

On the unit group of the semisimple group algebra $\mathcal{K}_q GL(2, \mathbb{Z}_s)$

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Abstract

In this article, we determine the unit group of the semisimple group algebra $\mathcal{K}_q GL(2, \mathbb{Z}_s)$ of general linear group of 2×2 invertible matrices over the finite field $\mathbb{Z}_s$. Here $\mathcal{K}_q$ represents a finite field of order $q = p^k$, where p is a prime and k is a positive integer.

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1. Introduction

Let $\mathcal{K}_q$ denote the finite field of characteristic p , where p is a prime and $q = p^k$ for some positive integer k . Let $\mathcal{K}_q G$ denote the group algebra of the finite group G over $\mathcal{K}_q$ (see [13]). Let $\mathcal{U}(\mathcal{K}_q G)$ be the group of units of the group algebra $\mathcal{K}_q G$. One of the well-known and important problems in the subject of group algebras is to determine the structure of $\mathcal{U}(\mathcal{K}_q G)$ of the group algebra $\mathcal{K}_q G$ (see [13]). This is because the characterization of units is helpful in the development of convolutional codes in coding theory (cf. [4, 8]) as well in the study of isomorphism problems ([13]) etc.

In the recent years, an extensive study has been conducted in order to characterize the structure of the unit group of the semisimple group algebra $\mathcal{K}_q G$ (see [13, 3, 2, 21, 11, 12, 19, 20, 14, 15, 16, 17]). One of the most important studies in this field is by Bakshi et al. [3], in which the authors investigated the unit groups of the semisimple group algebras of all metabelian groups. As a result, the majority of the research in this area is concentrated on the study of unit groups of non-metabelian semisimple group algebras. Mittal et al. [14] characterized the unit groups of the group algebras of non-metabelian groups up to order 72. Further, Sharma et al. [10, 18] determined the unit group of the semisimple group algebra of the respective groups $SL(2, \mathbb{Z}_3)$ and $SL(2, \mathbb{Z}_5)$ (special linear groups of 2×2 matrices over the finite fields $\mathbb{Z}_3$ and $\mathbb{Z}_5$). In continuation, Arvind et al. [1] studied the unit group of the semisimple group algebra of the group $SL(3, \mathbb{Z}_2)$. For the group $GL(2, \mathbb{Z}_3)$ (general linear group of 2×2 matrices over $\mathbb{Z}_3$), the unit group of its semisimple group algebra is studied in [14]. The group $GL(2, \mathbb{Z}_4)$ is isomorphic to the direct product of two lower order groups (i.e., $GL(2, 4) \cong A_5 \times C_3$, where A_5 is alternating group on 5 symbols and C_3 is a cyclic group of order 3). Consequently, the unit group of its associated group algebra can be easily characterized by using the tensor product formula (see [16]). In this paper, our goal is to deduce the unit group of the semisimple group algebra of the group $GL(2, \mathbb{Z}_5)$. We observe that as the size of the group increases, it is difficult to uniquely characterize the unit group of the semisimple group algebras.

Let $\mathcal{U}(\mathcal{K}_q GL(2, \mathbb{Z}_5))$ denote the unit group of the general linear group algebra $\mathcal{K}_q GL(2, \mathbb{Z}_5)$ of 2×2 matrices over $\mathbb{Z}_5$, where $\mathcal{K}_q$ is a finite field of characteristic p . Here p does not divide the order of the group $GL(2, \mathbb{Z}_5)$. To determine $\mathcal{U}(\mathcal{K}_q GL(2, \mathbb{Z}_5))$, our main task is to deduce the Wedderburn decomposition of $\mathcal{K}_q GL(2, \mathbb{Z}_5)$. From the knowledge of Wedderburn decomposition, it is trivial to obtain the unit group.

The rest of this paper is organized as follows. Section 2 deals with the preliminaries required for the article. In section 3, we characterize the unit group of the group algebra $\mathcal{K}_q GL(2, \mathbb{Z}_5)$ in theorem 3.1. Finally, section 4 concludes the paper.

2. Preliminaries

Throughout this article, $GL(n, \mathbb{Z}_p)$ denotes the general linear group of $n \times n$ invertible matrices over the field $\mathbb{Z}_p$. The order of the general linear group $GL(n, \mathbb{Z}_p)$ is given by

$$(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}). \quad (1)$$

Let $J(\mathcal{K}_q G)$ denote the Jacobson radical of $\mathcal{K}_q G$. Next, we discuss some notations and results from [6]. Let s be the exponent of the group G and let η be the primitive s^{th} root of unity over a finite field $\mathcal{K}$. Let

$$T_{G, \mathcal{K}} = \{t : \eta \rightarrow \eta^t \text{ is an automorphism of } \mathcal{K}(\eta) \text{ over } \mathcal{K}\}.$$

Since the Galois group $\text{Gal}(\mathcal{K}(\eta) : \mathcal{K})$ is cyclic, for any $\sigma \in \text{Gal}(\mathcal{K}(\eta) : \mathcal{K})$, there exists a positive integer s such that $\sigma(\eta) = \eta^s$. For any p -regular element $g \in G$ (i.e., p does not divide order of g), we define $\gamma_g = \sum_{h \in C_g} h$, where C_g is the conjugacy class of g . The cyclotomic $\mathcal{K}$ -class of γ_g is defined as

$$S\mathcal{K}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G, \mathcal{K}}\}.$$

The following theorem characterizes the set $T_{G, \mathcal{K}}$.

