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On Foster's theorem and positive para odd functions

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Abstract

Various tools in the literature are used to enhance the understanding of the Routh-Hurwitz stability theory. Among such tools are Foster's theorem and positive para odd functions. One of the objectives of this paper is to show strong correlations between these two tools. Such correlations enable us to create simple forms of such functions which can prove useful in tackling stability issues. Also, a simple derivation of Foster's Theorem is advanced using complex analysis techniques, avoiding the use of circuit network theory which led to Foster's theorem. A general form of positive and para odd functions is also advanced.

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1. Introduction

Historically, Foster's theorem is the starting point for the development of networks. It also instigated intensive research in the synthesis of filters.

In [1] and using Foster's Theorem, Rhodes derived important results on the frequency derivative of invariant systems and in [2], we showed that such systems are Routh-Hurwitz stable, and we established an alternating pattern of zeros of numerator and denominator of the frequency function on the real axis. The results established in [2] and [3] resolved a pertinent issue in network theory that is, finding conditions on the zeros of numerator and denominator for such functions to be a positive para odd. We also note that the approach adopted in [3] provides a simple derivation of the Hurwitz algorithm.

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On the other hand, it is a well-known fact that the impedance of a passive network is positive para odd. The converse is also true. A finite passive network can be constructed from such functions. In addition, positive para odd functions [4-6] are useful in tackling the stability of real and complex polynomials. Specifically, in [6] and [7] we explored special properties of Routh and Schur stable polynomials. In [8], we provided special continued fractions of systems of difference equations which are Schur stable.

Some important relationships between Foster's Theorem and positive para odd functions have been overlooked. It is one of the major objectives of this work to explore such relations. In Section 2, definitions and notations are introduced. Section 3 highlights important relations between Foster's Theorem and positive para odd functions which will be applied in Section 4 to generate some simple, yet important forms of para odd functions. In Section 5, we generate a general form of positive and para odd functions of which the results of Section 4 fall out as special cases. Concluding remarks are given in Section 6.

2. Definitions and Notations

Definition 1: If $f(z)$ is a rational function, the para-conjugate of f is defined by $f^*(z) = \overline{f(-\overline{z}\ )}$.

Definition 2: A rational function h is said to be positive if $\operatorname{Re} h(z) > 0$ whenever $\operatorname{Re} z > 0$.

Definition 3: A rational function h satisfying $h^*(z) = -h(z)$ is called para odd.

Definition 4: $h(z)$ is called a Foster function if it is real, positive and odd.

By odd, we mean $h(-z) = -h(z)$ and should not be confused with para-odd of Definition 3.

3. Main Results

The two lemmas below are required.

Lemma 1: *Every Foster function is a positive para odd function.*

***Proof*:** Suppose h is a Foster function, then h real, positive and odd.

$h^*(z) = \overline{h(-\overline{z})} = \overline{-h(\overline{z})} = -\overline{h(\overline{z})} = -h(z),$ since h is real and odd.

Hence, h is positive and para odd.

Lemma 2: *If $h(z)$ is positive para odd, then it can be written as*

$$h(z) = a_0 + d_0 z + \frac{d_1}{z - iw_1} + \frac{d_2}{z - iw_2} + \cdots + \frac{d_{n+1}}{z - iw_{n+1}},$$

where, Re $a_0 = 0$, $d_k \geq 0$ for $0 \leq k \leq n+1$, and the w_k are different real numbers.

Proof: By assumption, h is positive, therefore h cannot have any zeros when Re $z > 0$.

Since h is also para odd, then

$$h^*(z) = -h(z)$$

and therefore, $-h^*(z)$ is also positive and cannot have any zeros when Re $z < 0$.

Hence, the zeros of h must be pure imaginary.

These zeros must be simple for the following reason:

If z is a complex number different from zero, the real parts of z and $1/z$ have same signature.

So, if $1/h$ is positive, then h is also positive and visa versa, from which it follows that $h(z)$ cannot have any poles when Re $z > 0$.

In addition, any imaginary zeros of the denominator must be simple, proving our claim.

