Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 8, pp. 1745–1775
DOI : 10.47974/JIM-1319

Interval-valued picture fuzzy sets in UP-algebras

Nattacha Lapo [†]
Pimwaree Kankaew [§]
Sunisa Yuphaphin [‡]
Department of Mathematics
School of Science
University of Phayao
Mae Ka
Phayao 56000
Thailand

Ronnason Chinram [@]
Algebra and Applications Research Unit
Division of Computational Science
Faculty of Science
Prince of Songkla University
Hat Yai
Songkhla 90110
Thailand

Aiyared Iampan *
Fuzzy Algebras and Decision-Making Problems Research Unit
Department of Mathematics
School of Science
University of Phayao
Mae Ka
Phayao 56000
Thailand

[†] *E-mail:* nattachalapo.pp@gmail.com
[§] *E-mail:* pimwaree.kk@gmail.com
[‡] *E-mail:* sunisayuphaphin@gmail.com
[@] *E-mail:* ronnason.c@psu.ac.th
* *E-mail:* aiyared.ia@up.ac.th (Corresponding Author)

Abstract

We used the notion of interval-valued picture fuzzy sets (IvPFSs) in UP-algebras to introduce the eight new notions of IvPFSs. We also discuss about how the eight new notions of IvPFSs in UP-algebras relate to each other. This idea is extended to level subsets of IvPFSs in UP-algebras. Furthermore, we define an IvPFS in the same manner that a characteristic function is defined, and we examine its characterizations using the associated subset.

Subject Classification: *Primary 06F35, Secondary 08A72, 03E72.*

Keywords: *UP-algebra, Interval-valued picture fuzzy set, Level subset.*

1. Introduction

In 1965, Zadeh [25] was the first to consider the notion of fuzzy sets. The fuzzy set theories created by Zadeh and others have a wide range of applications in mathematics and beyond. After Zadeh [25] introduced the notion of fuzzy sets, Atanassov [2] proposed a new notion called an intuitionistic fuzzy set, which is a generalization of fuzzy sets. Cuong and Kreinovich [4] in 2013 proposed the notion of picture fuzzy sets (PFSs), which are direct expansions of fuzzy sets and intuitionistic fuzzy sets. In 2014, Cuong [3] published a paper in the J. Comput. Sci. Cybern. that introduced the notion of PFSs. On PFSs with certain properties, various operations are considered. Singh [16] proposed a geometrical interpretation of PFSs in 2015. The author suggested correlation coefficients for PFSs that take into account the degree of positive, neutral, negative, and refuse membership. Wei [21] presented another form of eight similarity measures between PFSs based on the cosine function between PFSs in 2017, taking into account the degree of positive, neutral, negative, and refusal membership in PFSs. Wei and Gao [23] presented some novel Dice similarity measures of PFSs as well as generalized Dice similarity measures of PFSs the following year, indicating that Dice similarity measures and asymmetric measures are special cases of generalized Dice similarity measures in some parameter values. Wei [22] described a novel method for determining the similarity of PFSs. These similarity measurements between PFSs were used by the author to recognize building materials and minerals fields. Ganie et al. [5] presented two picture fuzzy set correlation coefficients in the year 2020. These picture fuzzy set correlation coefficients are superior than previous ones in terms of reflecting the nature of correlation. In [3], the notion of IvPFSs was also proposed. The following was said by Liu et al. [12]: "In IvPFSs, the degrees of membership, abstinence and non-membership are given in closed sub-intervals of $[0,1]$ and have a condition that the sum of the supremum of all three subintervals must belong to a closed unit interval." Yuphaphin et al. [24] used the idea of PFSs to present eight new PFS concepts through a special type in UP-algebras this year. From the idea of this article led to this research.

In this article, we exploited the notion of IvPFSs in UP-algebras to present the eight new IvPFSs. We also discuss about how the eight new notions of IvPFSs in UP-algebras relate to each other. In UP-algebras, this notion is extended to level subsets of IvPFSs. Furthermore, we define an IvPFS in the same manner that a characteristic function is defined, and we examine its characterizations using the associated subset.