Theorem 2.1: [14, Theorem 2.3] *Let $\mathcal{K}$ be a finite field with prime power order d such that $\gcd(d, s) = 1$ and $e = \text{order}_s(d)$ is the multiplicative order of d modulo s , then*

$$T_{G, \mathcal{K}} = \{1, d, \dots, d^{e-1}\} \bmod s.$$

To uniquely identify the Wedderburn decomposition of the group algebra $\mathcal{K}_q GL(2, \mathbb{Z}_5)$, the following six results will play an important role.

Proposition 2.1: [6, Proposition 1.2] The number of non isomorphic simple components of $\frac{\mathcal{K}G}{J(\mathcal{K}G)}$ is equal to the number of cyclotomic $\mathcal{K}$ -classes in G .

Theorem 2.2: [6, Theorem 1.3] *Assume that G has t cyclotomic $\mathcal{K}$ -classes and $\text{Gal}(\mathcal{K}(\eta) : \mathcal{K})$ is a cyclic group, then $|S_i| = [\mathcal{K}_i : \mathcal{K}]$ with appropriate index*

ordering if $S_1, S_2, \dots, S_t$ are the cyclotomic $\mathcal{K}$ -classes of G and $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_t$ are the simple components of $\mathbb{Z}\left(\frac{\mathcal{K}G}{J(\mathcal{K}G)}\right)$.

Proposition 2.2: [13, Proposition 3.6.11] Let G' be the commutator subgroup of G and let $\mathcal{K}G$ be a semisimple group algebra, then

$$\mathcal{K}G \simeq \mathcal{K}(G/G') \oplus \Delta(G, G'),$$

where $\mathcal{K}(G/G')$ is the sum of all commutative simple components of $\mathcal{K}G$ and $\Delta(G, G')$ is the sum of all others.

Proposition 2.3: [13, Proposition 3.6.7] Let N be a normal subgroup of G and let $\mathcal{K}G$ be a semisimple group algebra, then

$$\mathcal{K}G \simeq \mathcal{K}(G/N) \oplus \Delta(G, N),$$

where $\Delta(G, N)$ is an ideal of $\mathcal{K}G$ generated by the set $\{n-1 : n \in N\}$.

Proposition 2.4: [5, Proposition 1] Let $\mathcal{K}G$ be a finite semisimple group algebra, where characteristics of $\mathcal{K}$ is p . Let

$$\mathcal{K}G \cong \bigoplus_{i=1}^r M_{n_i}(\mathcal{K}_i),$$

where $\mathcal{K}_i$ are finite extensions of $\mathcal{K}$ and r is a positive integer. Then p does not divide any of the n_i .

Proposition 2.5: [1, corollary 3.8] Let $\mathcal{K}G$ be a finite semisimple group algebra, where characteristics of $\mathcal{K}$ is p . Then if there exists an irreducible representations of degree n over $\mathcal{K}$, then one of the Wedderburn component of $\mathcal{K}G$ is $M(n, \mathcal{K})$. Also, if there exists k irreducible representations of degree n over $\mathcal{K}$, then $(M(n, \mathcal{K}))^k$ is a summand of the group algebra $\mathcal{K}G$.

Next, we discuss an important observation that play an important role in our main result (i.e., theorem 3.1).

Conjecture 2.1: Suppose that a group G has n conjugacy classes for some positive integer n . Let g_i , $1 \leq i \leq 2n$ be the distinct representatives of the conjugacy classes of G . Let q_1 be such that $|SK_{q_1}(\gamma_g)| = 1$ for m_1 representatives g of G and $|SK_{q_1}(\gamma_g)| = 2$ for m_2 representatives g of G , where $m_1 + m_2 = n$. Then, if $M(z, \mathcal{K}_{q_1})$ is a Wedderburn component of the semisimple group algebra $\mathcal{K}_{q_1}G$ for some $z \geq 2$, then $M(z, \mathcal{K}_q)$ must be a Wedderburn component of the semisimple group algebra $\mathcal{K}_q G$, where q is such that $|SK_q(\gamma_g)| = 1$ for all g in G .

The result is obviously true if $z=1$. Next, in the support of our conjecture, we give two non-trivial examples.

Example 2.1: Let $G = SL(2,3)$. This group has 7 conjugacy classes. For $q_1 = 5$, we note that $|SK_{q_1}(\gamma_g)| = 1$ for 3 representatives g of G and $|SK_{q_1}(\gamma_g)| = 2$ for 4 representatives g of G . Also, Using 10, we know that $M(2, \mathcal{K}_{5^2})$ is a Wedderburn component of the semisimple group algebra $\mathcal{K}_5 G$. For $q = 7$, we observe that $|SK_q(\gamma_g)| = 1$ for all g in G . In this case, again, 10 implies that $M(2, \mathcal{K}_7)$ is a Wedderburn component of the semisimple group algebra $\mathcal{K}_7 G$.

