

Solving systems of fractional differential equations by modified fractional differential transform method with adomian polynomials

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Abstract

In this article, we use a new algorithm as FDT with Adomian polynomials is carried out for solving systems of non-linear fractional differential equations. We approximate solutions to the system of nonlinear equations using this innovative method and discuss the suggested technique of convergence analysis. The theoretical findings and the performance of the numerical approximation are supported by numerical examples. Moreover, using the Python programming, we do numerical and graphical analysis.

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1. Introduction

The area of fractional calculus is one of the numerous applied mathematics disciplines. Instead of integer order, it deals with arbitrarily ordered derivatives and integrals, as well as their applications [1]. Fractional differential equations(FDEs) have been widely utilized to describe a variety of systems that might be associated with a variety of challenges, including viscoelastic systems [2]. Many researchers developed approximation & numerical methods to assess the solution of FDEs due to the difficulties in obtaining exact solutions. The ADM is one of these methods [3-7], beside the HAM [8], VIM [9], FDM [11], DTM [12-14], LADM [15, 16] and so many variety of transform methods [10, 17-19] to solved FDEs . The LADM is an analytical methodology that provides effective approximations for a set of nonlinear differential equations. The DTM [12] has been effectively used for a large class of differential equations that arise in a variety of scientific and engineering fields. Because FDEs more accurately reflect many physical events, Arikoglu and Ozkol devised the FDTM [13] for their adequate solution. The FDTM gives an iterative approach for solving FDEs in series. Unlike the classic series approach, which involves symbolic computing, the FDTM converts FDEs into algebraic approximate solutions by an iterative process. A. Elsaïda [21] recently introduced a more efficient method for solving nonlinear FDEs with the FDTM. L. Ghaffour and Z. Dahmani [22], explored a class of singular FDEs with two different orders of derivation, where the right hand side has an arbitrary singularity on a certain interval of the real axis. Furthermore, new findings on the existence and uniqueness of solutions were discussed by L. Ghaffour et al.

2. Basic Ideas of the FDTM

In this part, we review fractional calculus and fractional differential transform method(FDTM)

Definition 2.1: [23, 24] Riemann-Liouville fractional integral of order $\gamma \geq 0$, is defined by

$$\mathcal{I}^\gamma \varphi(\mathcal{G}) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^{\mathcal{G}} (\mathcal{G} - \tau)^{\gamma-1} \varphi(\tau) d\tau & \gamma > 0, \quad \mathcal{G} > 0. \\ \varphi(\mathcal{G}) & \gamma = 0. \end{cases} \quad (2.1)$$

Definition 2.2: [1, 25] Caputo fractional derivative of φ is defined as [1, 25]

$$\mathcal{D}^\mu \varphi(\mathcal{G}) = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \int_0^\mathcal{G} (\mathcal{G}-\tau)^{n-\gamma-1} \varphi^{(n)}(\tau) d\tau & n-1 < \gamma < n, \\ \frac{d^n \varphi(\mathcal{G})}{d\mathcal{J}^n} & \gamma = n. \end{cases} \quad (2.2)$$

In this case n is an integer. The integral operator of Caputo provides a number of advantages, including:

$$\mathcal{D}^\gamma \mathcal{I}^\gamma \varphi = \varphi(\mathcal{G})$$

$$\mathcal{I}^\gamma \mathcal{D}^\gamma \varphi(\mathcal{G}) = \varphi(\mathcal{G}) - \sum_{k=0}^{n-1} \varphi^{(k)}(0^+) \frac{\mathcal{G}^k}{k!}, \quad t \geq 0 \quad n-1 < \gamma \leq n.$$

In the Riemann-Liouville notion, fractional differentiation is defined by

$$\mathcal{D}_{\mathcal{G}_0}^\gamma \varphi(\mathcal{G}) = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{d\mathcal{G}^m} \left[\int_{\mathcal{G}_0}^\mathcal{G} \frac{\varphi(t)}{(\mathcal{G}-t)^{1+\gamma-m}} dt \right], \quad (2.3)$$

for $\gamma \in [m-1, m)$, & $m \in \mathbb{Z}^+$, $\mathcal{G} > \mathcal{G}_0$. Let's use a fractional power series to express the continuous and analytical function $\varphi(\mathcal{G})$:

$$\varphi(\mathcal{G}) = \sum_{k=0}^{\infty} \Phi(k) (\mathcal{G} - \mathcal{G}_0)^{k/\alpha}, \quad (2.4)$$

where $\Phi(k)$ be the FDT of $\varphi(\mathcal{G})$.

