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Linear programming problems and matrix games

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Abstract

In this article, the relationship between linear optimization and matrix games is discussed, and in particular, the transformation of a linear program to a matrix game is considered.

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1. Introduction

The relationship between linear programs and matrix games is considered, and in particular, the transformation of a linear program to a matrix game is discussed.

Consider a pair of a primal program and a dual program

I.

$$\max\left\{z(\xi)=\sum_{j=1}^{n}p_j\xi_j\right\}$$

subject to

$$\sum_{j=1}^{n}a_{ij}\xi_j \le a_i, \quad i=1,\ldots,k,$$

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$$\sum_{j=1}^{n} a_{ij}\xi_j = a_i, \quad i = k+1,\dots,m,$$

$$\xi_j \geq 0, \quad j = 1,\dots,l,$$

and

II.

$$\min\left\{w(u) = \sum_{i=1}^{m} a_i u_i\right\}$$

subject to

$$\sum_{i=1}^{m} a_{ij} u_i \geq p_j, \quad j = 1,\dots,l,$$

$$\sum_{i=1}^{m} a_{ij} u_i = p_j, \quad j = l+1,\dots,n,$$

$$u_i \geq 0, \quad i = 1,\dots,k,$$

where a_{ij}, $i = 1,\dots,m$; $j = 1,\dots,n$, are entries of the payment matrix of a matrix game.

The topics, considered in this article, are studied, for example, in [1–12].

The article is structured as follows. In section 2, a preliminary result is obtained, and transformation of a linear program to a matrix game is obtained. In section 3 *Conclusion*, content of the paper is summarized. In *References*, a list of titles related to matrix games, linear programming, and duality in linear programming, is presented.

2. Transformation of a linear program to a matrix game

Before transforming a linear program to a matrix game, the following auxiliary result is formulated and proved.

Theorem 1: *Each pair of a primal program and a dual program is equivalent to a system of linear equations and inequalities, consisting of all constraints of both problems, of all nonnegativity constraints of both problems, and of an additional condition, which is obtained by equating the objective functions of both linear programs.*

Proof: Consider the pair of a primal program and a dual program I. and II. of section 1. We will prove the equivalence of this pair of problems with the following system of equations and inequalities

$$\sum_{j=1}^{n} a_{ij}\xi_j \le a_i, \quad i = 1,\dots,k, \tag{1}$$

$$\sum_{j=1}^{n} a_{ij}\xi_j = a_i, \quad i = k+1,\dots,m, \tag{2}$$

$$\xi_j \ge 0, \quad j = 1,\dots,l, \tag{3}$$

$$\sum_{i=1}^{m} a_{ij}u_i \ge p_j, \quad j = 1,\dots,l, \tag{4}$$

$$\sum_{i=1}^{m} a_{ij}u_i = p_j, \quad j = l+1,\dots,n, \tag{5}$$

$$u_i \ge 0, \quad i = 1,\dots,k, \tag{6}$$

$$\sum_{j=1}^{n} p_j\xi_j = \sum_{i=1}^{m} a_i u_i. \tag{7}$$

a) Let $\xi^* = (\xi_1^*,\dots,\xi_n^*)$ and $u^* = (u_1^*,\dots,u_m^*)$ be optimal solutions to problems I. and II., respectively. Since, in particular, ξ^* and u^* are also feasible solutions to problems I. and II., respectively, these feasible solutions satisfy the system (1)–(6), and since ξ^* and u^* are optimal solutions to problems I. and II., according to Duality theorem (see, for example, [10]), objective function values of both problems, calculated for ξ^* and u^*, respectively, are equal, that is, ξ^* and u^* satisfy condition (7) as well.

b) Conversely, let $\xi' = (\xi_1',\dots,\xi_n')$ and $u' = (u_1',\dots,u_m')$ give a solution of system (1)–(7). Since ξ' and u' satisfy (1)–(6) by assumption, then ξ' and u' are feasible solutions to the primal and the dual linear programs I. and II., respectively, and since ξ' and u' also satisfy (7) by assumption, then objective functions of the pair of primal program I. and dual program II. take equal values. According to Proposition 1 ([10]), the feasible solutions ξ' and u' are optimal solutions to linear programs I. and II., respectively.

