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Solving Robin condition using a method based on integral equation

Mohsen Aghaeiboorkheili
Department of Mathematics and Computer Science
University of Technology
Lae 411
Papua New Guinea

Abstract

In this paper, Robin boundary value problem is solved numerically with an integral equations' based method. The suggested method has been compared with other existing methods or exact solution to approve the accuracy of this method.

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1. Introduction

Need to solve Robin problem happens in so many area of science and specially in engineering. In computational physics and mechanics these type of needs mostly appear as elliptic problems that require to be solved separately.

There are four types of BVPs, Dirichlet, Neumann, special mixed, mixed and Robin. The region of discussion for Dirichlet, Neumann, mixed and Robin condition are simply connected region while the region discussed for special mixed is multiply connected region. Robin condition is linear combination of Dirichlet and Neuman condition on the simply connected region [1].

Since as it has been mentioned, these conditions arise in many areas of engineering in practice so mathematicians and engineers are always looking for a method to find the solution for these conditions whether numerically or analytically [2].

E-mail: **mohsen.aghaeiboorkheili@pnguot.ac.pg**

In the real world when such problem arise, most of the time it can not be solved by analytical methods. And that is why scientist are always looking for numerical methods. Different numerical approaches have been introduced including BIE, FEM and FDM so far. Among these methods, BIE(boundary integral equation) is one of the most effective and popular methods [3].

Boundary integral equation method is a classical method for solving Robin condition [4]. Due to extensive appearances of Robin condition into real world problems obviously so many methods, both numerical and analytical, have been explored to solve this condition [5].

Robin condition appear in many are of engineering such as mechanical engineering. Suppose that the temperature of a rod(idealized 2-D) at time $t=0$ is given, what will be the temperature of mentioned rod after t seconds? Robin boundary value problem can be applied here to find temperature after t seconds [6]. Another application, let a region without any charge is given. To find electric potential for this region, this type of boundary value problem can be applied. There are some other application of Robin boundary value problem in fluid mechanic, air flow around an object and etc [7].

2. GNK

If $B(y)=\delta(y)-\alpha,$ then generalized Neumann kernel can be presented as [8, 9]

$$R(x,y):=\frac{1}{\pi}\mathrm{Im}\left(\frac{B(x)}{B(y)}\frac{\dot{\delta}(y)}{\delta(y)-\delta(x)}\right). \tag{2.1}$$

If R, that is kernel of the mentioned integral equation, is defined with the following form then it is continuous also,

$$R(y,y):=\frac{1}{\pi}\mathrm{Im}\left(\frac{1}{2}\frac{\ddot{\delta}(y)}{\dot{\delta}(y)}-\frac{\dot{B}(y)}{B(y)}\right). \tag{2.2}$$

So T, that is a kernel with real values, is [8, 9]

$$T(x,y):=\frac{1}{\pi}\mathrm{Re}\left(\frac{B(x)}{B(y)}\frac{\dot{\delta}(y)}{\delta(y)-\delta(x)}\right). \tag{2.3}$$

T can be represented as

$$T(x,y):=-\frac{1}{2\pi}\cos\frac{x-y}{2}+T_1(x,y), \tag{2.4}$$

in a way that continuous kernel T_1 on the diagonal is equal to

$$T_1(y,y) := \frac{1}{\pi}\mathrm{Re}\left(\frac{1}{2}\frac{\ddot{\delta}(y)}{\dot{\delta}(y)} - \frac{\dot{B}(y)}{B(y)}\right). \tag{2.5}$$

That means [8]

$$\mathbf{R}q(x) := \int_J R(x,y)q(y)dy, \quad x \in J, \tag{2.6}$$

can represented as Fredholm'type. Also

$$\mathrm{T}q(x) := \int_J T(x,y)q(y)dy, \quad x \in J, \tag{2.7}$$

can represented as singular' type.