The fractional derivative is utilised in the Caputo sense in order to prevent fractional initial & boundary conditions. The Riemann-Liouville and Caputo operators have the following relationship:

$$\mathcal{D}_{*\mathcal{G}_0}^\gamma \varphi(\mathcal{G}) = \mathcal{D}_{\mathcal{G}_0}^\gamma \left[\varphi(\mathcal{G}) - \sum_{k=0}^{m-1} \frac{1}{k!} (\mathcal{G} - \mathcal{G}_0)^k \varphi^{(k)}(\mathcal{G}_0) \right]. \quad (2.5)$$

Setting $\varphi(\mathcal{G}) = \varphi(\mathcal{G}) - \sum_{k=0}^{m-1} \frac{1}{k!} (\mathcal{G} - \mathcal{G}_0)^k \varphi^{(k)}(\mathcal{G}_0)$ in equation (2.1) & using equation (2.3), we obtain fractional derivative in the Caputo sense [25] as follows:

$$\mathcal{D}_{*\mathcal{G}_0}^\gamma \varphi(\mathcal{G}) = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{d\mathcal{G}^m} \left\{ \int_{\mathcal{G}_0}^\mathcal{G} \left[\frac{\varphi(t) - \sum_{k=0}^{m-1} \frac{1}{k!} (t - \mathcal{G}_0)^k \varphi^{(k)}(\mathcal{G}_0)}{(\mathcal{G}-t)^{1+\gamma-m}} \right] dt \right\}.$$

As the initial conditions are expressed as integer order derivatives, then

$$\Phi(l) = \begin{cases} \text{If } l/\alpha \in \mathbb{Z}^+, \frac{1}{(l/\alpha)!} \left[\frac{d^{l/\alpha} \varphi(g)}{d g^{l/\alpha}} \right]_{g=g_0} & \text{for } l = 0, 1, 2, \dots, (\gamma\alpha - 1) \\ \text{If } l/\alpha \notin \mathbb{Z}^+ & 0, \end{cases} \quad (2.6)$$

where γ denotes the order of the FDE under consideration. The following theorems may be inferred from equations (2.3) and (2.4), with proofs and details available in [13]:

Theorem 2.3: If $\varphi(\zeta) = \psi(\zeta) \pm w(\zeta)$, then $\Phi(l) = \Psi(l) \pm \omega(l)$.

Theorem 2.4: If $\varphi(\zeta) = \psi(\zeta)w(\zeta)$, then $\Phi(l) = \sum_{s=0}^l \Psi(s)\omega(l-s)$.

Theorem 2.5: If $\varphi(\zeta) = \psi_1(\zeta)\psi_2(\zeta)\dots\psi_{n-1}(\zeta)\psi_n(\zeta)$, then

$$\Phi(l) = \sum_{l_{n-1}=0}^l \sum_{l_{n-2}=0}^{l_{n-1}} \dots \sum_{l_2=0}^{l_3} \sum_{l_1=0}^{l_2} \Psi_1(l_1)\Psi_2(l_2-l_1)\dots\Psi_{n-1}(l_{n-1}-l_{n-2})\Psi_n(l-l_{n-1})$$

Theorem 2.6: If $\varphi(\zeta) = (\zeta - \zeta_0)^r$, then $\Phi(\eta) = \delta(\eta - \alpha r)$ where,

$$\delta(\eta) = \begin{cases} 1 & \text{if } \eta = 0 \\ 0 & \text{if } \eta \neq 0 \end{cases}$$

Theorem 2.7: If $\varphi(\zeta) = \mathcal{D}_{\zeta_0}^\gamma [\psi(\zeta)]$, then $\Phi(l) = \frac{\Gamma(\gamma+1+l/\alpha)}{\Gamma(1+l/\alpha)} \Psi(l+\alpha q)$.

2.1 Basic idea of FDTM with Adomian polynomials

Consider the FDE of the form

$$\mathcal{D}^\alpha y = \varphi(y, y^{(\alpha)}), \quad (2.1.1)$$

Here $\varphi(y, y^{(\alpha)})$ be a non-linear function. The recurrence strategy is in the form [15].

$$\frac{\Gamma\left(\alpha+1+\frac{k}{\theta}\right)}{\Gamma\left(1+\frac{k}{\theta}\right)} Y(k+\alpha\theta) = \tilde{A}_k. \quad (2.1.2)$$

Where $\tilde{A}_k$ is gained through the Adomian polynomials of $\varphi(y, y^{(\alpha)})$ by replacing each y_k and $\mathcal{D}^\beta y_k$ in the Adomian polynomial component

A_k by $Y(k)$ and $\frac{\Gamma\left(\beta+1+\frac{k}{\theta}\right)}{\Gamma\left(1+\frac{k}{\theta}\right)} Y(k+\beta\theta)$, respectively.

