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Some integral inequalities for Hadamard fractional moments of continuous random variables

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Abstract

Using Hadamard fractional integral operator, some new integral inequalities for the Hadamard fractional moments of continuous random variables are obtained.

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1. Introduction

Integral inequalities play a very important role in the theory of differential equations and applied sciences, several authors have carried out important studies on the theory of integral inequalities, see [2, 3, 16, 19]. Moreover, the study of the integral inequalities applying fractional integral operators is also of great importance, we refer the reader to [5, 12, 18] for further information and applications. Also, fractional integral inequalities have been used in probability theory and statistical problems for a long time and continue to attract the attention of authors. For more details, we refer to [4, 6, 7, 8, 9, 11, 14] and the references therein. Recently, many studies on fractional continuous random variable, involving fractional integral operators such as Riemann-Liouville operators [6, 7, 9, 10], generalized Riemann-Liouville fractional integrals [1], k–Riemann-Liouville fractional integral operators [16], $(k,s)-$fractional integral

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operators [14, 15, 20] and Hadamard fractional integral operators [17], have appeared during the past several years. Motivated by the results presented in [9, 10], in this present work, we introduce new Hadamard fractional concepts on $(r,\alpha)-$moments for continuous random variables. Then, we obtain new results for the Hadamard fractional function. We also present new fractional integral results for the Hadamard fractional moments.

2. Preliminaries

In this section, we present some definitions that will be used throughout this paper:

Definition 1 : [5, 19] The Hadamard fractional integral operator of order $\alpha > 0$, for a continuous function *f* on $[1,\infty[$ is defined as

$$H_1^\alpha[f(t)] = \frac{1}{\Gamma(\alpha)}\int_1^t \ln\left(\frac{t}{\tau}\right)^{\alpha-1}\frac{f(\tau)}{\tau}d\tau, \tag{1}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$.

For, $\alpha > 0, \beta > 0$, the following properties are valid:

$$H_a^\alpha H_a^\beta[f(t)] = H_a^{\alpha+\beta}[f(t)] \text{ and } H_a^\alpha H_a^\beta[f(t)] = H_a^\beta H_a^\alpha[f(t)] \tag{2}$$

Now, we introduce the following definitions:

Definition 2 : The Hadamard fractional moment function of order $(r,\alpha); (r > 0, \alpha > 0)$, for a continuous random variable X with a positive $p.d.f.$ f defined on $[a,b]$ is defined as

$${}_HM_{r,\alpha}(t) = H_a^\alpha[t^r f(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t \ln\left(\frac{t}{\tau}\right)^{\alpha-1}\tau^{r-1}f(\tau)d\tau;\ a < t \le b, a \ge 1. \tag{3}$$

For $t = b$, we introduce the Hadamard fractional moment of X.

Definition3: TheHadamardfractionalmomentoforder $(r,\alpha); (r > 0, \alpha > 0)$, for a continuous random variable X, is defined as

$${}_HM_{r,\alpha} = \frac{1}{\Gamma(\alpha)}\int_a^b \ln\left(\frac{b}{\tau}\right)^{\alpha-1}\tau^{r-1}f(\tau)d\tau;\ a \ge 1. \tag{4}$$

3. Main Results

In this section, we present some new fractional integral inequalities for the Hadamrad fractional moments of a continuous random variables having the probability density function $p.d.f.f$.

Theorem 4 : *Let X be a continuous random variable having a probability density function f defined on* $[a,b]$. *Then we have.*

(1) For all $\alpha > 0,\ \ a < t \le b$

$$H_a^{\alpha}[f(t)]H_a^{\alpha}[t^{r-1}(t-E(X))f(t)]-(H_a^{\alpha}[(t-E(X))f(t)])_H M_{r-1,\alpha}(t)$$
$$\le \| f\|_{\infty}^{2}\left[\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\alpha}[t^{r}]-H_a^{\alpha}[t]H_a^{\alpha}[t^{r-1}]\right], \tag{5}$$

where $f \in L_{\infty}[a,b]$.