Example 2.2: Let $G = A_5$ (alternating group on 5 symbols). This group has 9 conjugacy classes. For $q_1 = 7$, we note that $|SK_{q_1}(\gamma_g)| = 1$ for 5 representatives g of G and $|SK_{q_1}(\gamma_g)| = 2$ for 4 representatives g of G . Also, $m_1 + m_2 = 5 + 4 = 9$. Using 7, we know that $M(2, \mathcal{K}_{7^2})$ and $M(3, \mathcal{K}_{5^2})$ are the Wedderburn components of the semisimple group algebra $\mathcal{K}_7 G$. For $q = 11$, we observe that $|SK_q(\gamma_g)| = 1$ for all g in G . In this case, again, 7 implies that $M(2, \mathcal{K}_{11})$ and $M(3, \mathcal{K}_{11})$ are the Wedderburn components of the semisimple group algebra $\mathcal{K}_{11} G$.

3. Main result

Let us denote $G = GL(2, \mathbb{Z}_5)$. Clearly, the order of G is 480 due to (1). We take the prime $p > 5$. Since it does not divide the order of G , $J(\mathcal{K}_q G)$ is 0, by Maschke's theorem [13], the group algebra $\mathcal{K}_q GL(2, \mathbb{Z}_5)$ is semisimple. Also, G has 24 conjugacy classes. The representative, size and the order of the representatives of G are tabulated below.

Representative	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
Size	1	24	1	24	1	24	1	24	20	20	20	20
Order	1	10	4	20	2	5	4	20	3	6	8	24

g_{13}	g_{14}	g_{15}	g_{16}	g_{17}	g_{18}	g_{19}	g_{20}	g_{21}	g_{22}	g_{23}	g_{24}
20	20	20	20	20	20	30	30	30	30	30	30
24	12	12	8	24	24	4	2	4	4	4	4

Here, g_i 's are defined as follows:

$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$	$g_7 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$	$g_{13} = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix},$	$g_{19} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},$
$g_2 = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix},$	$g_8 = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix},$	$g_{14} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix},$	$g_{20} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix},$
$g_3 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},$	$g_9 = \begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix},$	$g_{15} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$	$g_{21} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$
$g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$	$g_{10} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix},$	$g_{16} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix},$	$g_{22} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$
$g_5 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix},$	$g_{11} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix},$	$g_{17} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix},$	$g_{23} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$
$g_6 = \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix},$	$g_{12} = \begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix},$	$g_{18} = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},$	$g_{24} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$

Also, we know from [7] that G has irreducible representations of degrees 1,4,5 and 6 whenever $|SK_q(\gamma_g)|=1, \forall g \in G$. Next, we know that G can be generated by two elements a and b , where

$$a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}. \quad (2)$$

In this section, we characterize the structure of the unit group of the group algebra $\mathcal{K}_q G$ for $p > 5$, where $q = p^k$. In the following theorem, we discuss the Wedderburn decomposition of the group algebra $\mathcal{K}_q G$.

Theorem 3.1: *The Wedderburn decomposition of $\mathcal{K}_q G$, for $p > 5$, where $\mathcal{K}_q$ is a finite field having $q = p^k$ elements is given as follows:*

- (1) for $p^k \equiv \{1, 29, 49, 53, 73, 77, 97, 101\} \pmod{120}$, we have

$$(\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^{10} \oplus (M(5, \mathcal{K}_q))^4 \oplus (M(6, \mathcal{K}_q))^6.$$

- (2) for $p^k \equiv \{7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, 91, 103, 107, 119\} \pmod{120}$, we have

$$(\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \oplus (M(4, \mathcal{K}_q))^2 \oplus (M(5, \mathcal{K}_q))^2 \oplus (M(6, \mathcal{K}_q))^2 \oplus (M(4, \mathcal{K}_{q^2}))^4 \\ \oplus M(5, \mathcal{K}_{q^2}) \oplus (M(6, \mathcal{K}_{q^2}))^2.$$

- (3) for $p^k \equiv \{13, 17, 37, 41, 61, 89, 109, 113\} \pmod{120}$, we have

$$(\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^6 \oplus (M(5, \mathcal{K}_q))^4 \oplus (M(6, \mathcal{K}_q))^6 \oplus (M(4, \mathcal{K}_{q^2}))^2.$$

Proof: The order of the group G is 480 (see (1)). From the description of the conjugacy classes of G given in the beginning of this section, it is clear that the exponent of G is 120. Since $\mathcal{K}_q G$ is semisimple, it follows from the Wedderburn decomposition theorem (cf. [13]) that

$$\mathcal{K}_q G \simeq \bigoplus_{i=1}^r M(n_i, \mathcal{K}_i),$$

where $\mathcal{K}_i$ are finite extensions of $\mathcal{K}_q$ and n_i, r are positive integers. The derived subgroup G' of the group G is $SL(2, 5)$ and the quotient group $G/G' \simeq C_4$. This along with proposition 2.2 imply that

$$\mathcal{K}_q G \simeq (\mathcal{K}_q)^4 \bigoplus_{i=1}^{r-4} M(n_i, \mathcal{K}_i), \quad n_i \geq 2. \quad (3)$$

Further, we note from theorem 2.1 that

$$\begin{aligned} T_{G, \mathcal{K}_q} = \{ & 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, \\ & 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, 97, 101, 103, 107, \\ & 109, 113, 119 \} \bmod 120. \end{aligned}$$

At this stage, we split the set $T_{G, \mathcal{K}_q}$ into three disjoint subsets and proceed through the following three cases.