From a) and b) it follows that the proof of Theorem 1 is complete.

Remark 1: The equation (7) can be replaced by the inequality

$$\sum_{j=1}^{n} p_j \xi_j \geq \sum_{i=1}^{m} a_i u_i. \tag{7'}$$

We have to prove only the fact that if ξ' and u' give a solution of system (1)–(6), (7′) then ξ' and u' satisfy the equation (7) as well.

Indeed, since ξ' and u' satisfy system (1)–(6) then they also satisfy the inequality

$$\sum_{j=1}^{n} p_j \xi_j \leq \sum_{i=1}^{m} a_i u_i,$$

which is obtained after summation of (4) and (5) with respect to *j*, and since ξ' and u' satisfy inequality (7′), then they satisfy equation (7) as well.

Example 1 below illustrates the approach described above.

Example 1. Given a two-person matrix game with the payment matrix

$$A = \begin{pmatrix} -2 & -1 & 5 \\ 1 & -2 & 3 \\ 3 & 1 & -4 \end{pmatrix}.$$

The equivalent pair of a primal program and a dual program consists in finding a nonnegative solution of the system

$$\begin{array}{rrrcl} -2u_1 & +u_2 & +3u_3 & \geq & 1 \\ -u_1 & -2u_2 & +u_3 & \geq & 1 \\ 5u_1 & +3u_2 & -4u_3 & \geq & 1, \end{array}$$

which *minimizes* the following linear function

$$w(u) = u_1 + u_2 + u_3,$$

and finding a nonnegative solution of the system

$$\begin{array}{rrrcl} -2\xi_1 & -\xi_2 & +5\xi_3 & \leq & 1 \\ \xi_1 & -2\xi_2 & +3\xi_3 & \leq & 1 \\ 3\xi_1 & +\xi_2 & -4\xi_3 & \leq & 1, \end{array}$$

which *maximizes* the following linear function

$$z(\xi) = \xi_1 + \xi_2 + \xi_3.$$

Solving this pair of a primal program and a dual program, we obtain

$$\max z(\xi) = \min w(u) = 11 = \frac{1}{v},$$

and these optimal values are attained for

$$\xi_1 = 0, \xi_2 = 9, \xi_3 = 2 \quad \text{and} \quad u_1 = 5, u_2 = 0, u_3 = 6,$$

respectively, so that

$$S_{\mathrm{I}}^* \equiv x^* = \frac{u^*}{w(u^*)} = \frac{1}{11}(5,0,6) = \left(\frac{5}{11}, 0, \frac{6}{11}\right),$$

$$S_{II}^* \equiv y^* = \frac{\xi^*}{z(\xi^*)} = \frac{1}{11}(0,9,2) = \left(0, \frac{9}{11}, \frac{2}{11}\right)$$

are the optimal strategies of the first and the second player, respectively, and $v = \frac{1}{11}$ is the value of this game.

Return to the transformation of a linear programming problem to a matrix game.

Consider the following pair of a primal program and a dual program

I′.

$$\max \left\{ z(\xi) = \sum_{j=1}^{n} p_j \xi_j \right\}$$

subject to

$$\sum_{j=1}^{n} a_{ij} \xi_j \le a_i, \quad i = 1, \dots, m,$$

$$\xi_j \ge 0, \quad j = 1, \dots, n,$$

and

II′.