3. Solving system of FDEs

In this part, we'll use FDTM with Adomian polynomials to solve a systems of FDEs.

Consider the FDEs in a nonlinear system,

$$\mathcal{D}^{\alpha_i} y_i(\zeta) = \mathcal{L}_i(y_1, \dots, y_n) + \mathcal{N}_i(y_1, \dots, y_n) \quad i \in N_n = \{1, 2, \dots, n\} \quad n_{i-1} \leq \alpha_i \leq n_i \quad (3.1)$$

$$y_i^{(k)}(0) = c_k, \quad k \in N_m \cup \{0\} \quad (3.2)$$

where $\mathcal{L}_i$ and $\mathcal{N}_i$ represents linear & nonlinear operators, respectively. We employ the FDTM with Adomian polynomials on system of equation 3.1 we get.

$$\frac{\Gamma\left(\alpha_i + 1 + \frac{k}{\theta_i}\right)}{\Gamma\left(1 + \frac{k}{\theta_i}\right)} Y_{i,k+\alpha_i\theta_i}(k + \alpha_i\theta_i) = Y_{i,k}(k) + \tilde{A}_{i,k}, \quad k \in N_m \cup \{0\} \quad \& \quad i \in N_n \quad (3.3)$$

where $\tilde{A}_{i,k}$ is derived from the Adomian polynomials of $\mathcal{N}_i(y_1, \dots, y_n)$ by substituting each y_k and $\mathcal{D}^{\beta_i} y_{i,k}$ in the Adomian polynomial component

$A_{i,k}$ by $Y_{i,k}(k)$ & $\frac{\Gamma\left(\beta_i + 1 + \frac{k}{\theta_i}\right)}{\Gamma\left(1 + \frac{k}{\theta_i}\right)} Y_{i,k+\beta_i\theta_i}(k + \beta_i\theta_i)$, respectively.

Using equation 2.6, an equation 3.2 can be written as follows,

$$Y_{ik}(k) = C_k, \quad k = 0, 1, 2, \dots, (\alpha_i\theta_i - 1), \quad i \in N_n.$$

Using an infinite series to illustrate the solution is given by in the second phase of the differential transform decomposition approach, we can:

$$y_i = \sum_{m=0}^{\infty} Y_{im}, \quad \& \quad i \in N_n \quad (3.4)$$

and

$$\mathcal{N}_i(y_1, \dots, y_n) = \sum_{m=0}^{\infty} A_{im}.$$

Adomian polynomials are defined as A_{im} and are specified by the following relations,

$$A_{im} = \left[\frac{1}{m!} \frac{d^m}{d\lambda^m} \mathcal{N}_i \left(\sum_{m=0}^{\infty} y_{1m} \lambda^m, \dots, \sum_{m=0}^{\infty} y_{nm} \lambda^m \right) \right]_{\lambda=0}.$$

and, $\tilde{A}_{im}$ is

$$\tilde{A}_{im} = \left[\frac{1}{m!} \frac{d^m}{d\lambda^m} \mathcal{N}_i \left(\sum_{m=0}^{\infty} Y_{1m} \lambda^m, \dots, \sum_{m=0}^{\infty} Y_{nm} \lambda^m \right) \right]_{\lambda=0}.$$

The following recursive formula gives:

$$\begin{aligned} Y_{ik}(k) &= C_k, \quad k = 0, 1, 2, \dots, (\alpha_i \theta_i - 1), \quad i \in N_n \\ Y_{i, (k + \alpha_i \theta_i)} &= \tilde{A}_{i,m}, \quad m = 0, 1, \dots \& \theta_i. \end{aligned} \quad (3.5)$$

Using inverse transformation rule in equation 2.4 we get

$$y_i(\zeta) = \sum_{m=0}^{\infty} Y_{im} \zeta^{k/\alpha_i}. \quad (3.6)$$

3.1 Analysis of Convergence and Error Estimate

Theorem 3.1: If $\mathcal{B}$ be a Banach space, then the series solution of the system (3.1) converges to $S_j \in \mathcal{B}$ for $j \in N_n$, if $\exists \sigma_j \in [0, 1)$ such that, $\|Y_{jn}\| \leq \sigma_j \|Y_{j(n-1)}\|$ $\forall n \in \mathbb{N}$.