Theorem 2.1: [8, 9] *Consider the Dirichlet problem* $\mathrm{Re}[F(\delta(y))] = p(y)$ *such that function F takes the values* $F = p + iq$ *on* Γ. *Moreover, suppose that* $\mathrm{Im}\, F(0) = 0$. *All of these yields*

$$q - \mathbf{R}q = -\mathrm{T}p, \tag{2.8}$$

or

$$q(x) - \int_0^{2\pi} R(x,y)q(y)dy = -\int_0^{2\pi} T(x,y)p(y)dy. \tag{2.9}$$

Theorem 2.2: [10, 11] *Let* $p \in H$. *Also q is an answer for the equation (2.8). Moreover let the function* Φ *to be defined as*

$$\Phi = \frac{1}{2\pi i}\int_\Gamma \frac{p+iq}{\delta - z} d\delta, \qquad z \notin \Gamma. \tag{2.10}$$

So by all of these, the function Φ will be equal to f and it is hold on the following equation

$$f = p + iq. \tag{2.11}$$

3. Robin condition

Let α and β to be either real number or continuous functions and also $l(x)$ is a continuous function on Γ. Then solving Robin condition will give a function like ω (harmonic) in the G^+ which

$$\alpha(x)\omega(x) + \beta(x)\frac{\partial \omega}{\partial \mathbf{n}}(x) = l(x), \qquad \alpha(x) \text{ and } \beta(x) \neq 0.$$

Obviously when $\alpha = 0$ or $\beta = 0$ we get respectively Neumann and Dirichlet problems.

4. Numerical Implementation

Suppose that $x_i = h(i-1)$ where $h = \frac{2\pi}{n}$. Using these notations and central difference approximation for $q'(y) = |\dot{\delta}(y)| \frac{\partial \omega}{\partial \mathbf{n}}(y)$ leads to the following results [12]:

$$\begin{cases} q'(x_i) = \left|\dot{\delta}(x_i)\right| \frac{\partial \omega}{\partial \mathbf{n}}(x_i), \\ q'(x_i) \approx \frac{q(x_{i+1}) - q(x_{i-1})}{2h}, \quad i = 1, 2, \cdots, n. \end{cases} \tag{4.1}$$

It is mention-able that since all functions assume to be periodic so it can be derive $q(x_0) = q(x_n)$ and $q(x_{n+1}) = q(x_1)$. So we have

$$\frac{q(x_{i+1}) - q(x_{i-1})}{2h} = \left|\dot{\delta}(x_i)\right| \frac{\partial \omega}{\partial \mathbf{n}}(x_i),$$

that means

$$\frac{\partial \omega}{\partial \mathbf{n}}(x_i) = \frac{1}{\left|\dot{\delta}(x_i)\right|} \frac{q(x_{i+1}) - q(x_{i-1})}{2h}. \tag{4.2}$$

Substitution equation (4.2) into Robin condition gives

$$\alpha(x_i) p(x_i) + \beta(x_i) \frac{1}{\left|\dot{\delta}(x_i)\right|} \frac{q(x_{i+1}) - q(x_{i-1})}{2h} = l(x_i), \qquad i = 1, 2, \cdots, n. \tag{4.3}$$

In the equation (4.3), there are n equations with $2n$ unknowns in the form

$$\begin{cases} \alpha(x_1)p(x_1) & + & \beta(x_1)\frac{1}{\left|\dot{\delta}(x_1)\right|}\frac{q(x_{1+1}) - q(x_{1-1})}{2h} & = & l(x_1), \\ \alpha(x_2)p(x_2) & + & \beta(x_2)\frac{1}{\left|\dot{\delta}(x_2)\right|}\frac{q(x_{2+1}) - q(x_{2-1})}{2h} & = & l(s_2), \\ \alpha(x_3)p(x_3) & + & \beta(x_3)\frac{1}{\left|\dot{\delta}(x_3)\right|}\frac{q(x_{3+1}) - q(x_{3-1})}{2h} & = & l(x_3), \\ \vdots & & \vdots & & \\ \alpha(x_n)p(x_n) & + & \beta(x_n)\frac{1}{\left|\dot{\delta}(x_n)\right|}\frac{q(x_{n+1}) - q(x_{n-1})}{2h} & = & l(x_n), \end{cases} \tag{4.4}$$

