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The efficacy of a new fixed point method in nonlinear equation solutions

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Abstract

The key objective of this research is to establish a new approach to the classical fixed point method for solving nonlinear equations by using the new theorem of α-fixed point established in [1] and the conformable fractional derivative defined in [6]. As an application, we establish a new α-fixed point algorithm as a numerical result.

Subject Classification: *[2010] 26A33, 54H25, 65Yxx.*

Keywords: *Conformable fractional derivative, α-contractive function, α-fixed-point method, numerical algorithm*

1. Introduction

The principle of the fixed point method in numerical analysis consists in transforming the nonlinear equation $f(x)=0$ in the form $g(x)=x$

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where g is a selected ''good'' auxiliary function. To approach the zeros of f is to approach the fixed points of g. The choice of the function g is motivated by the requirements of the fixed point theorem. It must be a contractive function in a neighbourhood I of the fixed point c, and so $g'(x) < 1$ on this neighborhood. In this case, we build the sequence $(x_n)_n$ defined by $x_0 \in I$ and the relation $x_{n+1} = g(x_n)$, and then we apply locally the fixed point theorem to demonstrate that $(x_n)_n$ converges to c.

Conformable fractional calculus, a generalization of classical fractional calculus, provides a versatile framework for capturing nonlocal dependencies in mathematical models. By incorporating the conformable fractional derivative into the fixed-point method, our proposed approach aims to exploit the benefits of both concepts, offering a robust solution strategy for nonlinear equations. The conformable fractional derivative possesses unique properties that allow for a more accurate representation of the underlying dynamics, especially in cases where standard derivatives may fall short [4, 7, 8, 10, 5].

In [9], Samet Erden and Mehmet Zeki presented some pre-Grüss type results. Afterward, inequalities and they use a special distribution function.

The Riemann-Liouville fractional derivative of order α for a function $f(t)$ is denoted by $D_t^\alpha f(t)$ and is defined as:

$$D_t^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{1.1}$$

where n is the smallest integer greater than α and Γ is the gamma function.

The Caputo fractional derivative of order α for a function $f(t)$ is denoted by $D_t^\alpha f(t)$ and is defined as:

$$D_t^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} f(\tau) d\tau, \tag{1.2}$$

Moreover, the most natural definition of conformable fractional derivative of a function $f : [0, +\infty[\to \mathbb{R}$ defined in [6] for each $\alpha \in]0.1[$ by

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad \forall t > 0;$$

The object of this paper is to generalize the principle of the fixed point method using the theorem of α-fixed point [1] and to establish a new α-fixed point algorithm as a numerical result. In some cases, our α-fixed point algorithm is more efficient than the classical algorithm and minimizes the number of iterations, which is useful to avoid wasting time and machine storage.

We use here the C language as a programming language, leaving the reader to use any other programming language.

2. Preliminaries

We recalll bellew the famous Banach theorem [3] and the classical fixed point method.

Let $f:[a,b]\to[a,b]$ be a contraction mapping on the closed interval $[a,b]$, i.e., there exists a constant $0\le k<1$ such that for all $x, y\in[a,b]$, we have

$$|f(x)-f(y)|\le k|x-y|.$$

Then, the Banach Fixed Point Theorem states that f has a unique fixed point $c\in[a,b]$, i.e., there exists a unique $c\in[a,b]$ such that $f(c)=c$.

Moreover, the sequence defined by $x_0\in I$ and the $x_{n+1}=f(x_n)$ converges to c.

Principle of the classical algorithm of the fixed point method:

The fixed point method is an iterative numerical algorithm used to approximate the solution of the equation $x=g(x)$, where g is a function. Here's the algorithm:

(1) Choose the first guess x_0.

(2) calculate the next approximation $x_1=g(x_0)$.

(3) Repeat step 2 using x_n to compute $x_{n+1}=g(x_n)$, and stopping if a desired number of iterations is achieved.

In other words, we start with an initial value x_0, and then we use the function g to compute a new value x_1. We then use x_1 as the input to g to get x_2, and continue this process until we obtain a value x_n that is ''close enough'' to the true solution.

Typically, we stop iterating when either a specified level of accuracy is achieved, such as when $|x_{n+1}-x_n|<\varepsilon$ for some small value of ε, or when a maximum number of iterations is reached to avoid infinite looping. The key idea behind the fixed point method is to find solutions to a given equation $x=g(x)$, then we can use this iterative method to approximate that solution. In practice, finding a suitable function g can be challenging, and the convergence of the method can be sensitive to the choice of initial guess x_0 and the properties of g.

The following generalized definitions and results are given and showen in [1, 2].

Definition 2.1: [1, 2] Let $f:[a,b]\to[a,b]$ and $\alpha\in]0,1[$.

(1) $c\in[a,b]$ is said a α-fixed point of f if $c^{\frac{1}{\alpha}}\in[a,b]$ and $c=f(c^{\frac{1}{\alpha}})$.