Case 1: $p^k \equiv \{1, 29, 49, 53, 73, 77, 97, 101\} \bmod 120$. In this case, it can be verified that the cardinality of cyclotomic $\mathcal{K}_q$ -class of γ_g is 1 for all g in G (i.e., $|S\mathcal{K}_q(\gamma_g)| = 1, \forall g \in G$). Therefore, employing proposition 2.1 and theorem 2.2 to write (3) as

$$\mathcal{K}_q G \simeq (\mathcal{K}_q)^4 \bigoplus_{i=1}^{20} M(n_i, \mathcal{K}_q) \Rightarrow 476 = \sum_{i=1}^{20} n_i^2, \quad n_i \geq 2. \quad (4)$$

Next, we note that the size of conjugacy class of g_5 is 1, which means that $N = \langle g_5 \rangle$ is a normal subgroup of G having order 2. Also, we observe that $G/N \simeq A_5 \rtimes C_4$. To this end, we recall from [15, theorem 4.3] that

$$\mathcal{K}_q (A_5 \rtimes C_4) \simeq (\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^4 \oplus (M(5, \mathcal{K}_q))^4 \oplus (M(6, \mathcal{K}_q))^2. \quad (5)$$

By utilizing (5) in proposition 2.3 for $N = \langle g_5 \rangle$ and plugging the result in (4) to arrive at

$$\mathcal{K}_q G \simeq (\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^4 \oplus (M(5, \mathcal{K}_q))^4 \oplus (M(6, \mathcal{K}_q))^2 \bigoplus_{i=1}^{10} M(n_i, \mathcal{K}_q), \quad (6)$$

with

$$240 = \sum_{i=1}^{10} n_i^2, n_i \geq 2. \quad (7)$$

One can see that there are 69 possible choices of n_i 's that fulfills (7). But we need to derive the unique choice. In the beginning of this section, we discussed that there are no irreducible representations of G having degree 2 and 3 whenever $|SK_q(\gamma_g)| = 1, \forall g \in G$. This, proposition 2.5, (6) and (7) derive that

$$\mathcal{K}_q G = (\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^4 \oplus (M(5, \mathcal{K}_q))^4 \oplus (M(6, \mathcal{K}_q))^2 \bigoplus_{i=1}^{10} M(n_i, \mathcal{K}_q), \quad (8)$$

with

$$240 = \sum_{i=1}^{10} n_i^2, n_i \geq 4. \quad (9)$$

The equation (6) has only two possibilities namely $(4^6, 6^4)$ and $(4^5, 5^3, 6, 7)$. Suppose that the second possibility is true. This means that $M(7, \mathcal{K}_q)$ is a Wedderburn component of the group algebra $\mathcal{K}_q G$. For $p=7$, we know that $7^4 \bmod 120 = 1$. This means that 7^4 is covered in the current case. As per proposition 2.4, we know that if characteristics of $\mathcal{K}_q$ is p and $\mathcal{K}G \cong \bigoplus_{i=1}^r M_{n_i}(\mathcal{K}_i)$, then p must not divide n_i . This means that $M(7, \mathcal{K}_q)$ can not be the Wedderburn component of the group algebra $\mathcal{K}_q G$. Therefore, the unique choice of n_i 's that fulfills (9) is $(4^6, 6^4)$. Hence, the required Wedderburn decomposition is

$$\mathcal{K}_q G = (\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^{10} \oplus (M(5, \mathcal{K}_q))^4 \oplus (M(6, \mathcal{K}_q))^6. \quad (10)$$

Case 2: $p^k \equiv \{7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, 91, 103, 107, 119\} \bmod 120$. The cyclotomic $\mathcal{K}_q$ classes of γ_g are

$$SK_q(\gamma_{g_1}) = \{\gamma_{g_1}\}, \text{ for } 1, 2, 5, 6, 9, 10, 20, 23, SK_q(\gamma_{g_3}) = \{\gamma_{g_3}, \gamma_{g_7}\}, SK_q(\gamma_{g_{12}}) = \{\gamma_{g_{12}}, \gamma_{g_{17}}\},$$

$$SK_q(\gamma_{g_{13}}) = \{\gamma_{g_{13}}, \gamma_{g_{18}}\}, SK_q(\gamma_{g_{14}}) = \{\gamma_{g_{14}}, \gamma_{g_{15}}\}, SK_q(\gamma_{g_{19}}) = \{\gamma_{g_{19}}, \gamma_{g_{21}}\},$$

$$SK_q(\gamma_{g_{22}}) = \{\gamma_{g_{22}}, \gamma_{g_{24}}\}, SK_q(\gamma_{g_4}) = \{\gamma_{g_4}, \gamma_{g_8}\}, SK_q(\gamma_{g_{11}}) = \{\gamma_{g_{11}}, \gamma_{g_{16}}\}.$$

