of the results. The outcome of the DMA is illustrated in Figure 2, which can be viewed above. This explains the frequency-dependent viscoelastic behaviour of the material by demonstrating the storage modulus (RED) and the loss modulus (BLUE) and how they change with frequencies. There is a correlation between the rate of change and a boost in both the storage and loss moduli over some time.

## 6. Comparative Analysis

The viscoelastic attributes of Polyurethane (PU), a polymer made from synthetic materials, have been contrasted in the present research with those of other VM models. Because of its adaptability, polyurethane has several applications in manufacturing goods, such as rubbery materials for tyres and wheels, thermal foams for buildings, and seat foams for vehicles. Viscoelastic modelling is advantageous because of its distinctive equilibrium between elasticity and viscosity. Two primary techniques for testing were stress relaxation tests and DMA, which were implemented to study the viscoelastic behaviour of Polyurethane in the context of

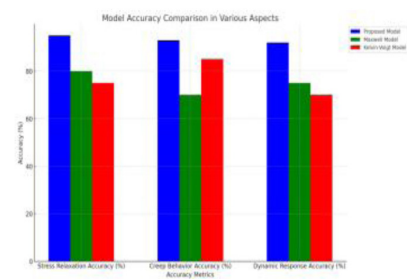


Figure 3

Predicting material behavior

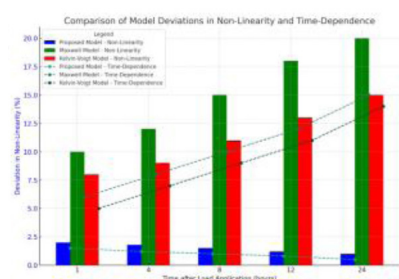


Figure 4

Capturing complex behaviors over time

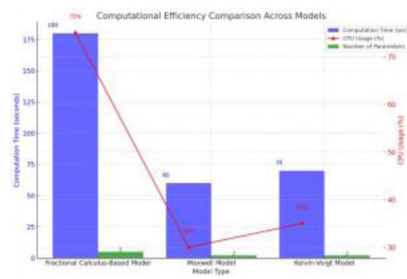


Figure 5

Computational efficiency

experimentation. To better comprehend the time-dependent elastic attributes of Polyurethane, the SRT monitored the material's stress decay under constant strain. Polyurethane's frequency-dependent behaviour has been investigated by employing DMA, which is a significant factor in dynamic loading uses. The proposed approach has been evaluated in comparison with two other models: the Maxwell and the Kelvin-Voigt models. The review was based on these factors:

- Accuracy in Predicting Material Behavior:* Step one was the comparison of the predictions made by the models with the outcomes of the Polyurethane stress relaxation and DMA testing (Figure 3).
- Ability to Capture Complex Behaviors:* An examination was performed in order to identify which mathematical approach was the most successful at describing the non-linear and time-dependent behaviour of polyurethane. The following has significance when stress and strain conditions change (Figure 4).

- iii. *Computational Efficiency*: This factor has been employed to estimate each model's computational resource needs due to Polyurethane's complexity (Figure 5).

However, the FC-based simulation involved higher computational resources, and it ended up being accurate to a greater extent and better at representing Polyurethane's complex viscoelastic behaviour when compared to the Maxwell and Kelvin-Voigt models. This thorough examination illustrates the critical nature of finding the right paradigm based on the material's viscoelastic properties as well as application demands.

## 7. Conclusion and Future Work

The research conducted discovered that the fractional calculus in the Viscoelastic Materials (VM) paradigm performed better than Maxwell and Kelvin-Voigt. With the Caputo fractional derivative in the core hypothesis, VM's complicated, time-dependent behaviour can be clearly demonstrated. Stress Relaxation Tests (SRT) along with Dynamic Mechanical Analysis (DMA) have verified the effectiveness of the model in non-linear and time-dependent response attributes. The preferred model's more significant processing period is a consequence of higher accuracy and forecast. Any subsequent research should focus on boosting the mathematical model's computational speed and developing its material applications. The work presented here advances the material modelling hypothesis as well as provides scientific researchers and engineers with a helpful resource for novel uses across sectors. The research enriches material modelling science and enables scientists and technologists something to use for innovative uses in numerous industries.

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