Proof: Let the sequences $S_{jn}, j \in N_n$ be a partial sums of the series given by the system (3.5) as

$$\begin{cases} S_{j0} = Y_{j0}(\xi) \\ S_{j1} = Y_{j0}(\xi) + Y_{j1}(\xi) \\ S_{j2} = Y_{j0}(\xi) + Y_{j1}(\xi) + Y_{j2}(\xi) \\ \vdots \\ S_{jn} = Y_{j0}(\xi) + Y_{j1}(\xi) + Y_{j2}(\xi) + \dots + Y_{jn}(\xi), j = 1, 2, \dots, n \end{cases} \quad (3.7)$$

Then we must prove that in Banach space $\mathcal{B}$, $\{S_{jn}\}$ are Cauchy sequences. We examine the following factors in this regard:

$$\begin{aligned} \|S_{j(n+1)} - S_{jn}\| &= \|Y_{j(n+1)}(\xi)\| \leq \sigma_j \|Y_{jn}(\xi)\| \leq \sigma_j^2 \|Y_{j(n-1)}(\xi)\| \leq \dots \\ &\leq \sigma_j^{n+1} \|Y_{j0}(\xi)\|, j \in N_n \end{aligned} \quad (3.8)$$

For $n \geq m$ & $\forall n, m \in \mathbb{N}$, by using the system (3.7) and triangle inequality successively, we have,

$$\begin{aligned}
\|S_{jn} - S_{jm}\| &= \|S_{j(m+1)} - S_{jm} + S_{j(m+2)} - S_{j(m+1)} + \cdots + S_{jn} - S_{j(n-1)}\| \\
&\leq \|S_{j(m+1)} - S_{jm}\| + \|S_{j(m+2)} - S_{j(m+1)}\| + \cdots + \|S_{jn} - S_{j(n-1)}\| \\
&\leq \sigma_j^{m+1} \|Y_{j0}(\xi)\| + \sigma_j^{m+2} \|Y_{j0}(\xi)\| + \cdots + \sigma_j^n \|Y_{j0}(\xi)\| \\
&= \sigma_j^{m+1} (1 + \sigma_j + \cdots + \sigma_j^{n-m-1}) \|Y_{j0}(\xi)\| \\
&\leq \sigma_j^{m+1} \left(\frac{1 - \sigma_j^{n-m}}{1 - \sigma_j} \right) \|Y_{j0}(\xi)\|.
\end{aligned}$$

As $0 < \sigma_j < 1$, so $1 - \sigma_j^{n-m} \leq 1$ then

$$\|S_{jn} - S_{jm}\| \leq \frac{\sigma_j^{m+1}}{1 - \sigma_j} \|Y_{j0}(\xi)\|.$$

Since $Y_{j0}(\xi)$ is bounded, then

$$\lim_{n, m \rightarrow \infty} \|S_{jn} - S_{jm}\| = 0, \quad j \in N_n.$$

As a result, the sequences $\{S_{jn}\}$ in the Banach space $\mathcal{B}$ are Cauchy sequences, and the series solution specified in system (3.6) converges.

Theorem 3.2: *The series solution (3.4) of the system (3.1) is determined to have a maximum absolute truncation error of (3.4).*

$$\sup_{\xi \in \Theta} \left| Y_j(\xi) - \sum_{k=0}^m Y_{jk}(\xi) \right| \leq \frac{\sigma_j^{m+1}}{1 - \sigma_j} \sup_{\xi \in \Theta} |Y_{j0}(\xi)|, \quad j \in N_n,$$

where the region $\Theta \subset \mathbb{R}^{n+1}$.

Proof: We deduce the following from Theorem 3.1:

$$\|S_{jn} - S_{jm}\| \leq \frac{\sigma_j^{m+1}}{1 - \sigma_j} \sup_{(\xi) \in \Theta} |Y_{j0}(\xi)|, \quad j \in N_n \quad (3.9)$$

However, we suppose that $S_{jn} = \sum_{k=0}^n Y_{jk}(\xi)$ for $j = 1, 2, \dots, n$, and since $n \rightarrow \infty$, we obtain $S_{jn} \rightarrow Y_j(\xi)$, so the system (3.9) can be rephrased as

$$\begin{aligned}
\|Y_j(\xi) - S_{jm}\| &= \left\| Y_j(\xi) - \sum_{k=0}^m Y_{jk}(\xi) \right\| \\
&\leq \frac{\sigma_j^{m+1}}{1 - \sigma_j} \sup_{\xi \in \Theta} |Y_{j0}(\xi)|, \quad j \in N_n.
\end{aligned}$$

As a result, in the Θ region, the maximum absolute truncation error is

$$\sup_{\xi \in \Theta} \left| Y_j(\xi) - \sum_{k=0}^m Y_{jk}(\xi) \right| \leq \frac{\sigma_j^{m+1}}{1 - \sigma_j} \sup_{(\xi) \in \Theta} |Y_{j0}(\xi)|, j \in N_n$$

and this completes the proof.