By incorporating proposition 2.1, theorem 2.2 and proposition 2.2 with $G/G' \simeq C_4$, $\mathcal{K}_q G/G' \simeq (\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2}$, we derive from (3) that

$$\mathcal{K}_q G \simeq (\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \bigoplus_{i=1}^6 \mathbf{M}(n_i, \mathcal{K}_q) \bigoplus_{i=7}^{13} \mathbf{M}(n_i, \mathcal{K}_{q^2}) 476 = \sum_{i=1}^6 n_i^2 + \sum_{i=7}^{13} 2n_i^2, n_i \geq 2. \quad (11)$$

Next, we observe that the Wedderburn decomposition of $\mathcal{K}_q(A_5 \rtimes C_4)$ in this case is (see [15, theorem 4.3])

$$\begin{aligned} \mathcal{K}_q(A_5 \rtimes C_4) \simeq & (\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \oplus (\mathbf{M}(4, \mathcal{K}_q))^2 \oplus (\mathbf{M}(5, \mathcal{K}_q))^2 \oplus (\mathbf{M}(6, \mathcal{K}_q))^2 \\ & \oplus \mathbf{M}(4, \mathcal{K}_{q^2}) \oplus \mathbf{M}(5, \mathcal{K}_{q^2}). \end{aligned}$$

This and (11) due to proposition 2.3 further derive that $\mathcal{K}_q G \simeq$

$$\begin{aligned} & (\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \oplus (\mathbf{M}(4, \mathcal{K}_q))^2 \oplus (\mathbf{M}(5, \mathcal{K}_q))^2 \oplus (\mathbf{M}(6, \mathcal{K}_q))^2 \oplus \mathbf{M}(4, \mathcal{K}_{q^2}) \oplus \\ & \mathbf{M}(5, \mathcal{K}_{q^2}) \bigoplus_{i=1}^5 \mathbf{M}(n_i, \mathcal{K}_{q^2}) \end{aligned}$$

with

$$240 = \sum_{i=1}^5 2n_i^2, n_i \geq 2. \quad (12)$$

The possibilities of n_i 's fulfilling (12) are $(2, 3, 3, 7, 7)$, $(3, 5, 5, 5, 6)$ and $(4, 4, 4, 6, 6)$. Again we note that $\mathbf{M}(7, \mathcal{K}_7)$ can not be the Wedderburn component of the group algebra $\mathcal{K}_7 G$ due to proposition 2.4. This means $(2, 3, 3, 7, 7)$ can not be the choice of n_i 's. Consequently, we need to find the unique choice among $(3, 5, 5, 5, 6)$ and $(4, 4, 4, 6, 6)$. Suppose that $(3, 5, 5, 5, 6)$ is the correct choice. This means that $\mathbf{M}(3, \mathcal{K}_{q^2})$ is one of the Wedderburn components of the group algebra $\mathcal{K}_q G$. In accordance with conjecture 2.1, it follows that $\mathbf{M}(3, \mathcal{K}_{q_1})$, for some q_1 such that $|SK_{q_1}(\gamma_g)| = 1 \forall g \in G$, must be the Wedderburn component of the group algebra $\mathcal{K}_{q_1} G$. Since $|SK_{q_1}(\gamma_g)| = 1 \forall g \in G$, q_1 is among one of $\{1, 29, 49, 53, 73, 77, 97, 101\} \bmod 120$. Consequently, the Wedderburn decomposition of $\mathcal{K}_{q_1} G$ is given by (10). Clearly, $\mathbf{M}(3, \mathcal{K}_{q_1})$ is not the Wedderburn component of the group algebra $\mathcal{K}_q G$ in (10). This means that our supposition is wrong. Hence, the unique choice of n_i 's that fulfills (12) is $(4, 4, 4, 6, 6)$. Therefore, the Wedderburn decomposition of $\mathcal{K}_q G$ is

$$\begin{aligned} & (\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \oplus \mathbf{M}(4, \mathcal{K}_q)^2 \oplus (\mathbf{M}(5, \mathcal{K}_q))^2 \oplus (\mathbf{M}(6, \mathcal{K}_q))^2 \\ & \oplus (\mathbf{M}(4, \mathcal{K}_{q^2}))^4 \oplus \mathbf{M}(5, \mathcal{K}_{q^2}) \oplus (\mathbf{M}(6, \mathcal{K}_{q^2}))^2. \end{aligned}$$

Case 3: $p^k \equiv \{13, 17, 37, 41, 61, 89, 109, 113\} \pmod{120}$. The cyclotomic $\mathcal{K}_q$ classes of γ_s are

$$S\mathcal{K}_q(\gamma_{s_i}) = \{\gamma_{s_i}\}, \text{ for } 1 \leq i \leq 11, 14 \leq i \leq 16, 19 \leq i \leq 24,$$

$$S\mathcal{K}_q(\gamma_{s_{12}}) = \{\gamma_{s_{12}}, \gamma_{s_{13}}\}, S\mathcal{K}_q(\gamma_{s_{17}}) = \{\gamma_{s_{17}}, \gamma_{s_{18}}\}.$$