2. Basic UP-algebra results

Let's look at the definition of UP-algebras before we get started.

Definition 2.1 : [7] Allow $\mathcal{U}$ to be a nonempty set, $*$ to be a binary operation on $\mathcal{U}$, and 0 to be a fixed element of $\mathcal{U}$. A *UP-algebra* is defined as an algebra $\mathcal{U} = (\mathcal{U}, *, 0)$ of type $(2,0)$ if the following axioms are true:

$$(\text{all}\, x, y, z \in \mathcal{U})((y*z)*((x*y)*(x*z)) = 0), \tag{2.1}$$

$$(\text{all}\, x \in \mathcal{U})(0*x = x), \tag{2.2}$$

$$(\text{all}\, x \in \mathcal{U})(x*0 = 0), \text{ and} \tag{2.3}$$

$$(\text{all}\, x, y \in \mathcal{U})(x*y = 0, y*x = 0 \Rightarrow x = y). \tag{2.4}$$

Unless otherwise indicated, let $\mathcal{U}$ signify a $(\mathcal{U}, *, 0)$ UP-algebra.

The binary relation $\leqslant$ is described as follows on $\mathcal{U}$:

$$(\text{all}\, x, y \in \mathcal{U})(x \leqslant y \Leftrightarrow x*y = 0) \tag{2.5}$$

and the claims that follow are true (see [7, 8]).

$$(\text{all}\, x \in \mathcal{U})(x \leqslant x), \tag{2.6}$$

$$(\text{all}\, x, y, z \in \mathcal{U})(x \leqslant y, y \leqslant z \Rightarrow x \leqslant z), \tag{2.7}$$

$$(\text{all}\, x, y, z \in \mathcal{U})(x \leqslant y \Rightarrow z*x \leqslant z*y), \tag{2.8}$$

$$(\text{all}\, x, y, z \in \mathcal{U})(x \leqslant y \Rightarrow y*z \leqslant x*z), \tag{2.9}$$

$$(\text{all}\, x, y, z \in \mathcal{U})(x \leqslant y*x, \text{inparticular}, y*z \leqslant x*(y*z)), \tag{2.10}$$

$$(\text{all}\, x, y \in \mathcal{U})(y * x \leqslant x \Leftrightarrow x = y * x), \tag{2.11}$$
$$(\text{all}\, x, y \in \mathcal{U})(x \leqslant y * y), \tag{2.12}$$
$$(\text{all}\, a, x, y, z \in \mathcal{U})(x * (y * z) \leqslant x * ((a * y) * (a * z))), \tag{2.13}$$
$$(\text{all}\, a, x, y, z \in \mathcal{U})(((a * x) * (a * y)) * z \leqslant (x * y) * z), \tag{2.14}$$
$$(\text{all}\, x, y, z \in \mathcal{U})((x * y) * z \leqslant y * z), \tag{2.15}$$
$$(\text{all}\, x, y, z \in \mathcal{U})(x \leqslant y \Rightarrow x \leqslant z * y), \tag{2.16}$$
$$(\text{all}\, x, y, z \in \mathcal{U})((x * y) * z \leqslant x * (y * z)), \text{ and} \tag{2.17}$$
$$(\text{all}\, a, x, y, z \in \mathcal{U})((x * y) * z \leqslant y * (a * z)). \tag{2.18}$$

For more studies and examples of UP-algebras, see [1, 8, 10, 11, 13, 14, 15, 19].

