

Existence results for a coupled system of fractional Caputo-Langevin equations involving two fractional orders

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Abstract

In this current article, we study a coupled system of fractional Langevin equations with two Caputo fractional derivatives. The fixed point techniques are applied to prove the existence of solutions for our problem. illustrative example is also presented.

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1. Introduction

Fractional differential equations involving a variety fractional derivatives have been investigated by many scholars, and this is due to the importance of this field and its applications in various problems of the applied science. For getting further information the reader can address the following articles [9, 11, 14, 15, 17]. Langevin equation is widely used to describe the evolution of physical phenomena in fluctuating environments [4]. Langevin equation of fractional-order as a generalization of classical one gives a fractional Gaussian process parametrized by two indices, which is more flexible for modeling fractal processes [18, 21]. Recently, fractional-order Langevin equations and coupled systems are extensively studied in literature for instance, see [1, 2, 3, 20, 25]. Several interesting results concerning the existence, uniqueness, by applying fixed point techniques are obtained, see [5, 6, 10, 19, 23] and the references therein. In [8, 12,

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13, 22, 24], the existence and uniqueness of solutions were investigated for a coupled system of nonlinear fractional Langevin equations with different fractional operators. So, in this paper, we study the existence and uniqueness of solutions for the following fractional-order Langevin system:

$$\begin{cases} D^{\alpha_1}(D^{\beta_1} + \lambda_1)u(t) = \varphi_1(t, u(t), v(t), D^{\alpha_1+\beta_1-1}v(t)), t \in [0, 1], \\ u(0) = \psi_1(u), u(1) = \delta_1, \delta_1 \in \mathbb{R}, \\ D^{\alpha_2}(D^{\beta_2} + \lambda_2)v(t) = \varphi_2(t, u(t), D^{\alpha_2+\beta_2-1}u(t), v(t)), t \in [0, 1], \\ v(0) = \psi_2(v), v(1) = \delta_2, \delta_2 \in \mathbb{R}, \end{cases} \quad (1)$$

where for $i = 1, 2$, $D^{\alpha_i}, D^{\beta_i}$ denote the Caputo fractional derivatives with $0 < \alpha_i, \beta_i \leq 1, \lambda_i \in \mathbb{R}$ and $\varphi_i : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $\psi_i : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ are given continuous functions. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function φ on $[0, \infty[$ is defined by

$$J^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \varphi(\tau) d\tau, \alpha > 0,$$

where $\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1} dx$. The fractional derivative of $\varphi \in C^n([0, \infty[)$ in the Caputo's sense is defined as:

$$D^\alpha \varphi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t - \tau)^{n-\alpha-1} \varphi^{(n)}(\tau) d\tau, n-1 < \alpha < n, n \in \mathbb{N}^*.$$

We give the following lemmas [15, 16]:

Lemma 1: Let $r, s > 0, \varphi \in L_1([a, b])$. Then $J^r J^s \varphi(t) = J^{r+s} \varphi(t)$, $D^s J^s \varphi(t) = \varphi(t)$, $t \in [a, b]$.

Lemma 2: Let $s > r > 0, \varphi \in L_1([a, b])$. Then $D^r J^s \varphi(t) = J^{s-r} \varphi(t)$, $t \in [a, b]$.

Also, we present the following two lemmas [15]:

Lemma 3: For $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha u(t) = 0$ is given by

$$u(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1, n = [\alpha] + 1$.

Lemma 4: Let $\alpha > 0$. Then

$$J^\alpha D^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1, n = [\alpha] + 1$.

Lemma 5: [7] *Let $P : E \rightarrow E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let $\Sigma(P) = \{u \in E : u = \rho P(u) \text{ for some } 0 < \rho < 1\}$. Then either the set $\Sigma(P)$ is unbounded, or P has at least one fixed point.*

We prove the following auxiliary lemma.

Lemma 6: *Suppose that $h \in C([0, 1])$ and consider the problem*

$$D^{\alpha_1}(D^{\beta_1} + \lambda_1)u(t) = h(t), t \in [0, 1], 0 < \alpha_1, \beta_1 \leq 1, \lambda_1 \in \mathbb{R}, \quad (2)$$

with the conditions

$$u(0) = \psi_1(u), u(1) = \delta_1, \delta_1 \in \mathbb{R}. \quad (3)$$

Then, we have

$$\begin{aligned} u(t) = & \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} h_1(s) ds - \lambda_1 \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} u(s) ds + \psi_1(u) \\ & + t^{\beta_1} \left[\delta_1 + \lambda_1 \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} u(s) ds - \int_0^1 \frac{(1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} h_1(s) ds - \psi_1(u) \right]. \end{aligned} \quad (4)$$

Proof: We have

$$u(t) = J^{\alpha_1+\beta_1} h_1(t) - \lambda_1 J^{\beta_1} u(t) + \frac{c_1 t^{\beta_1}}{\Gamma(\beta_1+1)} + c_2, \quad (5)$$

where $c_i \in \mathbb{R}, (i=1, 2)$, are arbitrary constants.

Using the boundary conditions (3), we obtain

$$C_2 = \psi_1(u),$$

and

$$c_1 = \Gamma(\beta_1 + 1) \left[\theta_1 + \lambda_1 \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} u(s) ds - \int_0^1 \frac{(1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} h_1(s) ds - \phi(u) \right].$$

Inserting the values of $c_i \in \mathbb{R}, (i=1, 2)$ in (5), we get the solution (4).