4. Solution of the System of FDEs

Example 4.1: Consider the system of FDEs

$$\begin{aligned} \mathcal{D}_\xi^\alpha y_1 &= y_1, & 0 < \alpha \leq 1, \\ \mathcal{D}_\xi^\beta y_2 &= 2y_1^2, & 0 < \beta \leq 1, \\ \mathcal{D}_\xi^\gamma y_3 &= 3y_1 y_2, & 0 < \gamma \leq 1, \end{aligned} \quad (4.1)$$

subject to

$$y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0. \quad (4.2)$$

Now apply the FDTM with adomian polynomials can be expressed as follows:

$$\begin{aligned} Y_{1,k+\alpha\theta_1}(k+\alpha\theta_1) &= \frac{\Gamma(1+k/\theta_1)}{\Gamma(\alpha+1+k/\theta_1)} Y_{1k} \\ Y_{2,k+\beta\theta_2}(k+\beta\theta_2) &= \frac{\Gamma(1+k/\theta_2)}{\Gamma(\beta+1+k/\theta_2)} \tilde{A}_{2k} \\ Y_{3,k+\gamma\theta_3}(k+\gamma\theta_3) &= \frac{\Gamma(1+k/\theta_3)}{\Gamma(\gamma+1+k/\theta_3)} \tilde{A}_{3k}, \end{aligned} \quad (4.3)$$

where $\theta_1, \theta_2, \theta_3$ are the fractions of order α, β, γ and $Y_1(k), Y_2(k)$ & $Y_3(k)$ are FDT of $y_1(\xi), y_2(\xi)$ and $y_3(\xi)$ respectively. Equation (2.6) can be used to transform the initial conditions in equation (4.2).

$$Y_{10}(0) = 1, \quad Y_{20}(0) = 1, \quad Y_{30}(0) = 0. \quad (4.4)$$

Now we obtained the Adomian polynomials that correspond $A_{ij}, i = 1, 2, 3$ & $j = 0, 1, \dots$

$$\begin{aligned} \tilde{A}_{20} &= 2Y_{10}^2, \\ \tilde{A}_{21} &= 4Y_{10}Y_{11}, \\ \tilde{A}_{22} &= 2Y_{11}^2 + 4Y_{10}Y_{12}, \\ \tilde{A}_{23} &= 4Y_{11}Y_{12} + 4Y_{10}Y_{13}, \end{aligned}$$

$$\begin{aligned}
\tilde{A}_{24} &= 2Y_{12}^2 + 4Y_{11}Y_{13} + 4Y_{10}Y_{14}, \\
\tilde{A}_{25} &= 4Y_{10}Y_{15} + 4Y_{11}Y_{14} + 4Y_{12}Y_{13}, \\
&\vdots \\
\tilde{A}_{30} &= 3Y_{10}Y_{20}, \\
\tilde{A}_{31} &= 3[Y_{11}Y_{20} + Y_{10}Y_{21}], \\
\tilde{A}_{32} &= 3[Y_{12}Y_{20} + Y_{11}Y_{21} + Y_{10}Y_{22}], \\
\tilde{A}_{33} &= 3[Y_{13}Y_{20} + Y_{12}Y_{21} + Y_{11}Y_{22} + Y_{10}Y_{23}], \\
\tilde{A}_{34} &= 3[Y_{14}Y_{20} + Y_{13}Y_{21} + Y_{12}Y_{22} + Y_{11}Y_{23} + Y_{10}Y_{24}], \\
&\vdots
\end{aligned}$$

If $\theta_1 = \frac{1}{\alpha}, \theta_2 = \frac{1}{\beta}, \theta_3 = \frac{1}{\gamma}$ then

