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Solvability and stability for fractional differential equations involving two Riemann-Liouville fractional orders

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Abstract

In this work we study existence, uniqueness and Ulam-type stability for nonlinear fractional differential equations involving two Riemann-Liouville fractional derivatives. The existence and uniqueness of solutions is established by Banach's fixed point theorem, whereas the existence of solutions is derived from Schaefer's fixed point theorem. Furthermore, we present and discuss different types of Ulam stability. At the end, some illustrative examples are presented.

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1. Introduction

Recently, many studies on fractional differential equations, involving different operators such as Riemann-Liouville operators [13, 23], Caputo operators [2, 6, 11, 22] and Hadamard operators [3, 19], have appeared during the past several years. Moreover, by using many classical fixed-point theorems, several authors have obtained results of the existence of solutions or positive solutions for various classes of fractional differential equations, see for example [4, 5, 9, 10, 16, 17, 20] and the references cited therein. The study of Ulam type stability problems has grown to be one of the most important subjects in the mathematical analysis area. Since then,

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a large number of papers have been published in connection with various generalizations of Ulam's type stability theory or the Ulam-Hyers stability theory. For the advanced contribution on Ulam's type stability, we refer to [1, 3, 8, 12, 14, 16, 18, 21].

Motivated by the above works, in this paper, we discuss the existence, uniqueness and the Ulam stabilities, generalized Ulam-Hyers stabilities and Ulam-Hyers-Rassias stable for boundary value problem of nonlinear fractional differential equations with two Riemann-Liouville fractional derivatives of the form

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)x(t)=f(t,x(t),D^{\delta}x(t)),t\in[0,T], \\ J^{1-\alpha}x(0)=0,D^{\delta}x(T)=\sum_{i=1}^{k}\mu_i D^{\delta}x(\eta_i)+\theta,0<\eta_i<T, \end{cases} \tag{1}$$

where D^{ϑ} is the Riemann-Liouville fractional derivative of order $\vartheta,\vartheta\in\{\alpha,\beta,\delta\}$, with $0<\alpha,\ \beta\le 1,\ 1<\alpha+\beta\le 2,\ \delta<\alpha,\quad \lambda,\mu_i\quad (1\le i\le k)$ are real constants, $J^{1-\alpha}$ is the Riemann-Liouville fractional integral of order $1-\alpha$ and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous functions on $[0,T]$. The operator J^{ϑ} is the Riemann-Liouville fractional integral, defined by

$$f(t)=\frac{1}{\Gamma(\vartheta)}\int_0^t(t-\tau)^{\vartheta-1}f(\tau)d\tau,\vartheta>0,$$

where $\Gamma(\vartheta):=\int_0^{\infty}e^{-u}u^{\vartheta-1}du.$ The operator D^{ϑ} is the fractional derivative in the sense of Riemann-Liouville, defined by

$$D^{\vartheta}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-\tau)^{n-\vartheta-1}f(\tau)d\tau,\vartheta>0,n=[\vartheta]+1.$$

For $\vartheta<0,$ we use the convention that $D^{\vartheta}h=J^{-\vartheta}h.$ Also for $0\le\rho<\vartheta,$ it is valid that $D^{\rho}J^{\vartheta}h=h^{\vartheta-\rho}.$

We note that for $\varepsilon>-1$ and $\varepsilon\ne\vartheta-1,\vartheta-2,...,\vartheta-n,$ we have

$$\begin{aligned} D^{\vartheta}t^{\varepsilon}&=\frac{\Gamma(\varepsilon+1)}{\Gamma(\varepsilon-\vartheta+1)}t^{\varepsilon-\vartheta}, \\ D^{\vartheta}t^{\vartheta-i}&=0,i=1,2,...,n. \end{aligned}$$

Lemma 1 : [15] *Let* $\vartheta>0$ *and* $x\in C(0,T)\cap L^1(0,T)$. *Then the fractional differential equation* $D^{\vartheta}x(t)=0$ *has a unique solution*

$$x(t)=\sum_{i=1}^{n}c_i t^{\vartheta-i},$$

where $c_i \in \mathbb{R},\ i = 1, 2, .., n,\ n = [\vartheta] + 1.$

Lemma 2 : [15] *Let* $\vartheta > 0.$ *Then the following equality holds for* $x \in C(0,T) \cap L^1(0,T),\ D^\vartheta x \in C(0,T) \cap L^1(0,T);$

$$J^\vartheta D^\vartheta x(t) = x(t) + \sum_{i=1}^{n} c_i t^{\vartheta - i},$$

for some $c_i \in \mathbb{R},\ i = 1, 2, ..., n,\ n - 1 < \vartheta \le n.$

Lemma 3 : *For a given* $h \in C([0,T], \mathbb{R}),$ *the fractional boundary value problem*