$$\min \left\{ w(u) = \sum_{i=1}^{m} a_i u_i \right\}$$

subject to

$$\sum_{i=1}^{m} a_{ij} u_i \ge p_j, \quad j = 1, \dots, n,$$

$$u_i \geq 0, \quad i = 1, \ldots, m.$$

According to Theorem 1, the pair of problems $\mathrm{I}'.-\mathrm{II}'.$ is equivalent to the following system of equations and inequalities

$$\sum_{j=1}^{n} a_{ij}\xi_j \leq a_i, \quad i = 1, \ldots, m,$$

$$\xi_j \geq 0, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} a_{ij}u_i \geq p_j, \quad j = 1, \ldots, n,$$

$$u_i \geq 0, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} p_j\xi_j \geq \sum_{i=1}^{m} a_i u_i.$$

Set

$$\tilde{\xi}_j = \xi_j t, \quad j = 1, \ldots, n,$$

$$\tilde{u}_i = u_i t, \quad i = 1, \ldots, m,$$

and as a result we get the system

$$\begin{array}{ccccccccc}
0.\tilde{u}_1 & +\cdots & +0.\tilde{u}_m & -a_{11}\tilde{\xi}_1 & -\cdots & -a_{1n}\tilde{\xi}_n & +a_1 t & \geq & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0.\tilde{u}_1 & +\cdots & +0.\tilde{u}_m & -a_{m1}\tilde{\xi}_1 & -\cdots & -a_{mn}\tilde{\xi}_n & +a_m t & \geq & 0 \\
a_{11}\tilde{u}_1 & +\cdots & +a_{m1}\tilde{u}_m & +0.\tilde{\xi}_1 & +\cdots & +0.\tilde{\xi}_n & -p_1 t & \geq & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1n}\tilde{u}_1 & +\cdots & +a_{mn}\tilde{u}_m & +0.\tilde{\xi}_1 & +\cdots & +0.\tilde{\xi}_n & -p_n t & \geq & 0 \\
-a_1\tilde{u}_1 & -\cdots & -a_m\tilde{u}_m & +p_1\tilde{\xi}_1 & +\cdots & +p_n\tilde{\xi}_n & +0.t & \geq & 0
\end{array} \tag{8}$$

$$\tilde{u}_i \geq 0, \quad i = 1, \ldots, m, \quad \tilde{\xi}_j \geq 0, \quad j = 1, \ldots, n.$$

Find a matrix game, whose optimal players' strategies satisfy system (8).

This requirement is satisfied by a symmetric matrix game, such that the payment matrix of the first player is a semi-symmetric matrix of coefficients of $m+n+1$ inequalities of system (8), multiplied by -1, that is, with the payment matrix

$$A=\begin{pmatrix} 0 & \dots & 0 & a_{11} & \dots & a_{1n} & -a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & \dots & 0 & a_{m1} & \dots & a_{mn} & -a_m \\ -a_{11} & \dots & -a_{m1} & 0 & \dots & 0 & p_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -a_{1n} & \dots & -a_{mn} & 0 & \dots & 0 & p_n \\ a_1 & \dots & a_m & -p_1 & \dots & -p_n & 0 \end{pmatrix}. \tag{9}$$

Indeed, let $(u^*,\tilde{\xi}^*,t^*)=(\tilde{u}_1^*,\dots,\tilde{u}_m^*;\tilde{\xi}_1^*,\dots,\tilde{\xi}_n^*;t^*)$ be an optimal strategy for both the first and the second players (due to the symmetry of the game, both players have the same optimal strategies, and the value of the game is $v=0$).