By incorporating proposition 2.1, theorem 2.2 and proposition 2.2 with $G/G' \simeq C_4$, $\mathcal{K}_q G/G' \simeq (\mathcal{K}_q)^4$, we derive from (3) that

$$\mathcal{K}_q G \simeq (\mathcal{K}_q)^4 \bigoplus_{i=1}^{16} M(n_i, \mathcal{K}_q) \oplus M(n_{17}, \mathcal{K}_{q^2}) \oplus M(n_{18}, \mathcal{K}_{q^2}) \quad (13)$$

with

$$476 = \sum_{i=1}^{16} n_i^2 + 2(n_{17}^2 + n_{18}^2), n_i \geq 2. \quad (14)$$

We observe that the Wedderburn decomposition of $\mathcal{K}_q(A_5 \rtimes C_4)$ in this case is same as in (5) (see [15]) This, proposition 2.3 along with (13), (14) yield

$$\begin{aligned} \mathcal{K}_q G \simeq (\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^4 \oplus (M(5, \mathcal{K}_q))^4 \\ \oplus (M(6, \mathcal{K}_q))^2 \bigoplus_{i=1}^6 M(n_i, \mathcal{K}_q) \bigoplus_{i=7}^8 M(n_i, \mathcal{K}_{q^2}), \end{aligned} \quad (15)$$

with

$$240 = \sum_{i=1}^6 n_i^2 + 2n_7^2 + 2n_8^2, n_i \geq 2. \quad (16)$$

Let $q = 13$. Suppose that $M(2, \mathcal{K}_{13})$ and $M(3, \mathcal{K}_{13})$ are the Wedderburn components of the semisimple group algebra $\mathcal{K}_{13}G$. This means that there exist $\mathcal{K}_{13}$ -algebra onto homomorphisms from $\mathcal{K}_{13}G$ to $M(2, \mathcal{K}_{13})$ and $M(3, \mathcal{K}_{13})$. The restrictions of these homomorphisms to G are non-trivial group homomorphisms from G to $GL(2, \mathcal{K}_{13})$ and $GL(3, \mathcal{K}_{13})$. Consequently, G has irreducible representations of degrees 2 and 3 over $\mathcal{K}_{13}$, which is not possible. Therefore, $n_i \geq 4$ for $1 \leq i \leq 6$ in (16). Also, conjecture 2.1 yields that $M(2, \mathcal{K}_{13^2})$ and $M(3, \mathcal{K}_{13^2})$ are not the Wedderburn components of the semisimple group algebra $\mathcal{K}_{13}G$. Thus, (15) and (16) deduce that

$$\mathcal{K}_q G \simeq (\mathcal{K}_q)^4 \oplus (\mathbf{M}(4, \mathcal{K}_q))^4 \oplus (\mathbf{M}(5, \mathcal{K}_q))^4 \oplus (\mathbf{M}(6, \mathcal{K}_q))^2 \\ \bigoplus_{i=1}^6 \mathbf{M}(n_i, \mathcal{K}_q) \bigoplus_{i=7}^8 \mathbf{M}(n_i, \mathcal{K}_{q^2}), \quad (17)$$

with

$$240 = \sum_{i=1}^6 n_i^2 + 2n_7^2 + 2n_8^2, \quad n_i \geq 4. \quad (18)$$

The possible choices of n_i 's fulfilling (18) are

$$(4, 4, 4, 4, 4, 4, 6, 6), (4, 4, 4, 4, 6, 6, 4, 6), (4, 4, 4, 5, 6, 7, 4, 5), \\ (4, 4, 6, 6, 6, 6, 4, 4) \text{ and } (4, 5, 5, 5, 6, 7, 4, 4). \quad (19)$$

We note that there are 6 irreducible representations of degree 6 over $\mathcal{K}_{13}$. These are as follows (here, a and b are same as in (2)):

(1)

$$a \rightarrow \begin{bmatrix} 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 \\ 2 & 6 & 12 & 0 & 0 & 0 \\ 5 & 10 & 8 & 10 & 1 & 1 \\ 10 & 8 & 2 & 2 & 1 & 8 \\ 2 & 2 & 2 & 8 & 0 & 7 \end{bmatrix}, b \rightarrow \begin{bmatrix} 3 & 5 & 8 & 0 & 0 & 0 \\ 9 & 11 & 4 & 5 & 0 & 0 \\ 5 & 10 & 9 & 5 & 9 & 0 \\ 4 & 6 & 7 & 10 & 3 & 9 \\ 11 & 5 & 5 & 5 & 5 & 0 \\ 6 & 3 & 4 & 12 & 11 & 1 \end{bmatrix};$$