Definition 2.2 : [6, 9, 7, 17] Let S be a nonempty subset of $\mathcal{U}$. Then S is called

(1) a *UP-subalgebra* (UPs) of $\mathcal{U}$ if
$$(\text{all}\, x, y \in S)(x * y \in S), \tag{2.19}$$

(2) a *near UP-filter* (nUPf) of $\mathcal{U}$ if
$$(\text{all}\, x, y \in \mathcal{U})(y \in S \Rightarrow x * y \in S), \tag{2.20}$$

(3) a *UP-filter* (UPf) of $\mathcal{U}$ if
$$0 \in S, \tag{2.21}$$
$$(\text{all}\, x, y \in \mathcal{U})(x * y \in S, x \in S \Rightarrow y \in S), \tag{2.22}$$

(4) an *implicative UP-filter* (iUPf) of $\mathcal{U}$ if (2.21) and
$$(\text{all}\, x, y, z \in \mathcal{U})(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S), \tag{2.23}$$

(5) a *comparative UP-filter* (cUPf) of $\mathcal{U}$ if (2.21) and
$$(\text{all}\, x, y, z \in \mathcal{U})(x * ((y * z) * y) \in S, x \in S \Rightarrow y \in S), \tag{2.24}$$

(6) a *shift UP-filter* (sUPf) of $\mathcal{U}$ if (2.21) and
$$(\text{all}\, x, y, z \in \mathcal{U})(x * (y * z) \in S, x \in S \Rightarrow ((z * y) * y) * z \in S), \tag{2.25}$$

(7) a *UP-ideal* (UPi) of $\mathcal{U}$ if (2.21) and
$$(\text{all}\, x, y, z \in \mathcal{U})(x * (y * z) \in S, y \in S \Rightarrow x * z \in S), \tag{2.26}$$

(8) a *strong UP-ideal* (sUPi) of $\mathcal{U}$ if (2.21) and

$$(\text{all}\, x, y, z \in \mathcal{U})((z * y) * (z * x) \in S, y \in S \Rightarrow x \in S). \tag{2.27}$$

Guntasow et al. [6] proved that the only sUPi of a UP-algebra $\mathcal{U}$ is $\mathcal{U}$. The following theorem is straightforward to prove.

Theorem 2.3 : *Given a nonempty family $\mathcal{F}$ of UPss (resp., nUPfs, UPfs, iUPfs, cUPfs, sUPfs, UPis, sUPis) of $\mathcal{U}$. Then $\cap\mathcal{F}$ is a UPs (resp., nUPf, UPf, iUPf, cUPf, sUPf, UPi, sUPi) of $\mathcal{U}$.*

3. IvPFSs in UP-algebras

The concept of an interval number is studied in [26] which is stated as follows:

We refer to a close subinterval $\tilde{a} = [a^-, a^+]$ of $[0,1]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$ as an *interval number*. The set of all interval numbers is denoted by $\text{int}[0,1]$. If $\tilde{a}, \tilde{b} \in \text{int}[0,1]$, we define the *refined minimum* and the *refined maximum* of $\tilde{a}$ and $\tilde{b}$, denoted by $\text{rmin}\{\tilde{a}, \tilde{b}\}$ and $\text{rmax}\{\tilde{a}, \tilde{b}\}$, respectively, as follows:

$$\text{rmin}\{\tilde{a}, \tilde{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}] \text{ and}$$
$$\text{rmax}\{\tilde{a}, \tilde{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}].$$

The following, in case of $\tilde{a}$ and $\tilde{b}$, are the definitions for the relations $\succeq$, $\preceq$, and $=$.

$$\tilde{a} \succeq \tilde{b} \Leftrightarrow a^- \geqslant b^- \text{ and } a^+ \geqslant b^+,$$

and similarly we may have $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} = \tilde{b}$. To say $\tilde{a} \succ \tilde{b}$ (resp., $\tilde{a} \prec \tilde{b}$) we mean $\tilde{a} \succeq \tilde{b}$ and $\tilde{a} \neq \tilde{b}$ (resp., $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \neq \tilde{b}$). The *complement of* $\tilde{a}$, denoted by $\tilde{a}^C$, is defined by $\tilde{a}^C = [1 - a^+, 1 - a^-]$.