$$\begin{cases} D^\beta (D^\alpha + \lambda) x(t) = h(t), t \in [0,T], 0 < \alpha, \beta \le 1 \\ J^{1-\alpha} x(0) = 0, D^\delta x(T) = \sum_{i=1}^{k} \mu_i D^\delta x(\eta_i) + \theta, \end{cases} \tag{2}$$

has a unique solution

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s) ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds \\ &+ \frac{\Gamma(\beta+\alpha-\delta) t^{\alpha+\beta-1}}{\Gamma(\beta)\Lambda} \left[\sum_{i=1}^{k} \mu_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} h(s) ds \right. \\ &- \lambda \sum_{i=1}^{k} \mu_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} x(s) ds - \int_0^T \frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} h(s) ds \\ &\left. + \lambda \int_0^T \frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} x(s) ds \right] + \frac{\Gamma(\beta+\alpha-\delta)\theta}{\Gamma(\beta)\Lambda} t^{\alpha+\beta-1}, \end{aligned} \tag{3}$$

where

$$\Lambda = \sum_{i=1}^{k} \mu_i \eta_i^{\alpha+\beta-\delta-1} - T^{\alpha+\beta-\delta-1} \neq 0. \tag{4}$$

Proof : By Lemma 1 and Lemma 2, the solution of (2) can be written as

$$x(t) = J^{\alpha+\beta} h(t) - \lambda J^\alpha x(t) + c_1 \tfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\alpha+\beta-1} + c_2 t^{\alpha-1}, \tag{5}$$

where c_0 and c_1 are arbitrary constants. The boundary condition $J^{1-\alpha} x(0) = 0$ implies that $c_2 = 0.$ Using the relation $D^\vartheta t^\varepsilon = \frac{\Gamma(\varepsilon+1)}{\Gamma(\varepsilon-\vartheta+1)} t^{\varepsilon-\vartheta},$ (5) reduces to

$$D^\delta x(t) = J^{\alpha+\beta-\delta} h(t) - \lambda J^{\alpha-\delta} x(t) + c_1 \tfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha-\delta)} t^{\alpha+\beta-\delta-1},$$

Using the boundary conditions $D^{\delta}x(T)=\sum_{i=1}^{k}\mu_i D^{\delta}x(\eta_i)+\theta$ we obtain

$$c_1=\frac{\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)\Lambda}\left[\sum_{i=1}^{k}\mu_i J^{\alpha+\beta-\delta}h(\eta_i)-\lambda\sum_{i=1}^{k}\mu_i J^{\alpha-\delta}x(\eta_i)\right.$$
$$\left.-J^{\alpha+\beta-\delta}h(T)+\lambda J^{\alpha-\delta}x(T)+\theta\right],$$

where Λ givens by (4). Substituting the value of c_0 and c_1 in (5), we obtain the solution (3). □

2. Existence and uniqueness of solutions

We denote by $C([0,T],\mathbb{R})$ the space of continuous functions defined on $[0,T]$. The space $X=\{x:x\in C([0,T],\mathbb{R}),D^{\delta}x\in C([0,T],\mathbb{R})\}$ equipped with the norm $\|x\|=\sup_{t\in[0,T]}|x(t)|+\sup_{t\in[0,T]}|D^{\delta}x(t)|$ is a Banach space.

In view of Lemma 3, we define an operator $O:X\to X$ as:

$$\begin{aligned}
&Ox(t)\\
=\;& \int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s),D^{\delta}x(s))ds-\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds\\
&+\frac{\Gamma(\beta+\alpha-\delta)t^{\alpha+\beta-1}}{\Gamma(\beta)\Lambda}\left[\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}f(s,x(s),D^{\delta}x(s))ds\right.\\
&-\lambda\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}x(s)ds-\int_0^T\frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}f(s,x(s),D^{\delta}x(s))ds\\
&\left.+\lambda\int_0^T\frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}x(s)ds\right]+\frac{\Gamma(\beta+\alpha-\delta)\theta}{\Gamma(\beta)\Lambda}t^{\alpha+\beta-1}. \qquad (6)
\end{aligned}$$

Observe that the existence of a fixed point for the operator O implies the existence of a solution for the fractional boundary value problem (1).