Let us now introduce the following space:

$$X := \{u : u \in C([0, 1], \mathbb{R}) \text{ and } D^{\alpha_1+\beta_1-1} u \in C([0, 1], \mathbb{R})\},$$

endowed with the norm

$$\|u\|_X = \|u\| + \|D^{\alpha_1+\beta_1-1} u\|,$$

where

$$\|u\| = \sup_{t \in [0,1]} |u(t)| \text{ and } \|D^{\alpha_1 + \beta_1 - 1} u\| = \sup_{t \in [0,1]} |D^{\alpha_1 + \beta_1 - 1} u(t)|.$$

Similarly, we can define the space

$$Y := \{v : v \in C([0,1], \mathbb{R}) \text{ and } D^{\alpha_2 + \beta_2 - 1} v \in C([0,1], \mathbb{R}),$$

with the norm

$$\|v\|_Y = \|v\| + \|D^{\alpha_2 + \beta_2 - 1} v\|,$$

where

$$\|v\| = \sup_{t \in [0,1]} |v(t)|, \|D^{\alpha_2 + \beta_2 - 1} v\| = \sup_{t \in [0,1]} |D^{\alpha_2 + \beta_2 - 1} v(t)|.$$

For $(u, v) \in X \times Y$, let

$$\|(u, v)\|_{X \times Y} = \|u\|_X + \|v\|_Y.$$

It is clear that the product space $(X \times Y, \|(u, v)\|_{X \times Y})$ is a Banach space.

2. Main Results

In this section, we present the main results of the article. For the sake of convenience, let us take:

$$\begin{aligned} \nabla_1 &: = \frac{\omega}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{|\lambda_1|}{\Gamma(\beta_1 + 1)} + \theta_1, \\ \Delta_1 &: = \frac{2M_1}{\Gamma(\alpha_1 + \beta_1 + 1)} + |\delta_1| \\ \nabla_2 &: = \frac{\omega}{\Gamma(\alpha_1 + \beta_1)} + \frac{|\lambda_1|}{\Gamma(\beta_1)} + \theta_1 + \beta_1 \nabla_1 \\ \Delta_2 &: = \frac{M_1}{\Gamma(\alpha_1 + \beta_1)} + \beta_1 |\delta_1| + \frac{M_1}{\Gamma(\alpha_1 + \beta_1 + 1)}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \Lambda_1 &: = \frac{\varpi}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{|\lambda_2|}{\Gamma(\beta_2 + 1)} + \theta_2, \\ \Pi_1 &: = \frac{2M_2}{\Gamma(\alpha_2 + \beta_2 + 1)} + |\delta_2|, \\ \Lambda_2 &: = \frac{\varpi}{\Gamma(\alpha_2 + \beta_2)} + \frac{|\lambda_2|}{\Gamma(\beta_2)} + \theta_2 + \beta_2 \nabla_2, \\ \Pi_2 &: = \frac{M_2}{\Gamma(\alpha_2 + \beta_2)} + \beta_2 |\delta_2| + \frac{M_2}{\Gamma(\alpha_2 + \beta_2 + 1)}. \end{aligned} \tag{7}$$

We also suppose:

(H_1) : The functions $\varphi_1, \varphi_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, are continuous.

(H_2) : There exist constants $\omega > 0, \varpi > 0$ such that for all $t \in [0, 1]$ and $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, we have:

$$|\varphi_1(t, x_1, x_2) - \varphi_1(t, y_1, y_2)| \leq \omega(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),$$

$$|\varphi_2(t, x_1, x_2) - \varphi_2(t, y_1, y_2)| \leq \varpi(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),$$

(H_3) : $\psi_1, \psi_2 : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ are given continuous functions with $\psi_1(0) = \psi_2(0) = 0$ and there exist constants $\theta_1 > 0, \theta_2 > 0$ such that

$$|\psi_1(u) - \psi_1(v)| \leq \theta_1 \|u - v\|, |\psi_2(u) - \psi_2(v)| \leq \theta_2 \|u - v\|.$$

In view of Lemma 6, we define an operator $T : X \times Y \rightarrow X \times Y$ by:

$$T(u, v)(t) := (T_1(u, v)(t), T_2(u, v)(t)),$$

where, for each $t \in [0, 1]$,

$$\begin{aligned} T_1(u, v)(t) = & \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} \varphi_1(s, u(s), v(s), D^{\alpha_1+\beta_1-1}v(s)) ds + \psi_1(u) \\ & - \lambda_1 \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} u(s) ds + t^{\beta_1} \left[\delta_1 + \lambda_1 \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} u(s) ds \right. \\ & \left. - \int_0^1 \frac{(1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} \varphi_1(s, u(s), v(s), D^{\alpha_1+\beta_1-1}v(s)) ds - \psi_1(u) \right], \end{aligned} \quad (8)$$

and

$$\begin{aligned} T_2(u, v)(t) = & \int_0^t \frac{(t-s)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} \varphi_2(s, u(s), D^{\alpha_2+\beta_2-1}u(s), v(s)) ds + \psi_2(v) \\ & - \lambda_2 \int_0^t \frac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)} v(s) ds + t^{\beta_2} \left[\delta_2 + \lambda_2 \int_0^1 \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} v(s) ds \right. \\ & \left. - \int_0^1 \frac{(1-s)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} \varphi_2(s, u(s), D^{\alpha_2+\beta_2-1}u(s), v(s)) ds - \psi_2(v) \right]. \end{aligned} \quad (9)$$

Our first result is based on Banach contraction principle. It is the following:

Theorem 7 : Assume that $(H_1), (H_2)$ and (H_3) hold. In addition it is assumed that

$$2\nabla_1 + \frac{\nabla_2}{\Gamma(3-\alpha_1-\beta_1)} < \frac{1}{2} \text{ and } 2\Lambda_1 + \frac{\Lambda_2}{\Gamma(3-\alpha_2-\beta_2)} < \frac{1}{2}, \quad (10)$$

where ∇_i, Λ_i ($i=1,2$) are given by (6) and (7) respectively. Then the fractional system (1) has a unique solution on $[0,1]$.