In the $\text{int}[0,1]$, the following assertions are valid (see [20]).

$$(\text{all}\, \tilde{a} \in \text{int}[0,1])((\tilde{a}^C)^C = \tilde{a}), \tag{3.1}$$
$$(\text{all}\, \tilde{a} \in \text{int}[0,1])(\text{rmax}\{\tilde{a}, \tilde{a}\} = \tilde{a}), \tag{3.2}$$
$$(\text{all}\, \tilde{a} \in \text{int}[0,1])(\text{rmin}\{\tilde{a}, \tilde{a}\} = \tilde{a}), \tag{3.3}$$
$$(\text{all}\, \tilde{a}_1, \tilde{a}_2 \in \text{int}[0,1])(\text{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \text{rmax}\{\tilde{a}_2, \tilde{a}_1\}), \tag{3.4}$$
$$(\text{all}\, \tilde{a}_1, \tilde{a}_2 \in \text{int}[0,1])(\text{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \text{rmin}\{\tilde{a}_2, \tilde{a}_1\}), \tag{3.5}$$
$$(\text{all}\, \tilde{a}_1, \tilde{a}_2 \in \text{int}[0,1])(\text{rmax}\{\tilde{a}_1, \tilde{a}_2\} \succeq \tilde{a}_1, \tilde{a}_2), \tag{3.6}$$
$$(\text{all}\, \tilde{a}_1, \tilde{a}_2 \in \text{int}[0,1])(\tilde{a}_1, \tilde{a}_2 \succeq \text{rmin}\{\tilde{a}_1, \tilde{a}_2\}), \tag{3.7}$$
$$(\text{all}\, \tilde{a}_1, \tilde{a}_2 \in \text{int}[0,1])(\tilde{a}_1 \succeq \tilde{a}_2 \Leftrightarrow \tilde{a}_1^C \preceq \tilde{a}_2^C), \tag{3.9}$$
$$(\text{all}\, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4 \in \text{int}[0,1])(\tilde{a}_1 \succeq \tilde{a}_2, \tilde{a}_3 \succeq \tilde{a}_4 \Rightarrow \text{rmax}\{\tilde{a}_1, \tilde{a}_3\} \succeq \text{rmax}\{\tilde{a}_2, \tilde{a}_4\}), \tag{3.9}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4 \in \text{int}[0,1])(\tilde{a}_1 \succeq \tilde{a}_2, \tilde{a}_3 \succeq \tilde{a}_4 \Rightarrow \text{rmin}\{\tilde{a}_1,\tilde{a}_3\} \succeq \text{rmin}\{\tilde{a}_2,\tilde{a}_4\}), \tag{3.10}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2,\tilde{a}_3 \in \text{int}[0,1])(\tilde{a}_2 \succeq \tilde{a}_1, \tilde{a}_2 \succeq \tilde{a}_3 \Leftrightarrow \tilde{a}_2 \succeq \text{rmax}\{\tilde{a}_1,\tilde{a}_3\}), \tag{3.11}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2,\tilde{a}_3 \in \text{int}[0,1])(\tilde{a}_1 \succeq \tilde{a}_2, \tilde{a}_3 \succeq \tilde{a}_2 \Leftrightarrow \text{rmin}\{\tilde{a}_1,\tilde{a}_3\} \succeq \tilde{a}_2), \tag{3.12}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2 \in \text{int}[0,1])(\tilde{a}_1 \succeq \tilde{a}_2 \Leftrightarrow \text{rmax}\{\tilde{a}_1,\tilde{a}_2\} = \tilde{a}_1), \tag{3.13}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2 \in \text{int}[0,1])(\tilde{a}_1 \succeq \tilde{a}_2 \Leftrightarrow \text{rmin}\{\tilde{a}_1,\tilde{a}_2\} = \tilde{a}_2), \tag{3.14}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2 \in \text{int}[0,1])(\text{rmin}\{\tilde{a}_1^C,\tilde{a}_2^C\} = \text{rmax}\{\tilde{a}_1,\tilde{a}_2\}^C), \tag{3.15}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2 \in \text{int}[0,1])(\text{rmax}\{\tilde{a}_1^C,\tilde{a}_2^C\} = \text{rmin}\{\tilde{a}_1,\tilde{a}_2\}^C), \tag{3.16}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2,\tilde{a}_3 \in \text{int}[0,1])(\tilde{a}_1 \preceq \text{rmax}\{\tilde{a}_2,\tilde{a}_3\} \Leftrightarrow \tilde{a}_1^C \succeq \text{rmin}\{\tilde{a}_2^C,\tilde{a}_3^C\}), \tag{3.17}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2,\tilde{a}_3 \in \text{int}[0,1])(\tilde{a}_1 \succeq \text{rmax}\{\tilde{a}_2,\tilde{a}_3\} \Leftrightarrow \tilde{a}_1^C \preceq \text{rmin}\{\tilde{a}_2^C,\tilde{a}_3^C\}), \tag{3.18}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2,\tilde{a}_3 \in \text{int}[0,1])(\tilde{a}_1 \preceq \text{rmin}\{\tilde{a}_2,\tilde{a}_3\} \Leftrightarrow \tilde{a}_1^C \succeq \text{rmax}\{\tilde{a}_2^C,\tilde{a}_3^C\}), \text{ and} \tag{3.19}$$