Similar observations apply on $1/h$ leading to the above partial fraction expansion except possibly for the term $a_0 + d_0 z$ which may be a polynomial of degree higher than one.

If that is the case, then for $|z|$ large enough Re $h(z)$ changes sign when Re $z > 0$.

Similar observation leads to $d_k \geq 0$ for $0 \leq k \leq n+1$ and Re $a_0 = 0$ and that completes to proof.

We now introduce a simple derivation of Foster's theorem.

The proof of the next theorem uses basic complex analysis techniques without any reference to circuit network theory.

Theorem 1 (Foster's Theorem): *Every Foster Function $h(z)$ can be written as*

$$h(z) = az + \frac{b}{z} + \sum_{k=1}^{n-1} \frac{c_k z}{z^2 + w_k^2}$$

where, $a \geq 0$, $b \geq 0$, $c_k \geq 0$, and $w_k \neq 0$ are all real.

Proof: By Lemma 1, every Foster function $h(s)$ is positive para odd.

Therefore, Lemma 2 implies that $h(z)$ takes the form

$$h(z) = a_0 + d_0 z + \frac{d_1}{z - iw_1} + \frac{d_2}{z - iw_2} + \cdots + \frac{d_{n+1}}{z - iw_{n+1}},$$

where, Re $a_0 = 0$, $d_k \geq 0$ for $0 \leq k \leq n + 1$.

Since a Foster Function is real, then a_0 is real and that makes $a_0 = 0$. So,

$$h(z) = d_0 z + \frac{d_1}{z - i\,w_1} + \frac{d_2}{z - i\,w_2} + \cdots + \frac{d_{n+1}}{z - i\,w_{n+1}}$$

In the proof of Lemma 2, it was mentioned that $h(z)$ is positive para odd is a necessary and sufficient condition for $1/h(z)$ to be positive para odd.

Since a Foster Function is real, the zeros of $1/h(z)$ must occur in conjugate pairs.

Consider two cases:

Case 1: n even, then $n + 1$ odd. Therefore, one of the w's must be 0 and the remaining are conjugate pairs. Suppose $w_1 = 0$, then

$$h(z) = d_0 z + \frac{d_1}{z} + \frac{d_2}{z - iw_2} + \frac{d_3}{z + iw_2} + \cdots + \frac{d_n}{z - iw_n} + \frac{d_{n+1}}{z + iw_n},$$

$$h(z) = d_0 z + \frac{d_1}{z} + \frac{d_2(z + iw_2) + d_3(z - iw_2)}{z^2 + w_2^{\,2}} + \cdots + \frac{d_n(z + iw_n) + d_{n+1}(z - iw_n)}{z^2 + w_n^{\,2}},$$

$$h(z) = d_0 z + \frac{d_1}{z} + \frac{(d_2 + d_3)z + iw_2(d_2 - d_3)}{z^2 + w_2^{\,2}} + \cdots + \frac{(d_n + d_{n+1})z + iw_n(d_n - d_{n+1})}{z^2 + w_n^{\,2}}.$$

A Foster Function is odd, then $h(-z) = -h(z)$. Therefore,

$$h(-z) = -d_0 z - \frac{d_1}{z} - \frac{(d_2 + d_3)z}{z^2 + w_2^{\,2}} + \frac{iw_2(d_2 - d_3)}{z^2 + w_2^{\,2}} - \cdots - \frac{(d_n + d_{n+1})z}{z^2 + w_n^{\,2}} + \frac{iw_n(d_n - d_{n+1})}{z^2 + w_n^{\,2}}.$$

$$-h(z) = -d_0 z - \frac{d_1}{z} - \frac{(d_2 + d_3)z}{z^2 + w_2^{\,2}} - \frac{iw_2(d_2 - d_3)}{z^2 + w_2^{\,2}} - \cdots - \frac{(d_n + d_{n+1})z}{z^2 + w_n^{\,2}} - \frac{iw_n(d_n - d_{n+1})}{z^2 + w_n^{\,2}}.$$

By putting $h(-z) = -h(z)$, we get

$d_2 - d_3 = 0,\ \ d_4 - d_5 = 0,\ \ \cdots,\ \ d_n - d_{n+1} = 0,$ and $h(z)$ takes the form

$$h(z) = d_0 z + \frac{d_1}{z} + \frac{b_2 z}{z^2 + w_2^{\,2}} + \cdots + \frac{b_n z}{z^2 + w_n^{\,2}}.$$

Case 2: n odd, then $n + 1$ even. Therefore, all w's are conjugate pairs, and the proof proceeds as in the previous case, and that completes the proof.