(2) For all $\alpha > 0,\ \ a < t \le b,$

$$H_a^{\alpha}[f(t)]H_a^{\alpha}[t^{r-1}(t-E(X))f(t)]-(H_a^{\alpha}[(t-E(X))f(t)])_H M_{r-1,\alpha}(t)$$
$$\le \frac{1}{2}(t-a)(t^{r-1}-a^{r-1})(H_a^{\alpha}[f(t)])^{2}, \tag{6}$$

where $E(X)=\int_a^b \tau f(\tau)d\tau$ *is the classical expectation of X.*

Proof : Taking a function $p:[a,b]\to\mathbb{R}^{+}$ and let us consider the quantity

$$\varphi(\tau,\rho)=(g(\tau)-g(\rho))(h(\tau)-h(\rho)),\tau,\rho\in(a,t),a<t\le b. \tag{7}$$

Multiplying (7) by $\dfrac{\ln\left(\frac{t}{\tau}\right)^{\alpha-1}}{\Gamma(\alpha)}p(\tau);\ \tau\in(a,t),$ then integrating the resulting identity with respect to τ over (a,t), we get

$$\frac{1}{\Gamma(\alpha)}\int_a^t \ln\left(\frac{t}{\tau}\right)^{\alpha-1} p(\tau)\varphi(\tau,\rho)\frac{d\tau}{\tau}$$
$$=H_a^{\alpha}[pgh(t)]-g(\rho)H_a^{\alpha}[ph(t)]-h(\rho)H_a^{\alpha}[pg(t)]+g(\rho)h(\rho)H_a^{\alpha}[p(t)]. \tag{8}$$

Now, multiplying (8) by $\dfrac{\left(\ln\frac{t}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)}p(\rho);\ \rho\in(a,t),$ and integrating the resulting identity with respect to ρ from a to t, we have

$$\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\ln\frac{t}{\rho}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\alpha-1} p(\tau)p(\rho)\varphi(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$
$$=2H_a^\alpha[p(t)]H_a^\alpha[pgh(t)]-2H_a^\alpha[pg(t)]H_a^\alpha[ph(t)]. \tag{9}$$

In (9) taking $p(t)=f(t), g(t)=t-E(X)$ and $h(t)=t^{r-1}, t\in(a,b)$, we have

$$\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\ln\frac{t}{\rho}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\alpha-1}(\tau-\rho)(\tau^{r-1}-\rho^{r-1})f(\tau)f(\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$
$$=2H_a^\alpha[f(t)]H_a^\alpha[t^{r-1}(t-E(X))f(t)]-2H_a^\alpha[(t-E(X))f(t)]H_a^\alpha[t^{r-1}f(t)]$$
$$=2H_a^\alpha[f(t)]H_a^\alpha[t^{r-1}(t-E(X))f(t)]-2H_a^\alpha[(t-E(X))f(t)]_H M_{r-1,\alpha}(t), \tag{10}$$

we also have

$$\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\ln\frac{t}{\rho}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\alpha-1}(\tau^{r-1}-\rho^{r-1})f(\tau)f(\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$
$$\leq \|f\|_\infty^2\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\ln\frac{t}{\rho}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\alpha-1}(\tau-\rho)(\tau^{r-1}-\rho^{r-1})\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$
$$\leq 2\|f\|_\infty^2\left[\frac{\left(\ln\frac{t}{a}\right)^\alpha}{\Gamma(\alpha+1)}H_a^\alpha[t^r]-H_a^\alpha[t]H_a^\alpha[t^{r-1}]\right]. \tag{11}$$

Thanks to (10) and (11), we obtain (5).

For (2), we have

$$\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\ln\frac{t}{\rho}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\alpha-1}(\tau-\rho)(\tau^{r-1}-\rho^{r-1})f(\tau)f(\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$
$$\leq \sup_{\tau,\rho\in[a,t]}|(\tau-\rho)||(\tau^{r-1}-\rho^{r-1})|H_a^\alpha[f(t)]^2=(t-a)(t^{r-1}-a^{r-1})H_a^\alpha[f(t)]^2. \tag{12}$$

By (11) and (12), we get the desired inequality. □

Our next result in the following theorem, in which we use two real positive parameters:

Theorem 5 : *Let X be a continuous random variable having a p.d.f. $f:[a,b]\to\mathbb{R}^+$. Then we have.*