$$\begin{aligned}
y_1(\xi) &= 1 + \frac{1}{\Gamma(\alpha+1)}\xi^\alpha + \frac{1}{\Gamma(2\alpha+1)}\xi^{2\alpha} + \frac{1}{\Gamma(3\alpha+1)}\xi^{3\alpha} + \frac{1}{\Gamma(4\alpha+1)}\xi^{4\alpha} + \dots \\
y_2(\xi) &= 1 + \frac{2}{\Gamma(\beta+1)}\xi^\beta + \frac{4\Gamma(\beta+1)}{\Gamma(2\beta+1)\Gamma(\alpha+1)}\xi^{2\beta} + \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)}\left(\frac{2}{(\Gamma(\alpha+1))^2} + \frac{4}{\Gamma(2\alpha+1)}\right)\xi^{3\beta} \\
&\quad + \frac{\Gamma(3\beta+1)}{\Gamma(4\beta+1)}\left(\frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{4}{\Gamma(3\alpha+1)}\right)\xi^{4\beta} + \dots \\
y_3(\xi) &= \frac{3}{\Gamma(\gamma+1)}\xi^\gamma + \xi + \frac{\Gamma(\gamma+1)(3\Gamma(\beta+1)+6\Gamma(\alpha+1))}{\Gamma(2\gamma+1)\Gamma(\alpha+1)\Gamma(\beta+1)}\xi^{2\gamma} \\
&\quad + \frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\left[\frac{3\Gamma(\alpha+1)\Gamma(\beta+1)+6\Gamma(2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{12\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(2\beta+1)}\right]\xi^{3\gamma} \\
&\quad + \frac{\Gamma(3\gamma+1)}{\Gamma(4\gamma+1)}\left[3\left(\frac{1}{\Gamma(3\alpha+1)} + \frac{2}{\Gamma(2\alpha+1)\Gamma(\beta+1)} + \frac{4\Gamma(\beta+1)}{(\Gamma(\alpha+1))^2\Gamma(2\beta+1)}\right)\right. \\
&\quad \left. + \frac{2\Gamma(2\beta+1)}{\Gamma(3\beta+1)(\Gamma(\alpha+1))^2} + \frac{4\Gamma(2\beta+1)}{\Gamma(3\beta+1)\Gamma(2\alpha+1)}\right]\xi^{4\gamma} + \dots
\end{aligned}$$

Taking $\alpha = \beta = \gamma = 1$ and $\theta_1 = \frac{1}{\alpha}, \theta_2 = \frac{1}{\beta}, \theta_3 = \frac{1}{\gamma}$

$$\begin{aligned}
y_1(\xi) &= 1 + \xi + \frac{1}{2!}\xi^2 + \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \dots \\
y_2(\xi) &= 1 + 2\xi + \frac{4\xi^2}{2!} + \frac{4\xi^3}{3} + \dots \\
y_3(\xi) &= 3\xi + \frac{\xi^2}{2} + \frac{\xi^3}{6} + \frac{5\xi^4}{24} + \frac{7\xi^5}{40} + \frac{61\xi^6}{720} \dots
\end{aligned} \tag{4.5}$$

that is

$$y_1(\xi) = e^\xi \quad y_2(\xi) = e^{2\xi} \quad y_3(\xi) = e^{3\xi} - 1$$

Table 1

For $\alpha, \beta, \gamma = 1$, the error estimate between the fifth order approximation & exact of $y_1(\xi)$.

ξ	Exact	Approximate	Absolute Error
0.1	1.10517092	1.10517083	8.47423145e-08
0.2	1.22140276	1.2214	2.75816017e-06
0.3	1.34985881	1.3498375	2.13075760e-05
0.4	1.4918247	1.49173333	9.13643079e-05
0.5	1.64872127	1.6484375	2.83770700e-04
0.6	1.8221188	1.8214	7.18800391e-04
0.7	2.01375271	2.01217083	1.58187414e-03
0.8	2.22554093	2.2224	3.14092849e-03
0.9	2.45960311	2.4538375	5.76561116e-03
1.0	2.71828183	2.70833333	9.94849513e-03

Table 2

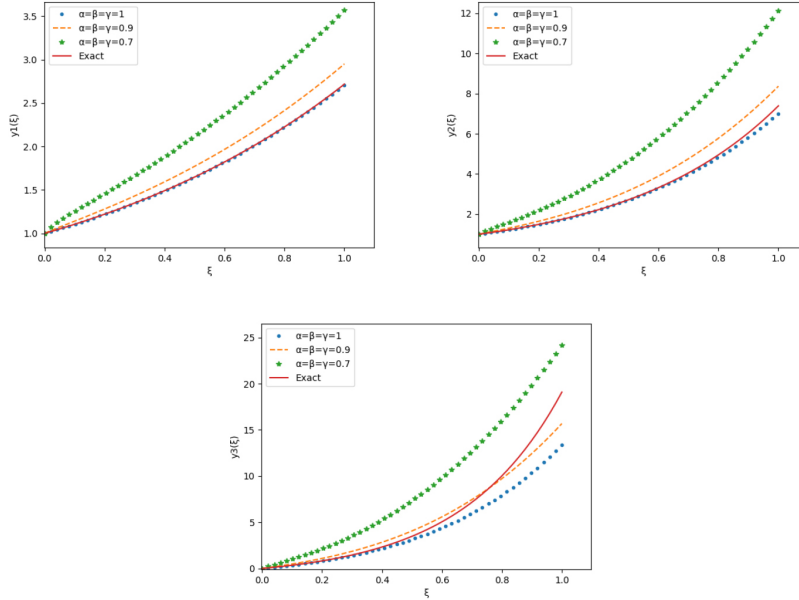
For $\alpha, \beta, \gamma = 1$, the error estimate between the fifth order approximation & exact of $y_2(\xi)$.