Proof: Define $M_1 = \sup_{t \in [0,1]} |\varphi_1(t,0,0,0)| < \infty$ and $M_2 = \sup_{t \in [0,1]} |\varphi_2(t,0,0,0)| < \infty$ such that

$$\max \left\{ \frac{\Delta_1 + \frac{\Lambda_2}{\Gamma(3-\alpha_1-\beta_1)}}{\frac{1}{2} \left(2\nabla_1 + \frac{\nabla_2}{\Gamma(3-\alpha_1-\beta_1)} \right)}, \frac{\Pi_1 + \frac{\Pi_2}{\Gamma(3-\alpha_2-\beta_2)}}{\frac{1}{2} \left(2\Lambda_1 + \frac{\Lambda_2}{\Gamma(3-\alpha_2-\beta_2)} \right)} \right\} \leq r.$$

We first show that $TB_r \subset B_r$, where $B_r = \{(u,v) \in X \times Y : \|(u,v)\|_{X \times Y} \leq r\}$. Using $(H_i), i=1, \dots, 3$, we can write

$$\begin{aligned} & |\varphi_1(t, u(t), v(t), D^{\alpha_1+\beta_1-1}v(t))| \\ & \leq |\varphi_1(t, u(t), v(t), D^{\alpha_1+\beta_1-1}v(t)) - \varphi_1(t, 0, 0, 0)| + |\varphi_1(t, 0, 0, 0)| \\ & \leq \omega(|u(t)| + |v(t)| + |D^{\alpha_1+\beta_1-1}v(t)|) + M_1 \\ & \leq \omega(\|u\|_X + \|v\|_Y) + M_1 \leq \omega\|(u,v)\|_{X \times Y} + M_1 \leq \omega r + M_1. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & |\varphi_2(t, u(t), D^{\alpha_1+\beta_1-1}u(t), v(t))| \\ & \leq |\varphi_2(t, u(t), D^{\alpha_1+\beta_1-1}u(t), v(t)) - \varphi_2(t, 0, 0, 0)| + |\varphi_2(t, 0, 0, 0)| \\ & \leq \omega(|u(t)| + |D^{\alpha_1+\beta_1-1}u(t)| + |v(t)|) + M_2 \\ & \leq \omega(\|u\|_X + \|v\|_Y) + M_2 \leq \omega\|(u,v)\|_{X \times Y} + M_2 \leq \omega r + M_2, \end{aligned}$$

and

$$|\psi_1(u)| \leq \theta_1 \|u\| \leq \theta_1 \|u\|_X \leq \theta_1 r, |\psi_2(v)| \leq \theta_2 \|v\| \leq \theta_2 \|v\|_Y \leq \theta_2 r.$$

By above estimates, we have

$$\begin{aligned} & |T_1(u,v)(t)| \\ & \leq \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} |\varphi_1(s, u(s), v(s), D^{\alpha_1+\beta_1-1}v(s))| ds + |\psi_1(u)| \\ & \quad + |\lambda_1| \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} |u(s)| ds + t^{\beta_1} \left[|\delta_1| + |\lambda_1| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} |u(s)| ds \right. \\ & \quad \left. + \int_0^1 \frac{(1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} |\varphi_1(s, u(s), v(s), D^{\alpha_1+\beta_1-1}v(s))| ds + |\psi_1(u)| \right] \\ & \leq 2 \left(\frac{\omega}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda_1|}{\Gamma(\beta_1+1)} + \theta_1 \right) r + \frac{2M_1}{\Gamma(\alpha_1+\beta_1+1)} + |\delta_1| \\ & = 2\nabla_1 r + \Delta_1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& |T_1(u, v)'(t)| \\
& \leq \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-2}}{\Gamma(\alpha_1+\beta_1-1)} |\varphi_1(s, u(s), v(s), D^{\alpha_1+\beta_1-1}v(s))| ds \\
& \quad + |\lambda_1| \int_0^t \frac{(t-s)^{\beta_1-2}}{\Gamma(\beta_1-1)} |u(s)| ds + \beta_1 t^{\beta_1-1} \left[|\delta_1| + |\lambda_1| \int_0^1 \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} |u(s)| ds \right. \\
& \quad \left. + \int_0^1 \frac{(1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} |\varphi_1(s, u(s), v(s), D^{\alpha_1+\beta_1-1}v(s))| ds + |\psi_1(u)| \right] \\
& \leq \left[\left(\frac{\omega}{\Gamma(\alpha_1+\beta_1)} + \frac{|\lambda_1|}{\Gamma(\beta_1)} \right) + \beta_1 \nabla_1 \right] r + \frac{M_1}{\Gamma(\alpha_1+\beta_1)} + \beta_1 \left(|\delta_1| + \frac{M_1}{\Gamma(\alpha_1+\beta_1+1)} \right) \\
& = \nabla_2 r + \Delta_2.
\end{aligned}$$

which implies that

$$\begin{aligned}
|D^{\alpha_1+\beta_1-1}T_1(u, v)(t)| & \leq \int_0^t \frac{(t-s)^{1-\alpha_1-\beta_1}}{\Gamma(2-\alpha_1-\beta_1)} |T_1(u, v)'(s)| ds \\
& \leq \frac{\nabla_2}{\Gamma(3-\alpha_1-\beta_1)} r + \frac{\Delta_2}{\Gamma(3-\alpha_1-\beta_1)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|T_1(u, v)\|_X & \leq \|T_1(x, y)\| + \|D^{\alpha+\beta-1}T_1(x, y)\|_X \\
& \leq \left(2\nabla_1 + \frac{\nabla_2}{\Gamma(3-\alpha_1-\beta_1)} \right) r + \Delta_1 + \frac{\Delta_2}{\Gamma(3-\alpha_1-\beta_1)} \leq \frac{r}{2}.
\end{aligned}$$