Since $\omega(z)$, the solution of Robin problem, can taken as $\text{Re}[F(z)]$ where $F(z)=\omega(z)+iv(z)$, so applying the integral equation (2.9) will give us another n equations with $2n$ unknowns in the following form

$$\begin{cases} q(x_1) & - & \int_0^{2\pi} R(x_1,y)q(y)dy & = & -\int_0^{2\pi} T(x_1,y)p(y)dy, \\ q(x_2) & - & \int_0^{2\pi} R(x_2,y)q(y)dy & = & -\int_0^{2\pi} T(x_2,y)p(y)dy, \\ q(x_3) & - & \int_0^{2\pi} R(x_3,y)q(y)dy & = & -\int_0^{2\pi} T(x_3,y)p(y)dy, \\ \vdots & & \vdots & & \\ q(x_n) & - & \int_0^{2\pi} R(x_n,y)q(y)dy & = & -\int_0^{2\pi} T(x_n,y)p(y)dy, \end{cases} \tag{4.5}$$

Combining equation (4.5) and (4.4) is resulted a system of equation ($2n$ unknown). Solution of this system will produce q and p. When p and q are founded,

$$F(z)=\frac{1}{2\pi i}\int_\Gamma \frac{p+iq}{\delta-z}d\delta, \qquad z\notin\Gamma, \tag{4.6}$$

and finally $\omega(z)=\text{Re}[F(z)]$.

5. Numerical Examples

Wolfarm Mathematica 10.4 has been used to solve obtained $2n\times 2n$ system of equations. $\omega(z)$ is the given answer to Robin condition, $\omega_n(z)$ is obtained answer by suggested method for for the Robin condition.

Example 5.1: Solve the following Robin problem on the given region,

$$\Gamma:\delta(y)=\cos(y)+i\sin(y)$$

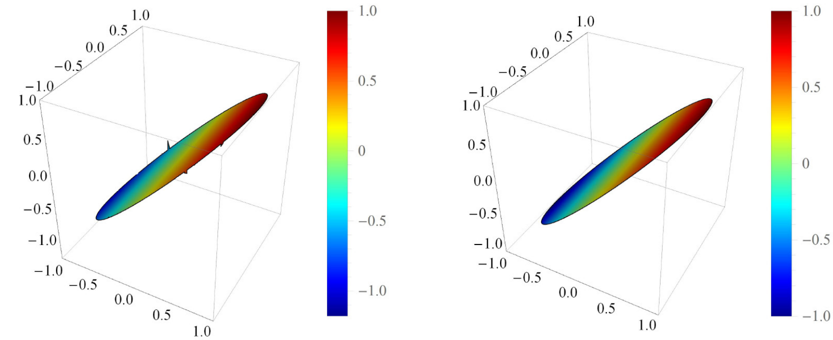
$$\cos(y)\omega(y)+2\sin(y)\frac{\partial\omega}{\partial\mathbf{n}}(y)=\frac{1}{2}(\cos(2y)+2\sin(2y)+1), \quad 0\le y<2\pi.$$

Exact solution for Example 5.1 is equal to

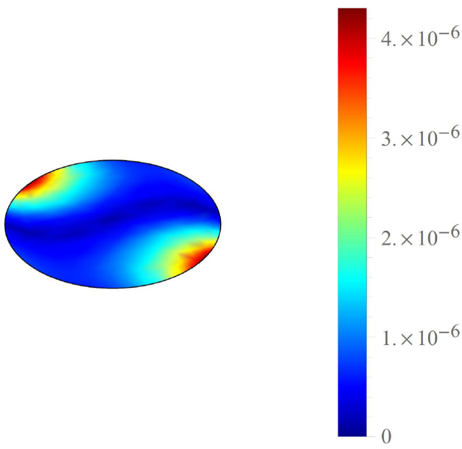
$$\omega(x,y)=\text{Re}[F(x,y)]=\text{Re}[x+iy]=x.$$