ξ	Exact	Approximate	Absolute Error
0.1	1.22140276	1.2214	2.75816017e-06
0.2	1.4918247	1.49173333	9.13643079e-05
0.3	1.8221188	1.8214	7.18800391e-04
0.4	2.22554093	2.2224	3.14092849e-03
0.5	2.71828183	2.70833333	9.94849513e-03
0.6	3.32011692	3.2944	2.57169227e-02
0.7	4.05519997	3.9974	5.77999668e-02
0.8	4.95303242	4.83573333	1.17299091e-01
0.9	6.04964746	5.8294	2.20247464e-01
1.0	7.3890561	7	3.89056099e-01

Table 3

For $\alpha, \beta, \gamma = 1$, the error estimate between the fifth order approximation & exact of $y_3(\xi)$.

ξ	Exact	Approximate	Absolute Error
0.1	0.34985881	0.3483375	1.52130758e-03
0.2	0.8221188	0.8094	1.27188004e-02
0.3	1.45960311	1.4133375	4.62656112e-02
0.4	2.32011692	2.1984	1.21716923e-01
0.5	3.48168907	3.2109375	2.70751570e-01
0.6	5.04964746	4.5054	5.44247464e-01
0.7	7.16616991	6.1443375	1.02183241e+00
0.8	10.02317638	8.1984	1.82477638e+00
0.9	13.87973172	10.7463375	3.13339422e+00
1.0	19.08553692	13.875	5.21053692e+00

**Figure 1**

Approximate solution of 4.1 at various α, β, γ values.

Example 4.2: Consider the system of FDEs

$$\begin{aligned}\mathcal{D}_\xi^\alpha y_1 &= \frac{1}{2}y_1, \quad 0 < \alpha \leq 1 \\ \mathcal{D}_\xi^\beta y_2 &= y_2 + y_1^2, \quad 0 < \beta \leq 1\end{aligned}\tag{4.6}$$

with

$$y_1(0) = 1, \quad y_2(0) = 0,\tag{4.7}$$

Now apply the FDTM with adomian polynomials can be expressed as follows:

$$\begin{aligned}Y_{1,k+\alpha\theta_1}(k+\alpha\theta_1) &= \frac{\Gamma(1+k/\theta_1)}{\Gamma(\alpha+1+k/\theta)} \left(\frac{1}{2} Y_{1k} \right) \\ Y_{2,k+\beta\theta_2}(k+\beta\theta_2) &= \frac{\Gamma(1+k/\theta_2)}{\Gamma(\beta+1+k/\theta_2)} (Y_{2k} + \tilde{A}_{2k})\end{aligned}\tag{4.8}$$

where θ_1, θ_2 are the fractions of order α, β and $Y_1(k)$ and $Y_2(k)$ are FDT of $y_1(\xi)$ and $y_2(\xi)$ respectively. Equation (2.6) can be used to transform the initial conditions in equation (4.7) as follows:

$$Y_{10}(0) = 1, \quad Y_{20}(0) = 0\tag{4.9}$$

Now we obtained the Adomian polynomials that correspond A_{2j} , and $j=0,1,\dots$

$$\begin{aligned}\tilde{A}_{20} &= Y_{10}^2, \\ \tilde{A}_{21} &= 2Y_{10}Y_{11}, \\ \tilde{A}_{22} &= Y_{11}^2 + 2Y_{10}Y_{12}, \\ \tilde{A}_{23} &= 2Y_{11}Y_{12} + 2Y_{10}Y_{13}, \\ &\vdots\end{aligned}$$