(i) For any $a<t\leq b, \alpha>0, \beta>0$

$$H_a^{\alpha}[f(t)]H_a^{\beta}[f(t)t^{r-1}(t-E(X))]+H_a^{\beta}[f(t)]H_a^{\alpha}[f(t)t^{r-1}(t-E(X))]$$
$$-H_a^{\alpha}[f(t)(t-E(X))]_H M_{r-1,\beta}(t)-H_a^{\beta}[f(t)(t-E(X))]_H M_{r-1,\alpha}(t)$$
$$\leq \|f\|_{\infty}^{2}\left[\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\beta}[t^r]+\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}H_a^{\alpha}[t^r]-H_a^{\alpha}[t]H_a^{\beta}[t^{r-1}]-H_a^{\beta}[t]H_a^{\alpha}[t^{r-1}]\right], \tag{13}$$

where $f\in L_{\infty}[a,b]$.

(ii) The inequality

$$H_a^{\alpha}[f(t)]H_a^{\beta}[f(t)t^{r-1}(t-E(X))]+H_a^{\beta}[f(t)]H_a^{\alpha}[f(t)t^{r-1}(t-E(X))]$$
$$-H_a^{\alpha}[f(t)(t-E(X))]_H M_{r-1,\beta}(t)-H_a^{\beta}[f(t)(t-E(X))]_H M_{r-1,\alpha}(t) \tag{14}$$
$$\leq (t-a)(t^{r-1}-a^{r-1})H_a^{\alpha}[f(t)]H_a^{\beta}[f(t)],$$

is valid for any $\alpha>0,\beta>0,a<t\leq b$.

Proof : Multiplying (8) by $\frac{\left(\ln\frac{t}{\rho}\right)^{\beta-1}}{\Gamma(\beta)}p(\rho);\ \rho\in(a,t),$ we have

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\beta-1}p(\tau)p(\rho)H(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$
$$=H_a^{\alpha}[p(t)]H_a^{\beta}[pgh(t)]+H_a^{\beta}[p(t)]H_a^{\alpha}[pgh(t)]$$
$$-H_a^{\alpha}[ph(t)]H_a^{\beta}[pg(t)]-H_a^{\beta}[ph(t)]H_a^{\alpha}[pg(t)]. \tag{15}$$

In (15), we take $p(t)=f(t),\ \ g(t)=t-E(X),\ \ h(t)=t^{r-1},t\in(a,b).$ So, we get

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\beta-1}(\tau-\rho)(\tau^{r-1}-\rho^{r-1})f(\tau)f(\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$
$$=H_a^{\alpha}[f(t)]H_a^{\beta}[t^{r-1}(t-E(X))f(t)]+H_a^{\beta}[f(t)]H_a^{\alpha}[t^{r-1}(t-E(X))f(t)]$$
$$-{}_H M_{r-1,\alpha}(t)H_a^{\beta}[f(t)(t-E(X))]-{}_H M_{r-1,\beta}(t)H_a^{\alpha}\left[f(t)(t-E(X))\right]. \tag{16}$$

We have also

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\left(\ln\frac{t}{\rho}\right)^{\beta-1}(\tau-\rho)(\tau^{r-1}-\rho^{r-1})f(\tau)f(\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}$$

$$\leq \| f \|_\infty^2 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} \left(\ln \frac{t}{\rho} \right)^{\beta-1} (\tau - \rho)(\tau^{r-1} - \rho^{r-1}) \frac{d\tau}{\tau} \frac{d\rho}{\rho}$$

$$= \| f \|_\infty^2 \left[\frac{\left(\ln \frac{t}{a} \right)^{\alpha}}{\Gamma(\alpha+1)} H_a^{\beta}[t^r] + \frac{\left(\ln \frac{t}{b} \right)^{\beta}}{\Gamma\left(\beta+1\right)} H_a^{\alpha}[t^r] - H_a^{\alpha}[t] H_a^{\beta}[t^{r-1}] - H_a^{\beta}[t] H^{\alpha}[t^{r-1}] \right]. \tag{17}$$

By (16) and (17), we obtain (13).