Starting with $n=1024$ boundary points for this example the the following results will gain:

$$\begin{cases} |\dot{\delta}(y)| = 1, \\ h = \dfrac{2\pi}{1024} = 6.1359 \times 10^{-3}, \\ \alpha(y) = \cos(y), \\ \beta(y) = 2\sin(y), \\ l(y) = \dfrac{1}{2}(\cos(2y) + 2\sin(2y) + 1), \end{cases}$$



(a) (b)



(c)

Figure 5.1

(a) represents a Plot3D related to $\omega_n(z)$ while (b) represents a Plot3D related to $\omega(z)$. Moreover (c) represents $|\omega_n(z) - \omega(z)|$.

and the system of equations (4.4) and (4.5) take the form

$$\begin{cases} \cos(x_i)p(x_i) + \dfrac{\sin(x_i)[q(x_{i+1}) - q(x_{i-1})]}{6.1359 \times 10^{-3}} = \dfrac{(\cos(2y) + 2\sin(2y) + 1)}{2}, \\ q(x_i) - \int_0^{2\pi} R(x_i, y)q(y)dy = -\int_0^{2\pi} T(x_i, y)p(y)dy. \end{cases} \tag{5.1}$$

Solving system of equations (5.1) for p and q, then using the equation (4.6) yields

Example 5.2: The purpose of this example is to show that the suggested method works perfectly when the exact solution is not given also. Solve the following Robin problem on the given region,

$$\Gamma : \delta(y) = 2\cos(y) + i\sin(y),$$

$$2\omega(y) + 5\frac{\partial \omega}{\partial \mathbf{n}}(y) = 4\cos(2y), \qquad 0 \le y < 2\pi.$$

Starting with $n = 1024$ boundary points for this example the the following results will gain:

$$\begin{cases} |\dot{\delta}(y)| = \sqrt{4\sin^2(y) + \cos^2(y)}, \\ h = \dfrac{2\pi}{1024} = 6.1359 \times 10^{-3}, \\ \alpha(y) = 2, \\ \beta(y) = 5, \\ l(y) = 4\cos(2y), \end{cases}$$

and the system of equations (4.4) and (4.5) take the forms

$$\begin{cases} 2p(x_i) + \dfrac{5[q(x_{i+1}) - q(x_{i-1})]}{2 \times 6.1359 \times 10^{-3} \times \sqrt{4\sin^2(y) + \cos^2(y)}} = 4\cos(2y), \\ q(x_i) - \int_0^{2\pi} R(x_i, y)q(y)dy = -\int_0^{2\pi} T(x_i, y)p(y)dy. \end{cases} \tag{5.2}$$

Solving system of equations (5.2) for p and q, then using the equation (4.6) gives

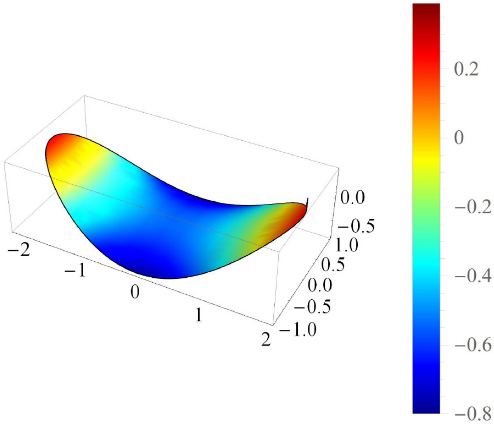


Figure 5.2

Plot3D for the function ω(*z*) on the given region

6. Conclusions

Suggested method for solving Robin problem has proved that it has good accuracy by the given examples. By noticing on the shown graph, it is obvious that how good accurate is the presented method. The procedure of solving Robin condition on bounded(or unbounded) multiply connected region is almost same and can be done in future papers.

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