In a similar manner, it can be shown that

$$\|T_2(u, v)\|_Y \leq \left(2\Lambda_1 + \frac{\Lambda_2}{\Gamma(3-\alpha_1-\beta_1)} \right) r + \Pi_1 + \frac{\Pi_2}{\Gamma(3-\alpha_1-\beta_1)} \leq \frac{r}{2}.$$

Consequently

$$\|T(u, v)\| \leq r,$$

which implies that $TB_r \subset B_r$. Now for $u_i, v_i \in B_r, i = 1, 2$ and for all $t \in [0, 1]$, we have

$$\begin{aligned}
& |T_1(u_1, v_1)(t) - T_1(u_2, v_2)(t)| \\
& \leq \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} |\varphi_1(s, u_1(s), v_1(s), D^{\alpha_1+\beta_1-1}v_1(s)) \\
& \quad - \varphi_1(s, u_2(s), v_2(s), D^{\alpha_1+\beta_1-1}v_2(s))| ds + |\psi_2(v_1) - \psi_2(v_2)| \\
& \quad + \lambda_1 \int_0^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} |u_1(s) - u_2(s)| ds + t^{\beta_1} \left[\lambda_1 \int_0^1 \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} |u_1(s) - u_2(s)| ds \right.
\end{aligned}$$

$$\begin{aligned}
& - \int_0^1 \frac{(t-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} \left| \varphi_1(s, u_1(s), v_1(s), D^{\alpha_1+\beta_1-1} v_1(s)) \right. \\
& \left. - \varphi_1(s, u_2(s), v_2(s), D^{\alpha_1+\beta_1-1} v_2(s)) \right| ds + |\psi_2(v_1) - \psi_2(v_2)| \big],
\end{aligned}$$

By $(H_i), i = 1, \dots, 3$, we obtain

$$\begin{aligned}
& \|T_1(u_1, v_1) - T_1(u_2, v_2)\| \\
& \leq 2 \left[\frac{\omega}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda|}{\Gamma(\beta_1+1)} + \theta_2 \right] (\|u_1 - u_2\|_X + \|v_1 - v_2\|_Y) \\
& = 2\nabla_1 (\|u_1 - u_2\|_X + \|v_1 - v_2\|_Y).
\end{aligned}$$

Also we have

$$|T_1(u_1, v_1)'(t) - T_1(u_2, v_2)'(t)| \leq \nabla_2 (\|u_1 - u_2\|_X + \|v_1 - v_2\|_Y).$$

Thus

$$\begin{aligned}
& |D^{\alpha_1+\beta_1-1} T_1(u_1, v_1)(t) - D^{\alpha_1+\beta_1-1} T_1(u_2, v_2)(t)| \\
& \leq \int_0^t \frac{(t-s)^{1-\alpha_1-\beta_1}}{\Gamma(2-\alpha_1-\beta_1)} |T_1(u_1, v_1)'(t) - T_1(u_2, v_2)'(t)| \\
& \leq \frac{\nabla_2}{\Gamma(3-\alpha_1-\beta_1)} (\|u_1 - u_2\|_X + \|v_1 - v_2\|_Y).
\end{aligned}$$

Hence

$$\begin{aligned}
& \|T_1(u_1, v_1) - T_1(u_2, v_2)\|_X \\
& \leq \left(2\nabla_1 + \frac{\nabla_2}{\Gamma(3-\alpha_1-\beta_1)} \right) (\|u_1 - u_2\|_X + \|v_1 - v_2\|_Y).
\end{aligned} \tag{11}$$

Similarly, we can have

$$\begin{aligned}
& \|T_2(u_1, v_1) - T_2(u_2, v_2)\|_Y \\
& \leq \left(2\Lambda_1 + \frac{\Lambda_2}{\Gamma(3-\alpha_2-\beta_2)} \right) (\|u_1 - u_2\|_X + \|v_1 - v_2\|_Y).
\end{aligned} \tag{12}$$

It follows from (11) and (12) that

$$\begin{aligned}
& \|T(u_1, v_1) - T(u_2, v_2)\| \\
& \leq \left[\left(2\nabla_1 + \frac{\nabla_2}{\Gamma(3-\alpha_1-\beta_1)} \right) + \left(2\Lambda_1 + \frac{\Lambda_2}{\Gamma(3-\alpha_2-\beta_2)} \right) \right] \| (u_1 - u_2, v_1 - v_2) \|_{X \times Y}.
\end{aligned}$$

By (10), we see that T is a contractive operator. Consequently, by Banach's fixed point theorem, the operator T has a unique fixed point which is the unique solution of fractional system (1). This completes the proof.

Now, we show the existence of solutions for the system (1) by using Lemma 5.