$$\text{If } \theta_2 = \frac{1}{\alpha}, \theta_2 = \frac{1}{\beta},$$

$$\begin{aligned}y_1(\xi) &= 1 + \xi^\alpha \frac{1}{2\Gamma(\alpha+1)} + \frac{1}{4\Gamma(2\alpha+1)} \xi^{2\alpha} + \frac{1}{2^3\Gamma(3\alpha+1)} \xi^{3\alpha} + \frac{1}{2^4\Gamma(4\alpha+1)} \xi^{4\alpha} + \dots \\ y_2(\xi) &= \frac{t^\beta}{\Gamma(\beta+1)} + \frac{\Gamma(1+\alpha)+\Gamma(1+\beta)}{\Gamma(2\beta+1)\Gamma(\alpha+1)} \xi^{2\beta} + \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} \left[\frac{\Gamma(1+\alpha)+\Gamma(1+\beta)}{\Gamma(2\beta+1)\Gamma(\alpha+1)} \right. \\ &\quad \left. + \frac{1}{(2\Gamma(\alpha+1))^2} + \frac{1}{2\Gamma(2\alpha+1)} \right] \xi^{3\beta} \\ &\quad + \frac{\Gamma(1+3\beta)}{\Gamma(1+4\beta)} \left[\frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)} \left(\frac{\Gamma(1+\alpha)+\Gamma(1+\beta)}{\Gamma(2\beta+1)\Gamma(\alpha+1)} + \frac{1}{(2\Gamma(\alpha+1))^2} + \frac{1}{2\Gamma(2\alpha+1)} \right) \right. \\ &\quad \left. + \frac{2}{8\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{2}{2^3\Gamma(3\alpha+1)} \right] \xi^{4\beta} \dots\end{aligned}\tag{4.10}$$

Table 4

For $\alpha, \beta = 1$, the error estimate between the fifth order approximation
& exact of $y_1(\xi)$.

ξ	Exact	Approximate	Absolute Error
0.1	1.0512711	1.05127109	2.62602406e-09
0.2	1.10517092	1.10517083	8.47423145e-08
0.3	1.16183424	1.16183359	6.48978283e-07
0.4	1.22140276	1.2214	2.75816017e-06
0.5	1.28402542	1.28401693	8.48960441e-06
0.6	1.34985881	1.3498375	2.13075760e-05
0.7	1.41906755	1.41902109	4.64548433e-05
0.8	1.4918247	1.49173333	9.13643079e-05
0.9	1.56831219	1.56814609	1.66091740e-04
1.0	1.64872127	1.6484375	2.83770700e-04

Table 5

For $\alpha, \beta = 1$, the error estimate between the fifth order approximation
& exact of $y_2(\xi)$.

ξ	Exact	Approximate	Absolute Error
0.1	0.11051709	0.11051806	-9.63747991e-07
0.2	0.24428055	0.24428889	-8.33725685e-06
0.3	0.40495764	0.4049625	-4.85772720e-06
0.4	0.59672988	0.59662222	1.07656834e-04
0.5	0.82436064	0.82378472	5.75913128e-04
0.6	1.09327128	1.0914	1.87128023e-03
0.7	1.4096269	1.40485139	4.77550634e-03
0.8	1.78043274	1.76995556	1.04771872e-02
0.9	2.2136428	2.1929625	2.06803000e-02
1.0	2.71828183	2.68055556	3.77262729e-02

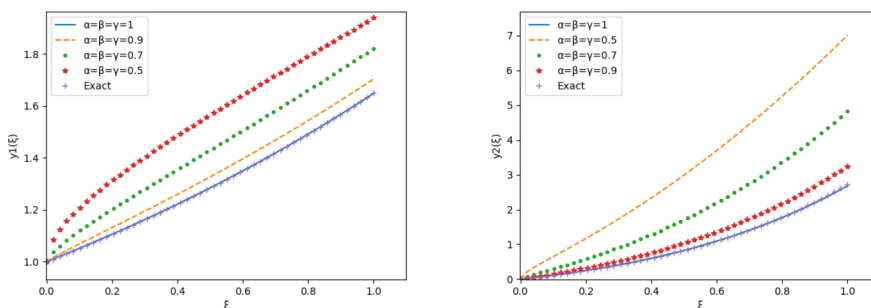


Figure 2

Approximate solution of 4.6 at various α, β values.

Conclusion

This work uses FDTM with Adomian polynomials to solve a system of nonlinear FDEs. We approximate solutions to the system of nonlinear equations using this innovative method and discussed this suggested technique of convergence analysis and Absolute error. To validate the theoretical findings and the effectiveness of the numerical approximation, certain numerical examples are provided. Moreover, we analyze graphically by using Python software.

List of Abbreviation

- FDEs Fractional Differential Equations.
- FDTM Fractional Differential Transform Method.
- DT Differential Transform.
- HAM Homotopy Analysis Method.
- ADM Adomian Decomposition Method.
- VIM Variational Iterational Method.
- LADM Laplace Adomian Decomposition Method.

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