(2) We say that f is strictly α-contractive if there exists a constant $0\le k<1$ such that for all $x,y\in[a,b]$, we have

$$|f(x)-f(y)|\le k|x^{\alpha}-y^{\alpha}|.$$

Theorem 2.2: [1, Theorem 3.5]

Let $f:[a,b]\to[a,b]$ such that f is strictly α-contractive for some $\alpha\in]0,1[$.

Then, f has a unique α-fixed point $c\in[a,b]$ and c is the limit of the sequence defined by $x_0^{\frac{1}{\alpha}}\in[a,b]$ and the relation $x_{n+1}=f(x_n^{\frac{1}{\alpha}})$.

Remarks 2.4: We remark at this level that the conclusion of the previous theorem holds if $x^{\frac{1}{\alpha}}\in[a,b]$ for every $x\in[a,b]$ and f is a mapping on the closed intervall $[a,b]\subset\mathbb{R}_+^*$, wich is α-differentiable for $\alpha\in]0,1[$ and satisfies $\max_{x\in[a,b]}|f^{(\alpha)}(x)|<\alpha$. Moreover, when $\alpha=1$ we recognize the calassical case $f(x)=x$.

3. Generalized α-fixed point method

The α-fixed point method is a numerical technique used to find the solution to a given equation by transforming it into an equivalent α-fixed point equation. The principle of the α-fixed point method revolves around finding an α-fixed point of a function.

To apply the α-fixed point method, we start with a given equation $f(x)=0$, where f is a function. The objective is to rearrange the equation into the form $g(x^{\frac{1}{\alpha}})=x$ where g is a new function derived from f in such a way that its α-fixed points coincide with the solutions of the original equation.

The choice of the function g is motivated by the requirements of the α-fixed point theorem. It must be α-contractive in a neighbourhood I of the α-fixed point c as well as $\max_{x\in I}|g^{(\alpha)}(x)|=k<\alpha$ on this neighborhood. So we can establish the sequence defined by $x_0^{\frac{1}{\alpha}}\in I$ and the relation

$x_{n+1} = g(x_n^{\frac{1}{\alpha}})$. It remains only to apply the α -fixed point theorem locally to demonstrate that $(x_n)_n$ converges to c.

4. Principle of the α -fixed point algorithm

The α -fixed point method is an iterative numerical algorithm used to approximate the solution of a given equation $x = g(x^{\frac{1}{\alpha}})$, where g is a function. Here's the algorithm:

(1) Start with an initial guess $x_0^{\frac{1}{\alpha}}$.

(2) Compute the next approximation $x_1 = g(x_0^{\frac{1}{\alpha}})$.

(3) Repeat step 2 using x_n to compute $x_{n+1} = g(x_n^{\frac{1}{\alpha}})$, and stopping if a desired number of iterations is achieved.

The convergence of the α -fixed point method depends on certain conditions. One important condition is that the function g must be α -contractive in a neighborhood of the α -fixed point.

If the function g satisfies the α -contraction mapping condition, then the α -fixed point method guarantees convergence to the α -fixed point. In practice, the method is often implemented with a termination criterion based on the desired number of iterations.

It's worth noting that finding the α -fixed point of g is equivalent to solving the equation $f(x) = 0$, and the α -fixed point method provides an iterative approach to approximate the solution of this equation.

5. Convergence Analysis

To analyze the convergence of the α -fixed point method, we can use the α -fixed point theorem, which states that if the iteration function f is an α -contraction mapping, then the iterative process converges to the unique α -fixed point.

One common method for estimating the rate of convergence is the a priori error estimate, which provides an upper bound on the error between the true solution and the estimate obtained after a certain number of iterations.

In practice, the convergence of the α -fixed point method may be slow or unstable if the α -contractive conditions of the theorem are not satisfied. To improve the convergence, we can use techniques such as acceleration methods, adaptive step sizes, and hybrid methods that combine different numerical techniques.

It is also important to choose a good initial guess that is close to the α -fixed point, to reduce the number of iterations.

In summary, the convergence analysis of the α -fixed point method involve choosing a suitable function f that satisfies the α -contraction condition and analyzing the value of the Lipschitz constant k to determine the convergence behavior of the method.

The α -fixed point method is a powerful numerical method to solve a wide range of problems, but it requires careful design and analysis to ensure its convergence and accuracy.