For the next result, we assume that:

(H_4) : There exist real constants $\eta_i, \vartheta_i \geq 0$ ($i = 1, \dots, 3$) and $\eta_0 > 0, \vartheta_0 > 0$ such that for any $(u_1, u_2, u_3) \in \mathbb{R}^3$, we have:

$$|\varphi_1(t, u_1, u_2, u_3)| \leq \eta_0 + \eta_1 |u_1| + \eta_2 |u_2| + \eta_3 |u_3|,$$

and

$$|\varphi_2(t, u_1, u_2, u_3)| \leq \vartheta_0 + \vartheta_1 |u_1| + \vartheta_2 |u_2| + \vartheta_3 |u_3|.$$

(H_5) : $\psi_1, \psi_2 : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ are continuous functions with $\psi_1(0) = \psi_2(0) = 0$ and there exist constants $k_1 > 0, k_2 > 0$ such that

$$|\psi_1(u)| \leq k_1 \|x\|, |\psi_2(v)| \leq k_2 \|v\| \text{ for all } u, v \in C([0, 1], \mathbb{R}).$$

For convenience, we set the constants:

$$\begin{aligned} F_0 &: = \frac{2\eta_0}{\Gamma(\alpha_1 + \beta_1 + 1)} + \left(1 + \frac{\beta_1}{\Gamma(3 - \alpha_1 - \beta_1)}\right) \delta_1 \\ &\quad + \frac{1}{\Gamma(3 - \alpha_1 - \beta_1)} \left(\frac{\eta_0}{\Gamma(\alpha_1 + \beta_1)} + \frac{\beta_1 \eta_0}{\Gamma(\alpha_1 + \beta_1 + 1)} \right), \\ F_1 &: = 2 \left(\frac{\eta_1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{|\lambda_1|}{\Gamma(\beta_1 + 1)} \right) + \left(1 + \frac{\beta_1}{\Gamma(3 - \alpha_1 - \beta_1)}\right) k_1 \\ &\quad + \frac{1}{\Gamma(3 - \alpha_1 - \beta_1)} \left(\frac{\eta_1}{\Gamma(\alpha_1 + \beta_1)} + \frac{|\lambda_1|}{\Gamma(\beta_1)} + \frac{\beta_1 |\lambda_1|}{\Gamma(\beta_1 + 1)} + \frac{\beta_1 \eta_1}{\Gamma(\alpha_1 + \beta_1 + 1)} \right), \\ F_2 &: = \frac{2(\eta_2 + \eta_3)}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{1}{\Gamma(3 - \alpha_1 - \beta_1)} \left(\frac{\eta_2 + \eta_3}{\Gamma(\alpha_1 + \beta_1)} + \frac{\beta_1(\eta_2 + \eta_3)}{\Gamma(\alpha_1 + \beta_1 + 1)} \right), \end{aligned} \quad (13)$$

$$\begin{aligned} \Omega_0 &: = \frac{2\vartheta_0}{\Gamma(\alpha_2 + \beta_2 + 1)} + \left(1 + \frac{\beta_2}{\Gamma(3 - \alpha_2 - \beta_2)}\right) \delta_2 \\ &\quad + \frac{1}{\Gamma(3 - \alpha_2 - \beta_2)} \left(\frac{\vartheta_0}{\Gamma(\alpha_2 + \beta_2)} + \frac{\beta_2 \vartheta_0}{\Gamma(\alpha_2 + \beta_2 + 1)} \right), \\ \Omega_1 &: = 2 \left(\frac{\vartheta_1}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{|\lambda_2|}{\Gamma(\beta_2 + 1)} \right) + \left(1 + \frac{\beta_2}{\Gamma(3 - \alpha_2 - \beta_2)}\right) k_2 \\ &\quad + \frac{1}{\Gamma(3 - \alpha_2 - \beta_2)} \left(\frac{\vartheta_1}{\Gamma(\alpha_2 + \beta_2)} + \frac{|\lambda_2|}{\Gamma(\beta_2)} + \frac{\beta_2 |\lambda_2|}{\Gamma(\beta_2 + 1)} + \frac{\beta_2 \vartheta_1}{\Gamma(\alpha_2 + \beta_2 + 1)} \right), \\ \Omega_2 &: = \frac{2(\vartheta_2 + \vartheta_3)}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{1}{\Gamma(3 - \alpha_2 - \beta_2)} \left(\frac{\eta_2 + \eta_3}{\Gamma(\alpha_2 + \beta_2)} + \frac{\beta_2(\vartheta_2 + \vartheta_3)}{\Gamma(\alpha_2 + \beta_2 + 1)} \right). \end{aligned} \quad (14)$$

Theorem 8: Assume that the hypotheses (H_1) , (H_4) and (H_5) hold. If the inequalities

$$F_1 + \Omega_1 < 1, F_2 + \Omega_2 < 1, \quad (15)$$

are valid, then the system (1) has at least one solution on $[0, 1]$.

Proof: Firstly, we show that the operator $T : X \times Y \rightarrow X \times Y$ is completely continuous. By continuity of the functions φ_i and ψ_i ($i = 1, 2$), the operator T is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants μ such that $\|(u, v)\|_{X \times Y} \leq \mu$ for all $(u, v) \in \Theta$. Also there exist constants L_i and N_i ($i = 1, 2$) such that

$$\begin{aligned} |\varphi_1(s, u(s), v(s), D^{\alpha_1 + \beta_1 - 1} v(s))| &\leq L_1, \\ |\varphi_2(s, u(s), D^{\alpha_2 + \beta_2 - 1} u(s), v(s),)| &\leq L_2, \forall (u, v) \in \Theta, \end{aligned}$$

and

$$|\psi_1(u)| \leq N_1, \quad |\psi_2(v)| \leq N_2 \text{ for all } u, v \in C([0, 1], \mathbb{R}).$$

Therefore, for any $(u, v) \in \Theta$, we have

$$\|T_1(u, v)\| \leq 2 \left(\frac{L_1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{|\lambda_1| \mu}{\Gamma(\beta_1 + 1)} + N_1 \right) + |\delta_1| := 2A_1 + |\delta_1|, \quad (16)$$

and

$$\begin{aligned} |T_1(u, v)'(t)| &\leq 2 \left(\frac{L_1}{\Gamma(\alpha_1 + \beta_1)} + \frac{|\lambda_1| \mu}{\Gamma(\beta_1)} \right) + \beta_1 A_1 + \beta_1 |\delta_1| \\ &:= 2B_1 + \beta_1 A_1 + \beta_1 |\delta_1|. \end{aligned}$$