To prove (ii), we remark that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} \left(\ln \frac{t}{\rho} \right)^{\beta-1} (\tau - \rho)(\tau^{r-1} - \rho^{r-1}) f(\tau) f(\rho) \frac{d\tau}{\tau} \frac{d\rho}{\rho}$$

$$\leq \sup_{\tau,\rho \in [a,t]} [\,|\tau - \rho|\,|\tau^{r-1} - \rho^{r-1}|\,] H_a^{\alpha}[f(t)] H_a^{\beta}[f(t)]$$

$$= (t-a)(t^{r-1} - a^{r-1}) H_a^{\alpha}[f(t)] H_a^{\beta}[f(t)]. \tag{18}$$

Therefore, by (16) and (18), we get (14). □

Remark 6 : Applying Theorem 5 for $\alpha = \beta$, we obtain Theorem 4.

We also give the following result for Hadamard fractional integrals:

Theorem 7 : *Let f be the p.d.f. of the random variable X on $[a;b]$. Then for $\alpha > 0$, we have*

$$H_a^{\alpha}[f(t)]\,{}_H M_{2r,\alpha}(t) - {}_H M_{r,\alpha}^2(t) \leq \frac{1}{4} (H_a^{\alpha}[f(t)])^2 (b^r - a^r)^2, a < t \leq b. \tag{19}$$

Proof : Using Theorem 3.2 of [5], we can write

$$\left| H_a^{\alpha}[p(t)] H_a^{\alpha}[pg^2(t)] - (H_a^{\alpha}[pg(t)])^2 \right| \leq \frac{1}{4} (H_a^{\alpha}[p(t)])^2 (M-m)^2. \tag{20}$$

Now, replacing $p(t) = f(t)$ and $g(t) = t^r, t \in (a,b)$, we obtain $m = a^r$, $M = b^r$. Hence, the inequality (20) allows us to obtain

$$H_a^{\alpha}[f(t)]\,{}_H M_{2r,\alpha}(t) - {}_H M_{r,\alpha}^2(t) \leq \frac{1}{4} (H_a^{\alpha}[f(t)])^2 (b^r - a^r)^2. \tag{21}$$

So, the theorem is proved. □

Using two fractional parameters, we propose the following generalization of the above results.

Theorem 8 : *Let X be a continuous random variable having a* $p.d.f.$ $f : [a,b] \to \mathbb{R}^+$: *Then for all* $a < t \le b, \alpha > 0, \beta > 0,$ *we have*

$$\begin{aligned} &H_a^\alpha[f(t)]_H M_{2r,\beta}(t) + H_a^\beta[f(t)]_H M_{2r,\alpha}(t) + 2a^r b^r H_a^\alpha[f(t)] H_a^\beta[f(t)] \\ &\qquad \le (a^r + b^r)[H_a^\alpha[f(t)]_H M_{r,\beta}(t) + H_a^\beta[f(t)]_H M_{r,\alpha}(t)]. \end{aligned} \tag{22}$$

Proof : By Theorem 3.5 of [5], we can write

$$\begin{aligned} &[H_a^\alpha[p(t)]H_a^\beta[pg^2(t)] + H_a^\beta[p(t)]H_a^\alpha[pg^2(t)] - 2H_a^\alpha[pg(t)]H_a^\beta[pg(t)]]^2 \\ &\qquad \le \Big[(MH_a^\alpha[p(t)] - H_a^\alpha[pg(t)])(H_a^\beta[pg(t)] - mH_a^\beta[p(t)]) \\ &\qquad +(H_a^\alpha[pg(t)] - mH_a^\alpha[p(t)])(MH_a^\beta[p(t)] - H_a^\beta[pg(t)])\Big]^2. \end{aligned} \tag{23}$$

If we take $p(t) = f(t), g(t) = t^r, t \in [a,b]$ and substituting the values of m and M in (23), we obtain

$$\begin{aligned} &H_a^\alpha[f(t)]_H M_{2r,\beta}(t) + H_a^\beta[f(t)]_H M_{2r,\alpha}(t) \\ &\le (a^r + b^r)H_a^\alpha[f(t)]_H M_{r,\beta}(t) + (a^r + b^r)H_a^\beta[f(t)]_H M_{r,\alpha}(t) \\ &\qquad -2a^r b^r H_a^\alpha[f(t)]H_a^\beta[f(t)]. \end{aligned} \tag{24}$$