For computational convenience, we set

$$\begin{aligned}
\Delta_1\;:\;&=\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\Gamma(\beta+\alpha-\delta)T^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|}\left(\sum_{i=1}^{k}\frac{|\mu_i|\eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}\right.\\
&\left.+\frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}\right),\\
\Delta_2\;:\;&=\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\Gamma(\beta+\alpha-\delta)T^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|}\left(\sum_{i=1}^{k}\frac{|\mu_i|\eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}+\frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right),
\end{aligned}$$

$$\Pi_1 \;:\; = \frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)} + \frac{\Gamma(\beta+\alpha)T^{\alpha+\beta-\delta-1}}{\Gamma(\beta)\left|\Lambda\right|}\left(\sum_{i=1}^{k}\frac{|\mu_i|\eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)} + \frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}\right),$$

$$\Pi_2 \;:\; = \frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} + \frac{\Gamma(\beta+\alpha)T^{\alpha+\beta-\delta-1}}{\Gamma(\beta)|\Lambda|}\left(\sum_{i=1}^{k}\frac{|\mu_i|\eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} + \frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right),$$

$$\Omega \;:\; = \left[\frac{\Gamma(\beta+\alpha-\delta)}{1-|\lambda|\Delta_2}T^{\alpha+\beta-1} + \frac{\Gamma(\beta+\alpha)}{1-|\lambda|\Pi_2}T^{\alpha+\beta-\delta-1}\right]. \tag{7}$$

Now, we present the existence and uniqueness of solutions of fractional boundary value problem (1) by using Banach's fixed point theorem.

Theorem 4 : *Let* $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Assume that*

(A_1) : There exists constant $0<\omega<\frac{1-|\lambda|(\Delta_2+\Pi_2)}{\Delta_1+\Pi_1}$ *shch that* $|f(t,x,y)-f(t,\overline{x},\overline{y})|\le\omega(|x-\overline{x}|+|y-\overline{y}|)$, *for* $t\in[0,T]$ *and* $(x,y),(\overline{x},\overline{y})\in\mathbb{R}^2$.

If

$$(\Delta_1+\Pi_1)\omega < 1-|\lambda|(\Delta_2+\Pi_2), \tag{8}$$

where $\Delta_j,\Pi_j (j=1,2)$ *given by (7). Then the fractional boundary value problem (1) has a unique solution.*

Proof : For $x,\overline{x}\in B_r, t\in[0,T]$, and by assumption (A_1), we have following estimates

$$\begin{aligned}
&|Ox(t)-O\overline{x}(t)| \\
\le\; & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|f(s,x(s),D^\delta x(s))-f(s,\overline{x}(s),D^\delta\overline{x}(s))\right|ds \\
& +|\lambda|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-\overline{x}(s)|ds \\
& +\frac{\Gamma(\beta+\alpha-\delta)t^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|}\left[\sum_{i=1}^{k}|\mu_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}\right. \\
& \times|f(s,x(s),D^\delta x(s))-f(s,\overline{x}(s),D^\delta\overline{x}(s))|ds \\
& +|\lambda|\sum_{i=1}^{k}|\mu_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}|x(s)-\overline{x}(s)|ds
\end{aligned}$$

$$+\int_0^T \frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} |f(s,x(s),D^\delta x(s)) - f(s,\overline{x}(s),D^\delta \overline{x}(s))|\, ds$$

$$+|\lambda| \int_0^T \frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)-\overline{x}(s)|\, ds\Bigg]. \tag{9}$$

$$\leq \Bigg[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\beta+\alpha-\delta)T^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|}\Bigg(\sum_{i=1}^{m-2} \frac{|\mu_i|\eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}$$

$$+\frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}\Bigg)\Bigg]\omega \|x-\overline{x}\|$$

$$+|\lambda| \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(\beta+\alpha-\delta)T^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|}\Bigg(\sum_{i=1}^{k} \frac{|\mu_i|\eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}$$

$$+\frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\Bigg)\Bigg] \|x-\overline{x}\|.$$

This means that,

$$\|Ox - O\overline{x}\| \leq (\Delta_1\omega + |\lambda|\Delta_2)\|x-\overline{x}\|. \tag{10}$$

On the other hand we have

$$|D^\delta Ox(t) - D^\delta O\overline{x}(t)|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} |f(s,x(s),D^\delta x(s)) - f(s,\overline{x}(s),D^\delta \overline{x}(s))|\, ds$$

$$+|\lambda| \int_0^t \frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)-\overline{x}(s)|\, ds$$

$$+\frac{\Gamma(\beta+\alpha)t^{\alpha+\beta-\delta-1}}{\Gamma(\beta)\Lambda}\Bigg[\sum_{i=1}^{m-2}|\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}$$

$$\times |f(s,x(s),D^\delta x(s)) - f(s,\overline{x}(s),D^\delta \overline{x}(s))|\, ds$$

$$+|\lambda| \sum_{i=1}^{m-2}|\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)-\overline{x}(s)|\, ds$$

$$+\int_0^T \frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} |f(s,x(s),D^\delta x(s)) - f(s,\overline{x}(s),D^\delta \overline{x}(s))|\, ds$$