Also, we have

$$|D^{\alpha_1 + \beta_1 - 1} T_1(u, v)(t)| \leq \int_0^t \frac{(t-s)^{1-\alpha_1-\beta_1}}{\Gamma(2-\alpha_1-\beta_1)} |T_1(u, v)'(s)| ds \leq \frac{[2B_1 + \beta_1 A_1 + \beta_1 |\delta_1|]}{\Gamma(3-\alpha_1-\beta_1)}$$

which implies that

$$\|D^{\alpha_1 + \beta_1 - 1} T_1(u, v)\| \leq \frac{[2B_1 + \beta_1 A_1 + \beta_1 |\delta_1|]}{\Gamma(3-\alpha_1-\beta_1)}. \quad (17)$$

Combining (16) and (17), yields to

$$\|T_1(u, v)\|_X \leq \|T_1(u, v)\| + \|D^{\alpha_1 + \beta_1 - 1} T_1(u, v)\| \leq 2A_1 + |\delta_1| + \frac{(2B_1 + \beta_1 A_1 + \beta_1 |\delta_1|)}{\Gamma(3-\alpha_1-\beta_1)}. \quad (18)$$

Similarly, we get

$$\|T_2(u, v)\|_Y \leq \|T_2(u, v)\| + \|D^{\alpha_1+\beta_1-1}T_2(u, v)\| \leq 2A_2 + |\delta_2| + \frac{(2B_2+\beta_1A_2+\beta_2|\delta_2|)}{\Gamma(3-\alpha_2-\beta_2)}, \quad (19)$$

with

$$A_2 := \frac{L_2}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda_2|\mu}{\Gamma(\beta_2+1)} + N_2,$$

and

$$B_2 := \frac{L_2}{\Gamma(\alpha_2+\beta_2)} + \frac{|\lambda_2|\mu}{\Gamma(\beta_2)}.$$

Thanks to (18) and (19), we have

$$\begin{aligned} \|T(u, v)\|_{X \times Y} &\leq 2(A_1 + A_2) + |\delta_1| + |\delta_2| + \frac{1}{\Gamma(3-\alpha_1-\beta_1)} \\ &\quad \times [(2B_1 + \beta_1A_1 + \beta_1|\delta_1|) + (2B_2 + \beta_1A_2 + \beta_2|\delta_2|)]. \end{aligned}$$

Consequently $\|T(u, v)\|_{X \times Y} < \infty$.

Next, we show that T is equicontinuous sets of X . For $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we get

$$\begin{aligned} &|T_1(u, v)(t_2) - T_1(u, v)(t_1)| \\ &\leq \left| \int_0^{t_1} \frac{[(t_2-s)^{\alpha_1+\beta_1-1} - (t_1-s)^{\alpha_1+\beta_1-1}]}{\Gamma(\alpha_1+\beta_1)} ds \right| + \left| \int_{t_1}^{t_2} \frac{L_1(t_2-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} ds \right| \\ &\quad \left| \int_0^{t_1} \frac{\mu|\lambda_1|[(t_2-s)^{\beta_1-1} - (t_1-s)^{\beta_1-1}]}{\Gamma(\beta_1)} ds \right| + \left| \int_{t_1}^{t_2} \frac{|\lambda_1|(t_2-s)^{\beta_1-1}}{\Gamma(\beta_1)} ds \right| \\ &\quad + |t_2^{\beta_1} - t_1^{\beta_1}| \left[|\delta_1| + \int_0^1 \frac{\mu|\lambda_1|(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} ds + \int_0^1 \frac{L_1(1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} ds + |\psi_1(u)| \right]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &|T_1(u, v)(t_2) - T_1(u, v)(t_1)| \\ &\leq \frac{L_1}{\Gamma(\alpha_1+\beta_1+1)} |t_2^{\alpha_1+\beta_1} - t_1^{\alpha_1+\beta_1}| + \frac{\mu|\lambda_1|}{\Gamma(\beta_1+1)} |t_2^{\beta_1} - t_1^{\beta_1}| \\ &\quad + |t_2^{\beta_1} - t_1^{\beta_1}| \left[|\delta_1| + \frac{\mu|\lambda_1|}{\Gamma(\beta_1+1)} + \frac{L_1}{\Gamma(\alpha_1+\beta_1+1)} + N_1 \right]. \end{aligned}$$

We can also show that

$$\begin{aligned} &|T_1(u, v)'(t_2) - T_1(u, v)'(t_1)| \\ &\leq \frac{L_1}{\Gamma(\alpha_1+\beta_1)} |t_2^{\alpha_1+\beta_1-1} - t_1^{\alpha_1+\beta_1-1}| + \frac{\mu|\lambda_1|}{\Gamma(\beta_1)} |t_2^{\beta_1-1} - t_1^{\beta_1-1}| \\ &\quad + \beta_1 |t_2^{\beta_1-1} - t_1^{\beta_1-1}| \left[|\delta_1| + \frac{\mu|\lambda_1|}{\Gamma(\beta_1+1)} + \frac{L_1}{\Gamma(\alpha_1+\beta_1+1)} + N_1 \right], \end{aligned}$$

and

$$\begin{aligned}
& |D^{\alpha_1+\beta_1-1}T_1(u,v)(t_2) - D^{\alpha_1+\beta_1-1}T_1(u,v)(t_1)| \\
&= \int_0^t \frac{(t-s)^{1-\alpha_1-\beta_1}}{\Gamma(2-\alpha_1-\beta_1)} |T_1(u,v)'(t_2) - T_1(u,v)'(t_1)| \\
&\leq \frac{1}{\Gamma(3-\alpha_1-\beta_1)} \left[\frac{L_1}{\Gamma(\alpha_1+\beta_1)} |t_2^{\alpha_1+\beta_1-1} - t_1^{\alpha_1+\beta_1-1}| + \frac{\mu|\lambda_1|}{\Gamma(\beta_1+1)} |t_2^{\beta_1-1} - t_1^{\beta_1-1}| \right. \\
&\quad \left. + \beta_1 |t_2^{\beta_1-1} - t_1^{\beta_1-1}| \left(|\delta_1| + \frac{\mu|\lambda_1|}{\Gamma(\beta_1+1)} + \frac{L_1}{\Gamma(\alpha_1+\beta_1+1)} + N_1 \right) \right].
\end{aligned}$$

Hence we have $\|T_1(u,v)(t_2) - T_1(u,v)(t_1)\| \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$.