(2)

$$a \rightarrow \begin{bmatrix} 4 & 11 & 0 & 0 & 0 & 0 \\ 9 & 11 & 1 & 1 & 0 & 0 \\ 12 & 5 & 8 & 0 & 0 & 0 \\ 4 & 12 & 8 & 3 & 0 & 0 \\ 3 & 4 & 5 & 5 & 3 & 0 \\ 1 & 10 & 1 & 7 & 1 & 5 \end{bmatrix}, b \rightarrow \begin{bmatrix} 2 & 3 & 8 & 0 & 0 & 0 \\ 10 & 6 & 9 & 0 & 0 & 0 \\ 10 & 8 & 5 & 0 & 0 & 0 \\ 0 & 11 & 6 & 2 & 3 & 0 \\ 8 & 1 & 11 & 5 & 5 & 5 \\ 3 & 2 & 6 & 7 & 10 & 6 \end{bmatrix};$$

(3)

$$a \rightarrow \begin{bmatrix} 7 & 5 & 2 & 9 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 \\ 10 & 2 & 4 & 10 & 2 & 5 \\ 8 & 8 & 0 & 5 & 3 & 9 \\ 5 & 2 & 10 & 1 & 7 & 5 \\ 5 & 0 & 10 & 2 & 11 & 6 \end{bmatrix}, b \rightarrow \begin{bmatrix} 9 & 5 & 2 & 2 & 8 & 0 \\ 8 & 12 & 6 & 0 & 0 & 0 \\ 8 & 4 & 2 & 4 & 2 & 11 \\ 6 & 0 & 9 & 11 & 4 & 7 \\ 8 & 10 & 6 & 8 & 10 & 5 \\ 9 & 10 & 8 & 5 & 12 & 8 \end{bmatrix};$$

(4)

$$a \rightarrow \begin{bmatrix} 12 & 6 & 7 & 1 & 0 & 0 \\ 5 & 8 & 0 & 0 & 0 & 0 \\ 9 & 7 & 6 & 2 & 0 & 0 \\ 11 & 0 & 4 & 0 & 0 & 0 \\ 6 & 12 & 10 & 12 & 8 & 0 \\ 9 & 1 & 9 & 4 & 7 & 12 \end{bmatrix}, b \rightarrow \begin{bmatrix} 8 & 5 & 0 & 0 & 0 & 0 \\ 4 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 2 & 4 & 1 & 8 & 4 & 0 \\ 3 & 6 & 7 & 6 & 11 & 6 \\ 11 & 10 & 5 & 0 & 0 & 9 \end{bmatrix};$$

(5)

$$a \rightarrow \begin{bmatrix} 12 & 11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 0 & 1 \\ 0 & 3 & 4 & 3 & 0 & 2 \\ 3 & 1 & 9 & 1 & 0 & 2 \\ 6 & 6 & 8 & 6 & 8 & 5 \\ 7 & 4 & 6 & 1 & 0 & 10 \end{bmatrix}, b \rightarrow \begin{bmatrix} 3 & 5 & 0 & 0 & 0 & 6 \\ 8 & 8 & 0 & 0 & 0 & 2 \\ 5 & 6 & 1 & 6 & 10 & 11 \\ 8 & 7 & 0 & 12 & 3 & 8 \\ 3 & 1 & 0 & 4 & 0 & 0 \\ 10 & 12 & 0 & 0 & 0 & 2 \end{bmatrix};$$

(6)

$$a \rightarrow \begin{bmatrix} 8 & 12 & 0 & 0 & 0 & 0 \\ 7 & 5 & 12 & 12 & 0 & 0 \\ 12 & 7 & 8 & 9 & 0 & 0 \\ 5 & 6 & 1 & 5 & 0 & 0 \\ 9 & 9 & 9 & 10 & 1 & 0 \\ 10 & 8 & 11 & 0 & 8 & 12 \end{bmatrix}, b \rightarrow \begin{bmatrix} 1 & 0 & 10 & 0 & 0 & 6 \\ 0 & 1 & 12 & 5 & 11 & 0 \\ 0 & 0 & 9 & 4 & 3 & 10 \\ 0 & 0 & 4 & 7 & 5 & 12 \\ 0 & 0 & 6 & 8 & 2 & 0 \\ 0 & 0 & 7 & 9 & 10 & 6 \end{bmatrix}.$$

Due to these 6 irreducible representations, proposition 2.5 derives that $(M(6, \mathcal{K}_{13}))^6$ is a summand of the group algebra $\mathcal{K}_{13}G$. Thus, out of the 5 choices of n'_i 's mentioned in (19), the correct choice is $(4, 4, 6, 6, 6, 6, 4, 4)$. Consequently, (17) implies that

$$\mathcal{K}_q G = (\mathcal{K}_q)^4 \oplus (M(4, \mathcal{K}_q))^6 \oplus (M(5, \mathcal{K}_q))^4 \oplus (M(6, \mathcal{K}_q))^6 \oplus (M(4, \mathcal{K}_{q^2}))^2.$$

This completes the proof.