$$+|\lambda| \int_0^T \frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)-\overline{x}(s)|\, ds\Bigg], \tag{11}$$

$$\leq \left[\frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}+\frac{\Gamma(\beta+\alpha)T^{\alpha+\beta-\delta-1}}{\Gamma(\beta)|\Lambda|}\right.$$
$$\left.\times\left(\sum_{i=1}^{m-2}\frac{|\mu_i|\eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}+\frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}\right)\right]\omega\|x-\overline{x}\|$$
$$+|\lambda|\left[\frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}+\frac{\Gamma(\beta+\alpha)T^{\alpha+\beta-\delta-1}}{\Gamma(\beta)|\Lambda|}\right.$$
$$\left.\times\left(\sum_{i=1}^{m-2}\frac{|\mu_i|\eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}+\frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right)\right]\|x-\overline{x}\|.$$

Consequently (11) becomes

$$\|D^\delta Ox-D^\delta Oy\|\leq(\Pi_1\omega+|\lambda|\Pi_2)\|x-\overline{x}\|. \tag{12}$$

Summing (8) and (12) we deduce

$$\|Ox-O\overline{x}\|\leq[(\Delta_1+\Pi_1)\omega+|\lambda|(\Delta_2+\Pi_2)]\|x-\overline{x}\|. \tag{13}$$

By (8), we conclude that O is contractive. As a consequence of Banach's fixed point theorem, we deduce that O has a unique fixed point which is a solution of the fractional boundary value problem (1). □

Now, we prove the existence of solutions of fractional boundary value problem (1) by applying Schaefer's fixed point theorem [7].

Theorem 5 : *Let* $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Suppose that*

(A_2) : There exists constant $M>0$ *shch that* $|f(t,x,\overline{x})|\leq M$, *for* $t\in[0,T]$ *and* $(x,\overline{x})\in\mathbb{R}^2$.

(A_3) : $|\lambda|\Delta_2<1, |\lambda|\Pi_2<1$.

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof : We will use Schaefer's fixed point theorem to prove this result. We divide the proof into four steps.

Step 1 : The continuity of O on X is trivial and hence it is omitted.

Step 2 : Next we prove that O maps bounded sets into bounded sets in X. For a number $r>0$, let $B_r=\{x\in X:\|x\|\leq r\}$ be a bounded set in X. Then, for $t\in[0,T]$ and (A_2), we have

$$
\begin{aligned}
&\| Ox \| \\
&\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s),D^{\delta}x(s))| ds + |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s)| ds \\
&+ \frac{\Gamma(\beta+\alpha-\delta)t^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|} \left[\sum_{i=1}^{m-2} |\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} \left| f(s,x(s),D^{\delta}x(s)) \right| ds \right. \\
&+ |\lambda| \sum_{i=1}^{m-2} |\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)| ds + \int_0^T \frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} |f(s,x(s),D^{\delta}x(s))| ds \\
&\left. + |\lambda| \int_0^T \frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)| ds \right] + \frac{|\theta|\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)|\Lambda|} t^{\alpha+\beta-1}, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\beta+\alpha-\delta)T^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|} \left(\sum_{i=1}^{m-2} \frac{|\mu_i| \eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)} \right. \right. \\
&\quad \left. \left. + \frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)} \right) \right] M \\
&\quad + |\lambda| \left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(\beta+\alpha-\delta)T^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|} \left(\sum_{i=1}^{m-2} \frac{|\mu_i| \eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right. \right. \\
&\quad \left. \left. + \frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right) \right] r + \frac{|\theta|\Gamma(\beta+\alpha-\delta)T^{\alpha+\beta-1}}{\Gamma(\beta)|\Lambda|}.
\end{aligned}
$$

Hence,

$$
\| Ox \| \leq \Delta_1 M + |\lambda| \Delta_2 r + \frac{|\theta|\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)|\Lambda|} T^{\alpha+\beta-1}, \qquad (15)
$$

Moreover, we have

$$
\begin{aligned}
&\| D^{\delta} Ox \| \\
&\leq \int_0^t \frac{(t-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} |f(s,x(s),D^{\delta}x(s))| ds + |\lambda| \int_0^t \frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)| ds \\
&+ \frac{\Gamma(\beta+\alpha)t^{\alpha+\beta-\delta-1}}{\Gamma(\beta)|\Lambda|} \left[\sum_{i=1}^{m-2} |\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} |f(s,x(s),D^{\delta}x(s))| ds \right. \\
&+ |\lambda| \sum_{i=1}^{m-2} |\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)| ds + \int_0^T \frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} |f(s,x(s),D^{\delta}x(s))| ds \\
&\left. + |\lambda| \int_0^T \frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |x(s)| ds \right] + \frac{|\theta|\Gamma(\beta+\alpha)}{\Gamma(\beta)|\Lambda|} t^{\alpha+\beta-\delta-1}, \qquad (16)
\end{aligned}
$$