Similarly, we get

$$\begin{aligned}
& |T_2(u,v)(t_2) - T_1(u,v)(t_2)| \\
&\leq \frac{L_2}{\Gamma(\alpha_2+\beta_2+1)} |t_2^{\alpha_2+\beta_2} - t_1^{\alpha_2+\beta_2}| + \frac{\mu|\lambda_2|}{\Gamma(\beta_2+1)} |t_2^{\beta_2} - t_1^{\beta_2}| \\
&\quad + |t_2^{\beta_2} - t_1^{\beta_2}| \left[|\delta_2| + \frac{\mu|\lambda_2|}{\Gamma(\beta_2+1)} + \frac{L_2}{\Gamma(\alpha_2+\beta_2+1)} + N_2 \right].
\end{aligned}$$

and

$$\begin{aligned}
& |D^{\alpha_2+\beta_2-1}T_2(u,v)(t_2) - D^{\alpha_2+\beta_2-1}T_2(u,v)(t_1)| \\
&\leq \frac{1}{\Gamma(3-\alpha_2-\beta_2)} \left[\frac{L_2}{\Gamma(\alpha_2+\beta_2)} |t_2^{\alpha_2+\beta_2-1} - t_1^{\alpha_2+\beta_2-1}| + \frac{\mu|\lambda_2|}{\Gamma(\beta_2+1)} |t_2^{\beta_2-1} - t_1^{\beta_2-1}| \right. \\
&\quad \left. + \beta_2 |t_2^{\beta_2-1} - t_1^{\beta_2-1}| \left(|\delta_2| + \frac{\mu|\lambda_2|}{\Gamma(\beta_2+1)} + \frac{L_2}{\Gamma(\alpha_2+\beta_2+1)} + N_2 \right) \right],
\end{aligned}$$

which imply that $\|T_2(u,v)(t_2) - T_2(u,v)(t_1)\| \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$. Therefore, $T : X \times Y \rightarrow X \times Y$ is completely continuous by application of the Arzela-Ascoli theorem.

Finally, we show that the set Φ defined by

$$\Phi = \{(u,v) \in X \times Y : (x,y) = \sigma T(u,v), 0 < \sigma < 1\},$$

is bounded. Let $(u,v) \in \Phi$, then $(u,v) = \sigma T(u,v)$. For any $t \in [0,1]$, we have

$$u(t) = \sigma T_1(u,v)(t), \quad y(t) = \sigma T_2(u,v)(t).$$

Then

$$|u(t)| \leq 2 \left(\frac{\eta_1}{\Gamma(\alpha_1+\beta_1+1)} + k_1 + \frac{|\lambda_1|}{\Gamma(\beta_1+1)} \right) \|u\|_X + \frac{2(\eta_2+\eta_3)}{\Gamma(\alpha_1+\beta_1+1)} \|v\|_Y + \frac{2\eta_0}{\Gamma(\alpha_1+\beta_1+1)} + \delta_1.$$

On the other hand, we have

$$\begin{aligned}
& |u'(t)| \\
& \leq \left[\frac{\eta_1}{\Gamma(\alpha_1+\beta_1)} + \frac{|\lambda_1|}{\Gamma(\beta_1)} + \beta_1 \left(\frac{|\lambda_1|}{\Gamma(\beta_1+1)} + \frac{\eta_1}{\Gamma(\alpha_1+\beta_1+1)} + k_1 \right) \right] \|u\|_X \\
& \quad \left(\frac{\eta_2+\eta_3}{\Gamma(\alpha_1+\beta_1)} + \frac{\beta_1(\eta_2+\eta_3)}{\Gamma(\alpha_1+\beta_1+1)} \right) \|v\|_Y + \frac{\eta_0}{\Gamma(\alpha_1+\beta_1)} + \beta_1 \left(\frac{\eta_0}{\Gamma(\alpha_1+\beta_1+1)} + \delta_1 \right),
\end{aligned}$$

and

$$\begin{aligned}
& |D^{\alpha_1+\beta_1-1}u(t)| \\
& \leq \frac{1}{\Gamma(3-\alpha_2-\beta_2)} \left(\frac{\eta_2+\eta_3}{\Gamma(\alpha_1+\beta_1)} + \frac{\beta_1(\eta_2+\eta_3)}{\Gamma(\alpha_1+\beta_1+1)} \right) \|v\|_Y \\
& \quad + \frac{1}{\Gamma(3-\alpha_1-\beta_1)} \left[\frac{\eta_1}{\Gamma(\alpha_1+\beta_1)} + \frac{|\lambda_1|}{\Gamma(\beta_1)} + \beta_1 \left(\frac{|\lambda_1|}{\Gamma(\beta_1+1)} + \frac{\eta_1}{\Gamma(\alpha_1+\beta_1+1)} + k_1 \right) \right] \|u\|_X \\
& \quad + \frac{1}{\Gamma(3-\alpha_1-\beta_1)} \left[\frac{\eta_0}{\Gamma(\alpha_1+\beta_1)} + \beta_1 \left(\frac{\eta_0}{\Gamma(\alpha_1+\beta_1+1)} + \delta_1 \right) \right],
\end{aligned}$$