4. Generation of Positive Para odd Functions

There have been several attempts to generate positive para-odd functions. In [9], Adler et al generated positive para-odd functions using Cauchy-Riemann equations and the positivity of the reactance gradient. In this section, we will achieve similar objectives as in [9] and beyond by using the results we already established in the current work, namely that every Foster function is positive para odd, and the expansion of a Foster Function as expressed in Theorem 1.

Theorem 2: *If* $\alpha = a + ib$ *is a non-zero complex number, then*

$$h(z) = \frac{a}{z-ib}$$

is positive para odd if and only if $a = \mathrm{Re}\,\alpha > 0$.

Proof:

$$h(z) = \frac{a}{z-i\,b} = \frac{a}{\mathrm{Re}\,z + i(\mathrm{Im}\,z-b)} = \frac{a[\mathrm{Re}\,z - i\ (\mathrm{Im}\,z-b)]}{(\mathrm{Re}\,z)^2+(\mathrm{Im}\,z-b)^2}.$$

Therefore,

$$h(z) = \frac{a}{z-ib} = \frac{a\,\mathrm{Re}\,z}{(\mathrm{Re}\,z)^2+(\mathrm{Im}\,z-b)^2} - i\,\frac{a(\mathrm{Im}\,z-b)}{(\mathrm{Re}\,z)^2+(\mathrm{Im}\,z-b)^2}.$$

Hence,

$$\mathrm{Re}\,h(z) = \frac{a\ \mathrm{Re}\,z}{(\mathrm{Re}\,z)^2+(\mathrm{Im}\,z-b)^2},$$

from which it follows that for $h(z)$ to be positive it is necessary and sufficient that $a = \mathrm{Re}\ \alpha > 0$.

To show that $h(z)$ is para-odd, consider

$$h^*(z) = \overline{h(-\overline{z})} = \overline{\left(\frac{a}{-\overline{z}-ib}\right)} = -\overline{\left(\frac{a}{\overline{z}+ib}\right)} = -\left(\frac{a}{z-ib}\right) = -h(z),$$

and the proof is complete.

Obviously, for $\frac{1}{h}$ to be positive para odd,it is necessary and sufficient that h is positive para odd.

It should be noted here that when

$$h(z)=\frac{a}{z-ib},$$

then

$$\frac{1}{h(z)}=\frac{z}{a}-i\frac{b}{a}$$

which matches with the expansion of Lemma 2, namely

$$h(z)=a_0+d_0z+\frac{d_1}{z-iw_1}+\frac{d_2}{z-iw_2}+\cdots+\frac{d_{n+1}}{z-iw_{n+1}},$$

where $d_k=0$ for $1\le k\le n+1$, $d_0=\frac{1}{a}$ and $a_0=-i\frac{b}{a}$.

Obviously, the coefficient of z and the constant coefficient in

$$\frac{1}{h(z)}=\frac{z}{a}-i\frac{b}{a}$$

satisfy the requirements of Lemma 2.

Theorem 3: *Given a non-zero complex number $\alpha = a + ib$, the function*

$$h(z)=\frac{az}{z^2+|\alpha|^2}$$

is positive para-odd if and only if $a = Re\ \alpha > 0$.

This function is an illustration on how Talbot theorem applies to Foster reactance functions [10].