Using (A_2), we can write

$$
\begin{aligned}
&\| D^{\delta} Ox \| \\
\leq\ & \left[\frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)} + \frac{\Gamma(\beta+\alpha) T^{\alpha+\beta-\delta-1}}{\Gamma(\beta)|\Lambda|} \left(\sum_{i=1}^{m-2} \frac{|\mu_i| \eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)} \right.\right. \\
& \left.\left. + \frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)} \right) \right] M \\
& + |\lambda| \left[\frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} + \frac{\Gamma(\beta+\alpha) T^{\alpha+\beta-\delta-1}}{\Gamma(\beta)|\Lambda|} \left(\sum_{i=1}^{m-2} \frac{|\mu_i| \eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right.\right. \\
& \left.\left. + \frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right) \right] r + \frac{|\theta| \Gamma(\beta+\alpha)}{\Gamma(\beta)|\Lambda|} T^{\alpha+\beta-\delta-1}. \qquad (17)
\end{aligned}
$$

This implies that

$$\| D^{\delta} Ox \| \leq \Pi_1 M + |\lambda| \Pi_2 r + \frac{|\theta| \Gamma(\beta+\alpha)}{\Gamma(\beta)|\Lambda|} T^{\alpha+\beta-\delta-1}. \qquad (18)$$

From (15) and (18), we get

$$\| Ox \| \leq (\Delta_1 + \Pi_1) M + |\lambda| (\Delta_2 + \Pi_2) r + \frac{|\theta|}{\Gamma(\beta)|\Lambda|} \Omega. \qquad (19)$$

Consequently

$$\| Ox \| < \infty.$$

Thus, O maps bounded sets into bounded sets in X.

Step 3 : Now, we show that O maps bounded sets into equicontinuous sets of X.

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $x \in B_r$. Then, thanks to (A_2), we can write

$$
\begin{aligned}
&|Ox(t_2) - Ox(t_1)| \\
\leq\ & \frac{M}{\Gamma(\alpha+\beta)} \int_0^{t_1} [(t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1}] ds \\
& + \frac{M}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} ds \\
& + \frac{|\lambda| r}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds + \frac{|\lambda| r}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds
\end{aligned}
$$

$$+\frac{\Gamma(\beta+\alpha-\delta)(t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1})}{\Gamma(\beta)|\Lambda|}\Bigg[M\sum_{i=1}^{m-2}|\mu_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}ds$$
$$+|\lambda|r\sum_{i=1}^{m-2}|\mu_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}ds+M\int_0^T\frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}ds$$
$$+|\lambda|r\int_0^T\frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}ds\Bigg]+\frac{|\theta|\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)|\Lambda|}(t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}), \tag{20}$$

Thus,

$$\begin{aligned}
&|Ox(t_2)-Ox(t_1)|\\
\le\ & \frac{M}{\Gamma(\alpha+\beta+1)}(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})+\frac{|\lambda|r}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})\\
&+\Bigg[\frac{M\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)|\Lambda|}\left(\sum_{i=1}^{m-2}\frac{|\mu_i|\eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}+\frac{T^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}\right)\\
&+\frac{|\lambda|r\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)|\Lambda|}\left(\sum_{i=1}^{m-2}\frac{|\mu_i|\eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}+\frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right)\\
&+\frac{|\theta|\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)|\Lambda|}\Bigg](t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\|D^{\delta}Ox\|\\
\le\ & \frac{M}{\Gamma(\alpha+\beta-\delta+1)}(t_2^{\alpha+\beta-\delta}-t_1^{\alpha+\beta-\delta})+\frac{|\lambda|r}{\Gamma(\alpha-\delta+1)}(t_2^{\alpha-\delta}-t_1^{\alpha-\delta})\\
&+\Bigg[\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)|\Lambda|}\left(\sum_{i=1}^{m-2}\frac{|\mu_i|\eta_i^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}+\frac{(T-s)^{\alpha+\beta-\delta}}{\Gamma(\alpha+\beta-\delta+1)}\right)\\
&+\frac{|\lambda|r\Gamma(\beta+\alpha)}{\Gamma(\beta)|\Lambda|}\left(\sum_{i=1}^{m-2}\frac{|\mu_i|\eta_i^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}+\frac{T^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right)\\
&+\frac{|\theta|\Gamma(\beta+\alpha)}{\Gamma(\beta)|\Lambda|}\Bigg](t_2^{\alpha+\beta-\delta-1}-t_1^{\alpha+\beta-\delta-1}),
\end{aligned} \tag{22}$$

Thanks to (21) and (22), we can state that $\|Ox(t_2)-Ox(t_1)\|\to 0$ as $t_2-t_1\to 0$. By Arzela-Ascoli theorem, we conclude that O is completely continuous operator.