Corollary 3.1: *The unit group of $\mathcal{K}_q G$ is as follows:*

(1) for $p^k \equiv \{1, 29, 49, 53, 73, 77, 97, 101\} \pmod{120}$, we have

$$(\mathcal{K}_q^*)^4 \oplus (GL(4, \mathcal{K}_q))^{10} \oplus (GL(5, \mathcal{K}_q))^4 \oplus (GL(6, \mathcal{K}_q))^6.$$

(2) for $p^k \equiv \{7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, 91, 103, 107, 119\} \pmod{120}$, we have

$$(\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \oplus (GL(4, \mathcal{K}_q))^2 \oplus (GL(5, \mathcal{K}_q))^2 \oplus (GL(6, \mathcal{K}_q))^2 \oplus (GL(4, \mathcal{K}_{q^2}))^4 \\ \oplus GL(5, \mathcal{K}_{q^2}) \oplus (GL(6, \mathcal{K}_{q^2}))^2.$$

- (3) for $p^k \equiv \{13, 17, 37, 41, 61, 89, 109, 113\} \pmod{120}$, we have
 $(\mathcal{K}_q)^4 \oplus (GL(4, \mathcal{K}_q))^6 \oplus (GL(5, \mathcal{K}_q))^4 \oplus (GL(6, \mathcal{K}_q))^6 \oplus (GL(4, \mathcal{K}_{q^2}))^2$.

4. Conclusion

Our research has focused on the group of units of semisimple group algebra $\mathbb{K}_q GL(2, \mathbb{Z}_5)$. In order to do so, we computed the Wedderburn decomposition using a number of findings from the classical theory of group algebras and representative theory. We have also proposed a conjecture in this direction and discussed several examples in the support of our conjecture. It is evident that as the group size increases, it becomes harder to characterize the Wedderburn decomposition in a distinctive manner due to many choices for Wedderburn components.

References

- [1] N. Arvind, S. Panja, The unit group of $\mathcal{K}_q SL(3, 2), p > 11$, arXiv:2106.07261.
- [2] Y. Bai, Y. Li, J. Peng, Unit groups of finite group algebras of abelian groups of order 17 to 20, *AIMS Mathematics* 6, 7305-7317 (2021).
- [3] G. K. Bakshi, S. Gupta, I. B. S. Passi, The algebraic structure of finite metabelian group algebras, *Comm. Algebra* 43, 2240-2257 (2015).
- [4] A. Bovdi, J. Kurdics, Lie properties of the group algebra and the nilpotency class of the group of units, *J. Algebra* 212, 28-64 (1999).
- [5] C. Dietzel, G. Mittal, Summands of finite group algebras, *Czech. Math. J.* 71, 1011-1014 (2021).
- [6] R. Ferraz, Simple components of center of $\mathbb{F}G / J(\mathbb{F}G)$, *Comm. Algebra* 36, 3191-3199 (2008).
- [7] M. Hill, Irreducible representation of $GL_2(\mathbb{F}_q)$, link: www.math.harvard.edu/archive/126_fall_98/papers/mahill.pdf.
- [8] T. Hurley, Convolutional codes from units in matrix and group rings, *Int. J. Pure Appl. Math.* 50, 431-463 (2009).
- [9] M. Khan, R. K. Sharma, J. Srivastava, The unit group of FS_4 , *Acta Math. Hungar.* 118, 105-113 (2008).
- [10] S. Maheshwari, R. K. Sharma, The unit group of the group algebra $\mathcal{K}_q SL(2, \mathbb{Z}_3)$, *J. Algebra Comb. Discrete Appl.* 3, 1-6 (2015).

- [11] N. Makhijani, R. K. Sharma, The unit group of algebra of circulant matrices, *Int. J. Group Theory* 3, 13-16 (2014).
- [12] N. Makhijani, R. K. Sharma, J. B. Srivastava, The unit group of some special semisimple group algebras, *Quaest. Math.* 39, 9-28 (2016).
- [13] C. P. Milies, S. K. Sehgal, An Introduction to group rings, Springer Dordrecht (2002).
- [14] G. Mittal, R. K. Sharma, On unit group of finite group algebras of non-metabelian groups up to order 72, *Math. Bohemica* 146, 1-27 (2021).
- [15] G. Mittal, R. K. Sharma, Wedderburn decomposition of a semisimple group algebra $\mathbb{F}_q G$ from a subalgebra of factor group of G , *Int. Elect. J. Algebra* 32, 91-100 (2022).
- [16] G. Mittal, R. K. Sharma, Unit group of semisimple group algebras of some non-metabelian groups of order 120, *Asian-European J. Math.* 15, 2250059 (2022).
- [17] G. Mittal, R. K. Sharma Computation of Wedderburn decomposition of groups algebras from their subalgebra, *Bull. Korean Math. Soc.* 59, 781-787 (2022).
- [18] R. K. Sharma, G. Mittal, On the unit group of a semisimple group algebra $K_q SL(2, \mathbb{Z}_5)$, *Math. Bohemica* 147, 1-10 (2022).
- [19] R. K. Sharma, J. B. Srivastava, M. Khan, The unit group of KS_3 , *Acta Math. Acad. Paedagog. Nyhazi. (N.S.)* 23, 129-142 (2007).
- [20] R. K. Sharma, P. Yadav, Unit group of algebra of circulant matrices, *Int. J. Group Theory* 2, 1-6 (2013).
- [21] G. Tang, Y. Wei, Y. Li, Unit groups of group algebras of some small groups, *Czech. Math. J.* 64, 149-157 (2014).

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