□

Theorem 9 : *Let X be a continuous random variable having a* $p.d.f.f : [a,b] \to \mathbb{R}^+$ *and* $m \le f \le M.$ *Then,for* $a < t \le b, \alpha > 0,$ *we have*

$$\left| \frac{\left(\ln \frac{t}{a}\right)^\alpha}{\Gamma(\alpha+1)}{}_H M_{r,\alpha}(t) - H_a^\alpha[f(t)]H_a^\alpha[t^r] \right|$$

$$\le \frac{(M-m)\left(\ln \frac{t}{a}\right)^\alpha}{2\Gamma(\alpha+1)} \left(\frac{\left(\ln \frac{t}{a}\right)^\alpha}{\Gamma(\alpha+1)} H_a^\alpha[t^{2r}] - (H_a^\alpha[t^r])^2 \right)^{\frac{1}{2}}. \tag{25}$$

Proof : To prove the above theorem, we use Theorem 7 of [13]. We find that

$$\left|\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\alpha}[fg(t)]-H_a^{\alpha}[f(t)]H_a^{\alpha}[g(t)]\right|$$

$$\leq\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{2\Gamma(\alpha+1)}(M-m)\left(\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\alpha}[g^2(t)]-(H_a^{\alpha}[g(t)])^2\right)^{\frac{1}{2}}. \tag{26}$$

In (26), taking $g(t)=t^r, a<t\leq b,$ we have

$$\left|\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\alpha}[t^r f(t)]-H_a^{\alpha}[f(t)]H_a^{\alpha}[t^r]\right|$$

$$\leq\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{2\Gamma(\alpha+1)}(M-m)\left(\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\alpha}[t^{2r}]-(H_a^{\alpha}[t^r])^2\right)^{\frac{1}{2}}. \tag{27}$$

□

Using two fractional parameters, we consider the following generalization of the above results:

Theorem 10 : *Let f be the* $p.d.f.$ *of X on* $[a,b]$ *and* $m\leq f\leq M$. *Then for all* $a<t\leq b,\ \alpha>0,\ \beta>0,$ *we have*

$$\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}{}_H M_{r,\alpha}(t)+\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}{}_H M_{r,\beta}(t)-H_a^{\alpha}[f(t)]H_a^{\beta}[t^r]-H_a^{\beta}[f(t)]H_a^{\alpha}[t^r]$$

$$\leq\left[\left(M\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}-H_a^{\alpha}[f(t)]\right)\left(H_a^{\beta}[f(t)]-m\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}\right)\right.$$

$$\left.+\left(H_a^{\alpha}[f(t)]-m\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}-H_a^{\beta}[f(t)]\right)\right]$$

$$\times\left[\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\beta}[t^{2r}]+\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}H_a^{\alpha}[t^{2r}]-2H_a^{\alpha}[t^r]H_a^{\beta}[t^r]\right]^{\frac{1}{2}}. \qquad (28)$$

Proof : Thanks to Theorem 8 of [13], we can state that

$$\left|\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\beta}[t^r f(t)]+\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}H_a^{\alpha}[t^r f(t)]-H_a^{\alpha}[f(t)]H_a^{\beta}[t^r]-H_a^{\beta}[f(t)]H_a^{\alpha}[t^r]\right|$$

$$\leq\left[\left(M\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}-H_a^{\alpha}[f(t)]\right)\left(H_a^{\beta}[f(t)]-m\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}\right)\right.$$

$$\left.+\left(H_a^{\alpha}[f(t)]-m\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}-H_a^{\beta}[f(t)]\right)\right]$$

$$\times\left(\frac{\left(\ln\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}H_a^{\beta}[t^{2r}]+\frac{\left(\ln\frac{t}{a}\right)^{\beta}}{\Gamma(\beta+1)}H_a^{\alpha}[t^{2r}]-2H_a^{\alpha}[t^r]H_a^{\beta}[t^r]\right)^{\frac{1}{2}}.$$

Remark 11 : If we take $\alpha=\beta$ in Theorem 10, we obtain Theorem 9.

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