Step 4 : Now, it remains to show that the set $\Sigma = \{x \in X : x = \rho O(x),\ 0 < \rho < 1\}$ is bounded. Let $x \in \Sigma$, then $x = \rho O(x), 0 < \rho < 1$, Thus, for each $t \in [0,T]$, we have

$$\begin{aligned}
&x(t)\\
=\ & \rho\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s),D^\delta x(s))ds - \rho\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds\\
&+\frac{\rho\Gamma(\beta+\alpha-\delta)t^{\alpha+\beta-1}}{\Gamma(\beta)\Lambda}\left[\sum_{i=1}^{m-2}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} f(s,x(s),D^\delta x(s))ds\right.\\
&-\lambda\sum_{i=1}^{m-2}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}x(s)ds - \int_0^T \frac{(T-s)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} f(s,x(s),D^\delta x(s))ds\\
&\left.+\lambda\int_0^T\frac{(T-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}x(s)ds\right] + \frac{\rho\Gamma(\beta+\alpha-\delta)\theta}{\Gamma(\beta)\Lambda}t^{\alpha+\beta-1}, \qquad (23)
\end{aligned}$$

Using (A_2) and (A_3), we obtain

$$\| x \| \le \frac{1}{1-|\lambda|\Delta_2}\left[\Delta_1 M + \frac{|\theta|\Gamma(\beta+\alpha-\delta)}{\Gamma(\beta)|\Lambda|}T^{\alpha+\beta-1}\right], \qquad (24)$$

and

$$\| D^\delta x \| \le \frac{1}{1-|\lambda|\Pi_2}\left[\Pi_1 M + \frac{|\theta|\Gamma(\beta+\alpha)}{\Gamma(\beta)|\Lambda|}T^{\alpha+\beta-\delta-1}\right]. \qquad (25)$$

From (24) and (25), we get

$$\begin{aligned}
\| x \| \quad \le\ & \left(\frac{\Delta_1}{1-|\lambda|\Delta_2} + \frac{\Pi_1}{1-|\lambda|\Pi_2}\right)M\\
&+\frac{|\theta|}{\Gamma(\beta)|\Lambda|}\left[\frac{\Gamma(\beta+\alpha-\delta)}{1-|\lambda|\Delta_2}T^{\alpha+\beta-1} + \frac{\Gamma(\beta+\alpha)}{1-|\lambda|\Pi_2}T^{\alpha+\beta-\delta-1}\right]\Omega \qquad (26).
\end{aligned}$$

This shows that the set Σ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that O has a fixed point which is a solution of boundary value problem (1). The proof is completed. □

3. Ulam-Hyers stability

We adopt the definitions [12]: Ulam-Hyers stability, generalized Ulam-Hyers stability and Ulam-Hyers-Rassias stable for the fractional boundary value problem (1). Let ε a positive real numbers and the function $h \in X$, we consider the following fractional differential inequalities:

$$\left|D^{\alpha}(D^{\beta}+\lambda)x(t)-f(t,x(t),D^{\delta}x(t))\right|\leq \varepsilon, t\in[0,T], \tag{27}$$

and

$$\left|D^{\alpha}(D^{\beta}+\lambda)x(t)-f(t,x(t),D^{\delta}x(t))\right|\leq \varepsilon\phi(t), t\in[0,T]. \tag{28}$$

Definition 6 : The fractional boundary value problem (1) is Ulam-Hyers stable if there exists a real number $\gamma>0$ such that for each $\varepsilon>0$ and for each solution $x\in X$ of the inequality (27) there exists a solution $\overline{x}\in X$ of fractional boundary value problem (1) with

$$\left|x(t)-\overline{x}(t)\right|\leq \gamma\,\varepsilon, t\in[0,T].$$

Definition 7 : The fractional boundary value problem (1) is generalised Ulam-Hyers stable if there exists $\varphi\in C(\mathbb{R}_{+},\mathbb{R}_{+}), \varphi(0)=0,$ such that for each solution $x\in X$ of the inequality (27) there exists a solution $\overline{x}\in X$ of the fractional boundary value problem (1) with

$$\left|x(t)-\overline{x}(t)\right|\leq \varphi(\varepsilon), t\in[0,T].$$

Definition 8 : The fractional boundary value problem (27) is Ulam-Hyers-Rassias stable with respect to $\phi\in C([0,T],\mathbb{R}_{+})$ if there exists a real number $\gamma>0$ such that for each $\varepsilon>0$ and for each solution $x\in X$ of the inequality (27) there exists a solution $\overline{x}\in X$ of fractional boundary value problem (1) with

$$\left|x(t)-\overline{x}(t)\right|\leq \gamma\,\varepsilon\,\phi(t), t\in[0,T].$$

Remark 9 : An element $x\in X$ is a solution of the inequality (27) if and only if there exists a function $\psi:[0,T]\to\mathbb{R}$ such that

(a): $|\psi(t)|\leq\varepsilon, t\in[0,T]$.