Proof: As mentioned before, for $\frac{1}{h}$ to be positive para odd, it is necessary and sufficient that h is positive para odd.

$$\frac{1}{h(z)}=\frac{z^2+|\alpha|^2}{az}=\frac{1}{a}z+\frac{|\alpha|^2}{a}\frac{1}{z}.$$

From the proof of Theorem 3.2 in [11]

$$\frac{1}{h(z)}=\frac{1}{a}z+\frac{|\alpha|^2}{a}\frac{1}{z}$$

is positive para odd if and only if

$a>0$, and $\frac{|\alpha|^2}{a}\frac{1}{z}$ is positive para odd

which itself is equivalent to

$\frac{a}{|\alpha|^2} z$ is positive para odd.

The last proposition is equivalent to a = Re $\alpha > 0$, as Talbot's theorem predicts.

Theorem 4: *If a and b are real numbers, then for the function*

$$h(z) = \frac{(a+b)z}{z^2+ab}$$

to be positive para odd, it is necessary and sufficient that both a and b are positive.

Proof: As mentioned earlier, for $\frac{1}{h}$ to be positive para odd, it is necessary and sufficient that h is positive para odd.

$$\frac{1}{h(z)} = \frac{z^2+ab}{(a+b)z} = \frac{1}{a+b} z + \frac{ab}{(a+b)} \frac{1}{z}.$$

From the proof of Theorem 3.2 in [11]

$$\frac{1}{h(z)} = \frac{1}{a+b} z + \frac{a\,b}{(a+b)} \frac{1}{z}$$

is positive para odd if and only if $a+b>0$ and $\frac{(a+b)z}{ab}$ is positive para odd, which is equivalent to $\frac{a+b}{ab} > 0$.

$a+b>0$ and $\frac{a+b}{ab} > 0$ if and only if $a>0$ and $b>0$. That completes the proof.

The function $h(z)$ stated in the current theorem matches with the expansion of Theorem 1 with the proper conditions on the coefficients. Therefore, it is a Foster function.

By Lemma 1, every Foster function is positive para-odd leading to the desired conclusion.

5. A General Positive Para-odd Function

The results we established in this work take us far beyond the functions of the previous section and enable us to provide a more general positive para odd function in the way we describe below. The results established in Section 4 will fall as special cases.

Let $$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-2} z^2 + a_{n-1} z + a_n,$$

then the para-conjugate of f is given by

$$f^*(z) = (-1)^n z^n + (-1)^{n-1} \overline{a_1} z^{n-1} + \cdots + \overline{a_{n-2}} z^2 - \overline{a_{n-1}} z + \overline{a_n}$$

$f(z)$ and $f^*(z)$ can also be written in the factored forms as

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n),$$

and

$$f^*(z) = (-1)^n (z + \overline{z_1})(z + \overline{z_2}) \cdots (z + \overline{z_n}).$$

Define

$$h(z) = \begin{cases} \dfrac{f - f^*}{f + f^*} & \text{if } n \text{ odd} \\ \dfrac{f + f^*}{f - f^*} & \text{if } n \text{ even.} \end{cases}$$

$h(z)$ can also be expressed as

$$h(z) = \frac{z^n + i \operatorname{Im} a_1 z^{n-1} + \operatorname{Re} a_2 z^{n-2} + i \operatorname{Im} a_3 z^{n-3} + \operatorname{Re} a_4 z^{n-4} + \cdots}{\operatorname{Re} a_1 z^{n-1} + i \operatorname{Im} a_2 z^{n-2} + \operatorname{Re} a_3 z^{n-3} + i \operatorname{Im} a_4 z^{n-4} + \cdots}.$$

In [11], it was established that h is positive para odd is a necessary and sufficient condition for f to be a Hurwitz polynomial (having all its zeros in the left-half plane).

It is an easy exercise to show that the functions of Section 4 are special cases of that defined in Section 5.

6. Conclusion

The central theme of this paper dwells on the classical Foster reactance theorem. A simple proof of this historic theorem is delivered using simple complex analysis techniques. Some intriguing relationships between Foster's functions and positive para-odd ones are explored paving the way to generate different types of positive para-odd functions. A very general form of positive and para odd functions is provided which encompasses all the special cases mentioned herein. Finally, it should be mentioned that both Foster's functions and positive para-odd ones are efficient tools in tackling Schur-Cohn stability problems.

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