Hence

$$\begin{aligned}
& \|u\|_X \\
& \leq \left[2 \left(\frac{\eta_1}{\Gamma(\alpha_1+\beta_1+1)} + k_1 + \frac{|\lambda_1|}{\Gamma(\beta_1+1)} \right) + \frac{1}{\Gamma(3-\alpha_1-\beta_1)} \right. \\
& \quad \left[\left(\frac{\eta_1}{\Gamma(\alpha_1+\beta_1)} + \frac{|\lambda_1|}{\Gamma(\beta_1)} + \frac{\beta_1|\lambda_1|}{\Gamma(\beta_1+1)} + \frac{\beta_1\eta_1}{\Gamma(\alpha_1+\beta_1+1)} + \beta_1k_1 \right) \|u\|_X \right. \\
& \quad \left. \left[\frac{2(\eta_2+\eta_3)}{\Gamma(\alpha_1+\beta_1+1)} + \frac{1}{\Gamma(3-\alpha_1-\beta_1)} \left(\frac{\eta_2+\eta_3}{\Gamma(\alpha_1+\beta_1)} + \frac{\beta_1(\eta_2+\eta_3)}{\Gamma(\alpha_1+\beta_1+1)} \right) \right] \|v\|_Y \right. \\
& \quad \left. + \frac{2\eta_0}{\Gamma(\alpha_1+\beta_1+1)} + \left(1 + \frac{\beta_1}{\Gamma(3-\alpha_1-\beta_1)} \right) \delta_1 + \frac{1}{\Gamma(3-\alpha_1-\beta_1)} \left(\frac{\eta_0}{\Gamma(\alpha_1+\beta_1)} + \beta_1 \frac{\beta_1\eta_0}{\Gamma(\alpha_1+\beta_1+1)} \right) \right],
\end{aligned}$$

which implies that

$$\|u\|_X \leq F_1 \|u\|_X + F_2 \|v\|_Y + F_0. \quad (20)$$

Analogously, we can obtain

$$\|v\|_Y \leq \Omega_1 \|u\|_X + \Omega_2 \|v\|_Y + \Omega_0. \quad (21)$$

It follows from (20) and (21), that

$$\|u\|_X + \|v\|_Y \leq (F_1 + \Omega_1) \|u\|_X + (F_2 + \Omega_2) \|v\|_Y + F_0 + \Omega_0.$$

Consequently,

$$\|(u, v)\|_{X \times Y} \leq \frac{F_0 + \Omega_0}{\min(1 - (F_1 + \Omega_1), 1 - (F_2 + \Omega_2))}.$$

This shows that Φ is bounded. Hence, by Lemma 5, the operator T has at least one fixed point. Hence, the fractional differential system (1) has at least one solution on $[0, 1]$. The proof is complete.

3. Application

Consider the following fractional differential system:

$$\left\{ \begin{array}{l} D^{\frac{3}{4}} \left(D^{\frac{5}{6}} + \frac{1}{19} \right) u(t) = \frac{|u(t)| + |v(t)| + |D^{\frac{7}{12}} v(t)|}{(t^2 + 39\pi)(e^t + |u(t)| + |v(t)| + |D^{\frac{7}{12}} v(t)|)} + \ln(2 + 5t^2), t \in [0, 1], \\ D^{\frac{4}{5}} \left(D^{\frac{2}{3}} + \frac{1}{15} \right) v(t) = \frac{\sin u(t) + \sin D^{\frac{7}{15}} u(t) + \sin v(t)}{32\sqrt{\pi}(t^2 + 1)} + \arctan(2 + 3t + 2t^2), t \in [0, 1], \\ u(0) = \frac{2}{37} \sin u(t), \quad u(1) = \sqrt{2}, \\ v(0) = \frac{1}{23} \cos v(t), \quad v(1) = \frac{2}{3}. \end{array} \right. \quad (22)$$

Then, for $t \in [0, 1]$ and $u_i, v_i \in \mathbb{R}$ ($i = 1, \dots, 3$), we have

$$\begin{aligned} |\varphi_1(t, u_1, u_2, u_3) - \varphi_1(t, v_1, v_2, v_3)| &\leq \frac{1}{(t^2 + 39\pi)} (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|), \\ |\varphi_2(t, u_1, u_2, u_3) - \varphi_1(t, v_1, v_2, v_3)| &\leq \frac{1}{32\sqrt{\pi}(t^2 + 1)} (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|), \\ \psi_1(u_1) - \psi_1(v_1) &\leq \frac{2}{37} |u_1 - v_1|, \quad \psi_2(u_1) - \psi_2(v_1) \leq \frac{1}{23} |u_1 - v_1|. \end{aligned}$$

So, we can take

$$\omega = \frac{1}{39\pi}, \varpi = \frac{1}{32\sqrt{\pi}}, \theta_1 = \frac{2}{37}, \theta_2 = \frac{1}{23}.$$

Then,

$$\nabla_1 = 0.11579, \nabla_2 = 0.20632, \Lambda = 0.1309, \Lambda_2 = 0.19989,$$

we have also

$$2\nabla_1 + \frac{\nabla_2}{\Gamma(3 - \alpha_1 - \beta_1)} \simeq 0.49409 < \frac{1}{2}, \quad 2\Lambda_1 + \frac{\Lambda_2}{\Gamma(3 - \alpha_2 - \beta_2)} \simeq 0.48696 < \frac{1}{2}.$$

Hence, by Theorem 7, the system (22) has a unique solution on $[0, 1]$.

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