(b): $D^{\alpha}(D^{\beta}+\lambda)x(t)=f(t,x(t),D^{\delta}x(t))+\psi(t), t\in[0,T]$.

Theorem 10 : *Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that the assumptions $(A_1),(A_2)$ and (A_3) hold. If*

$$D^{\alpha}(D^{\beta}+\lambda)x(t)\geq(\Delta_1+\Pi_1)M+|\lambda|(\Delta_2+\Pi_2)r+\frac{|\theta|}{\Gamma(\beta)|\Lambda|}\Omega. \tag{29}$$

Then the fractional boundary value problem (1) is Ulam-Hyers stable.

Proof : Let $x \in X$ be a solution of the inequation (27) satisfying

$$D^{\alpha}(D^{\beta}+\lambda)x(t) = f(t, x(t), D^{\delta}x(t)),$$

with

$$x(0) = 0, D^{\delta}x(T) = \sum_{i=1}^{m-2} \mu_i D^{\delta}x(\eta_i) + \theta.$$

Let us denote by $\overline{x} \in X$ the unique solution of the fractional boundary value problem

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)\overline{x}(t) = f(t, \overline{x}(t), D^{\delta}\overline{x}(t)), t \in [0, T], 0 < \alpha, \beta \le 1, \delta < \alpha, \\ \qquad x(0) = \overline{x}(0) =, D^{\delta}x(T) = D^{\delta}\overline{x}(T). \end{cases}$$

For each $\varepsilon > 0, x \in X,$ we let

$$\left| D^{\alpha}(D^{\beta}+\lambda)x(t) - f(t, x(t), D^{\delta}x(t)) \right| \le \varepsilon.$$

Using the hypoteses (A_2) and (A_3), we get

$$|x(t)| \le (\Delta_1 + \Pi_1)M + (\Delta_2 + \Pi_2)r + \frac{|\theta|}{\Gamma(\beta)|\Lambda|}\Omega. \tag{30}$$

By (29) and (30), we have

$$\sup_{t\in[0,T]} |x(t)| \le \sup_{t\in[0,T]} |D^{\alpha}(D^{\beta}+\lambda)x(t)|. \tag{31}$$

Using (31), we get

$$\begin{aligned} \sup_{t\in[0,T]} \left| x(t) - \overline{x}(t) \right| &\le \sup_{t\in[0,T]} \left| D^{\alpha}(D^{\beta}+\lambda)(x(t) - \overline{x}(t)) \right| \\ &\le \sup_{t\in[0,T]} \Big| D^{\alpha}(D^{\beta}+\lambda)x(t) - f(t, x(t), D^{\delta}x(t)) \\ &\quad - D^{\alpha}(D^{\beta}+\lambda)\overline{x}(t) + f(t, \overline{x}(t), D^{\delta}\overline{x}(t)) \\ &\quad + f(t, x(t), D^{\delta}x(t)) - f(t, \overline{x}(t), D^{\delta}\overline{x}(t)) \Big| \\ &\le \quad \varepsilon + \omega \| x - \overline{x} \|. \end{aligned} \tag{32}$$

This yields that

$$(1-\omega)\| x - \overline{x} \| \le \varepsilon. \tag{33}$$

Then, for each $t \in [0, T]$

$$\left|x(t)-\overline{x}(t)\right| \leq \frac{1}{1-\omega}\varepsilon := \gamma\varepsilon. \tag{34}$$

So, the fractional boundary value problem (1) is Ulam--Hyers stable.

By putting $\varphi(\varepsilon)=\gamma\varepsilon, \varphi(0)=0$ yields that the fractional boundary value problem (1) generalised Ulam-Hyers stable. □

Theorem 11 : *Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that the assumptions $(A_1),(A_2)$ and (A_3) hold . In addition we assume that*

(A_4) : There exists a function $\phi \in C([0,T],\mathbb{R}_+)$ that satisfies (28).

Then the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable.