6. Numerical Applications

(1) On the interval $I = [\frac{1}{10}, 1]$, we consider the equation

$$\cos(x^{\frac{10}{9}}) - x = 0.$$

For $\alpha = \frac{9}{10}$ and $f(x) = \cos(x^{\frac{10}{9}}) - x$, we have :

$$f(x) = 0 \Leftrightarrow g(x^{\frac{1}{\alpha}}) = x$$

such that $g(x^{\frac{1}{\alpha}}) = \cos(x^{\frac{10}{9}})$. And by [6, Theorem 2.2] we have

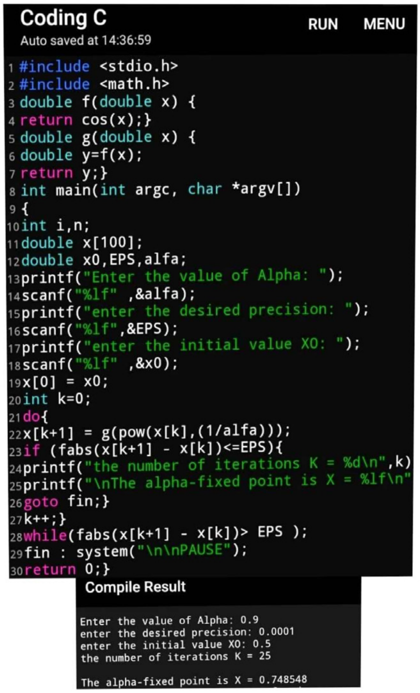
$$g^{\left(\frac{9}{10}\right)}(x) = -x^{\frac{1}{10}} \sin(x).$$

So, $\max_{x \in I} |g^{\left(\frac{9}{10}\right)}(x)| \approx 0.84 < \frac{9}{10}$.

And by approximation, $c \approx 0{,}749$; that is to say

$$g((0{,}749)^{\frac{10}{9}}) \approx 0{,}749.$$

Finally, the equation $g(x^{\frac{1}{\alpha}}) = x$ has a single solution in $[\frac{1}{10}, 1]$.




```c
#include <stdio.h>
#include <math.h>
double f(double x) {
return cos(x);}
double g(double x) {
double y=f(x);
return y;}
int main(int argc, char *argv[])
{
int i,n;
double x[100];
double x0,EPS,alfa;
printf("Enter the value of Alpha: ");
scanf("%lf" ,&alfa);
printf("enter the desired precision: ");
scanf("%lf",&EPS);
printf("enter the initial value X0: ");
scanf("%lf" ,&x0);
x[0] = x0;
int k=0;
do{
x[k+1] = g(pow(x[k],(1/alfa)));
if (fabs(x[k+1] - x[k])<=EPS){
printf("the number of iterations K = %d\n",k)
printf("\nThe alpha-fixed point is X = %lf\n"
goto fin;}
k++;}
while(fabs(x[k+1] - x[k])> EPS );
fin : system("\n\nPAUSE");
return 0;}
```

```
Enter the value of Alpha: 0.9
enter the desired precision: 0.0001
enter the initial value X0: 0.5
the number of iterations K = 25

The alpha-fixed point is X = 0.748548
```

(2) On the interval $[\frac{1}{10}, \frac{1}{2}]$, we consider the equation

$$\frac{e^{(-x^{\left(\frac{\pi}{e}\right)})}}{x} - 1 = 0.$$

For $\alpha = \frac{e}{\pi}$ and $f(x) = \dfrac{e^{(-x^{\left(\frac{\pi}{e}\right)})}}{x} - 1$, we have

$$f(x) = 0 \Leftrightarrow g(x^{\frac{1}{\alpha}}) = x$$

such that $g(x^{\frac{1}{\alpha}}) = e^{(-x^{\left(\frac{\pi}{e}\right)})}$.

And by [6, Theorem 2.2] we obtain

$$g^{\left(\frac{e}{\pi}\right)}(x) = -x^{\frac{\pi - e}{\pi}} e^{-x}.$$

So, we have for $\alpha = \frac{e}{\pi}$, $\max_{x \in I} |g^{\left(\frac{e}{\pi}\right)}(x)| = 0.667 < \frac{e}{\pi}$.

By approximation, we have $c \approx 0{,}584$, and so

$$g((0{,}584)^{\frac{\pi}{e}}) \approx 0{,}584.$$

Finally, the equation $g(x^{\frac{1}{\alpha}}) = x$ has a single solution in $[\frac{1}{10}, \frac{1}{2}]$.

```
Coding C                                RUN   MENU
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#include <stdio.h>
#include <math.h>
double f(double x) {
return exp(-x);}
double g(double x) {
double y=f(x);
return y;}
int main(int argc, char *argv[])
{
int i,n;
double x[100];
double x0,EPS,alfa;
printf("Enter the value of Alpha: ");
scanf("%lf" ,&alfa);
printf("enter the desired precision: ");
scanf("%lf",&EPS);
printf("enter the initial value X0: ");
scanf("%lf" ,&x0);
x[0] = x0;
int k=0;
do{
x[k+1] = g(pow(x[k],(1/alfa)));
if (fabs(x[k+1] - x[k])<=EPS){
printf("the number of iterations K = %d\n",k)
printf("\nThe alpha-fixed point is X = %lf\n"
goto fin;}
k++;}
while(fabs(x[k+1] - x[k])> EPS );
fin : system("\n\nPAUSE");
return 0;}
```

```
Compile Result

Enter the value of Alpha: 0.865
enter the desired precision: 0.0001
enter the initial value X0: 0.5
the number of iterations K = 16

The alpha-fixed point is X = 0.584342
```

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