Therefore, for each strategy $(\tilde{u},\tilde{\xi},t)$ of the second player we have

$$0=E(\tilde{u}^*,\tilde{\xi}^*,t^*;\tilde{u}^*,\tilde{\xi}^*,t^*)\le E(\tilde{u}^*,\tilde{\xi}^*,t^*;\tilde{u},\tilde{\xi},t).$$

If instead of $(\tilde{u},\tilde{\xi},t)$ we take successively all pure strategies

$$y^{(j)}=(\underbrace{0,\dots,0,1}_{j},0,\dots,0)_n,\quad j=1,\dots,n,$$

of the second player, we obtain the system

$$\begin{array}{ccccccccc} 0.\tilde{u}_1^* & +\cdots & +0.\tilde{u}_m^* & -a_{11}\tilde{\xi}_1^* & -\cdots & -a_{1n}\tilde{\xi}_n^* & +a_1t^* & \ge & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0.\tilde{u}_1^* & +\cdots & +0.\tilde{u}_m^* & -a_{m1}\tilde{\xi}_1^* & -\cdots & -a_{mn}\tilde{\xi}_n^* & +a_mt^* & \ge & 0 \\ a_{11}\tilde{u}_1^* & +\cdots & +a_{m1}\tilde{u}_m^* & +0.\tilde{\xi}_1^* & +\cdots & +0.\tilde{\xi}_n^* & -p_1t^* & \ge & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1n}\tilde{u}_1^* & +\cdots & +a_{mn}\tilde{u}_m^* & +0.\tilde{\xi}_1^* & +\cdots & +0.\tilde{\xi}_n^* & -p_nt^* & \ge & 0 \\ -a_1\tilde{u}_1^* & -\cdots & -a_m\tilde{u}_m^* & +p_1\tilde{\xi}_1^* & +\cdots & +p_n\tilde{\xi}_n^* & +0.t^* & \ge & 0 \end{array} \tag{10}$$

$$\tilde{u}_i^*\ge 0,\quad i=1,\dots,m;\quad \tilde{\xi}_j^*\ge 0,\quad j=1,\dots,n;\quad t^*\ge 0$$

$$\tilde{u}_1^*+\ldots+\tilde{u}_m^*+\tilde{\xi}_1^*+\ldots+\tilde{\xi}_n^*+t^*=1.$$

From the last equation of system (10) it follows that each optimal solution $(\tilde{u}^*,\tilde{\xi}^*,t^*)$ of the considered matrix game is nontrivial, that is, it is not a vector with zero components.

We have shown that each optimal solution of the matrix game is a solution of system (8), which is equivalent to the pair of primal and dual linear programming problems $\mathrm{I}'.$ and $\mathrm{II}'.$ Optimal solutions of programs $\mathrm{I}'.$ and $\mathrm{II}'.$ are obtained as follows

$$u_i^* = \frac{\tilde{u}_i^*}{t^*}, \quad i = 1, \ldots, m; \qquad \xi_j^* = \frac{\tilde{\xi}_j^*}{t^*}, \quad j = 1, \ldots, n, \tag{11}$$

respectively, with $t^* > 0.$

If $t = 0$ for each optimal strategy, then the corresponding primal and dual linear programming problems $\mathrm{I}'.$ and $\mathrm{II}'.$ are unsolvable.

Since $\tilde{u}_i^* = u_i^* t^*$, $\tilde{\xi}_j^* = \xi_j^* t^*$ according to (11), then from the last equality of system (10) it follows that

$$t^* = \frac{1}{\sum_{j=1}^{n} \xi_j^* + \sum_{i=1}^{m} u_i^* + 1.}$$

Using the optimal solutions of programs $\mathrm{I}'.$ and $\mathrm{II}'.$, , we can obtain the optimal solution of the corresponding matrix game with payment matrix (9) as follows

$$\tilde{\xi}_j^* = \xi_j^* t^* = \frac{\xi_j^*}{\sum_{j=1}^{n} \xi_j^* + \sum_{i=1}^{m} u_i^* + 1}, \qquad \tilde{u}_i^* = u_i^* t^* = \frac{u_i^*}{\sum_{j=1}^{n} \xi_j^* + \sum_{i=1}^{m} u_i^* + 1}. \tag{12}$$

3. Conclusion

In this article, the relationship between linear programs and matrix games is discussed, and in particular, the transformation of a linear program to a matrix game is considered.

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