Proof : Let $x\in X$ be a solution of the inequation (27) satisfying

$$\left|D^{\alpha}(D^{\beta}+\lambda)x(t)-f(t,x(t),D^{\delta}x(t))\right| \leq \varepsilon\phi(t), t\in[0,T],$$

Let us denote by $\overline{x}\in X$ the unique solution of the fractional boundary value problem

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)\overline{x}(t)=f(t,\overline{x}(t),D^{\delta}\overline{x}(t)), t\in[0,T], 0<\alpha,\beta\leq 1, \delta<\alpha, \\ x(0)=\overline{x}(0)=, D^{\delta}x(T)=D^{\delta}\overline{x}(T). \end{cases}$$

In virtue of Remark 11 and by some easy calculations, we get

$$\| x-\overline{x} \| (1-[(\Delta_1+\Pi_1)\omega+|\lambda|(\Delta_2+\Pi_2)]) \leq \varepsilon\phi(t). \tag{35}$$

Then, for each $t\in[0,T]$

$$\left|x(t)-\overline{x}(t)\right| \leq \frac{1}{1-[(\Delta_1+\Pi_1)\omega+|\lambda|(\Delta_2+\Pi_2)]}\varepsilon\phi(t) := \gamma\varepsilon\phi(t). \tag{36}$$

So, the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable. □

4. Application

To illustrate our main results, we treat the following examples.

Example 12 : Let us consider the following fractional boundary value problem

$$\begin{cases} D^{\frac{3}{4}}\left(D^{\frac{2}{3}}+\frac{1}{10}\right)x(t)=\frac{|x(t)|}{32\pi(te^{t^2}+1)}+\frac{\sin|D^{\frac{1}{2}}x(t)|}{32\pi(te^{t^2}+1)}+\arctan(t+1), t\in[0,1], \\ J^{1-\frac{1}{2}}x(0)=0, D^{\frac{1}{2}}x(1)=\sqrt{2}D^{\frac{1}{2}}x\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{2}D^{\frac{1}{2}}x\left(\frac{3}{4}\right)+e, \end{cases} \tag{37}$$

Here we have $\alpha=\frac{3}{4}$, $\beta=\frac{2}{3}$, $\lambda=\frac{1}{10}$, $\delta=\frac{1}{2}$, $\mu_1=\sqrt{2}$, $\mu_2=\frac{\sqrt{3}}{2}$, $\eta_1=\frac{1}{3}$, $\eta_2=\frac{3}{4}$ and

$$f(t,x,y)=\frac{x}{32\pi(te^{t^2}+1)}+\frac{\sin|y|}{32\pi(te^{t^2}+1)}+\arctan(t+1), t\in[0,1], x,y\in\mathbb{R}.$$

For each $x,y,\overline{x},\overline{y}\in\mathbb{R}$ and $t\in[0,1]$, we have

$$\left|f(t,x,y)-f(t,\overline{x},\overline{y})\right|\le\frac{1}{32\pi}|x-x|+|y-\overline{y}|.$$

Hence condition (A_1) is satisfied with $\omega=\frac{1}{32\pi}$. By computation, $(\Delta_1+\Pi_1)\omega=0.04059<1-|\lambda|(\Delta_2+\Pi_2)=0.46366.$ Thus Theorem 4 guarantees the uniqueness of a solution for the boundary value problem (37).

Example 13 : Consider the following problem

$$\begin{cases} D^{\frac{3}{4}}\left(D^{\frac{4}{5}}+\frac{1}{18}\right)x(t)=\frac{\cos x(t)+\sin D^{\frac{2}{5}}x(t)}{15\pi}, t\in[0,1], \\ J^{1-\frac{2}{5}}x(0)=0, D^{\frac{2}{5}}x(1)=\frac{6}{7}D^{\frac{2}{5}}x\left(\frac{2}{9}\right)+\frac{1}{2}D^{\frac{2}{5}}x\left(\frac{3}{4}\right)+D^{\frac{2}{5}}x\left(\frac{1}{4}\right)+\frac{e}{3}, \end{cases} \tag{38}$$

where $\alpha=\frac{3}{4}$, $\beta=\frac{4}{5}$, $\lambda=\frac{1}{18}$, $\delta=\frac{2}{5}$, $\mu_1=\frac{6}{7}$, $\mu_2=\frac{1}{2}$, $\mu_3=1$, $\eta_1=\frac{2}{9}$, $\eta_2=\frac{3}{4}$, $\eta_3=\frac{1}{4}$ and $f(t,x,y)=\frac{\cos x+\sin y}{15\pi}$, $t\in[0,1]$, $x,y\in\mathbb{R}$. With the given data, it is found that $M=\frac{2}{15\pi}$, $\Delta_2=3.4619$, $\Pi_2=2.5512$ as $|f(t,x,y)|\le\frac{2}{15\pi}$, $|\lambda|\Delta_2<1$ and $|\lambda|\Pi_2<1$.

It is clear that the conditions of Theorem 5 hold. Then the problem (38) has at least one solution on $[0,1]$.

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