$$(\text{all}\,\tilde{a}_1,\tilde{a}_2,\tilde{a}_3 \in \text{int}[0,1])(\tilde{a}_1 \succeq \text{rmin}\{\tilde{a}_2,\tilde{a}_3\} \Leftrightarrow \tilde{a}_1^C \preceq \text{rmax}\{\tilde{a}_2^C,\tilde{a}_3^C\}). \tag{3.20}$$

Cuong and Kreinovich [4] established the notion of PFSs in 2013, defining it as follows.

In a nonempty set $\mathcal{U}$, a *picture fuzzy set* (PFS) is a structure of the form:

$$P = \{(x, r_P(x), g_P(x), b_P(x)) \mid x \in \mathcal{U}\},$$

where $r_P : \mathcal{U} \to [0,1]$, $g_P : \mathcal{U} \to [0,1]$, and $b_P : \mathcal{U} \to [0,1]$ fulfill the following:

$$(\text{all}\, x \in \mathcal{U})(r_P(x) + g_P(x) + b_P(x) \leqslant 1).$$

We'll refer to a PFS as $P = (\mathcal{U}, r_P, g_P, b_P) = \{(x, r_P(x), g_P(x), b_P(x)) \mid x \in \mathcal{U}\}$ for the sake of simplicity.

Cuong [3] proposed the notion of IvPFSs in 2014, and provided the formal definition.

In a nonempty set $\mathcal{U}$, an *interval-valued picture fuzzy set* (IvPFS) is a structure of the form:

$$I = \{(x, R_I(x), G_I(x), B_I(x)) \mid x \in \mathcal{U}\},$$

where $R_I : \mathcal{U} \to \text{int}[0,1]$ is a *membership function* sending $x \mapsto [\inf R_I(x), \sup R_I(x)]$, $G_I : \mathcal{U} \to \text{int}[0,1]$ is a *abstinence function* sending $x \mapsto [\inf G_I(x), \sup G_I(x)]$, and $B_I : \mathcal{U} \to \text{int}[0,1]$ is a *non-membership function* sending $x \mapsto [\inf B_I(x), \sup B_I(x)]$, fulfill the following:

$$(\text{all}\, x \in \mathcal{U})(\sup R_I(x) + \sup G_I(x) + \sup B_I(x) \leqslant 1).$$

If we write $R_I(x) = \widetilde{R_I(x)}, G_I(x) = \widetilde{G_I(x)},$ and $B_I(x) = \widetilde{B_I(x)},$ then $(R_I(x))^+ + (G_I(x))^+ + (B_I(x))^+ \leqslant 1$. An IvPFS shall be denoted as $I = (\mathcal{U}, R_I, G_I, B_I) = \{(x, R_I(x), G_I(x), B_I(x)) \mid x \in \mathcal{U}\}$ for our convenience.

Now we'll present the eight new IvPFSs notions: IvPFUPss, IvPFnUPfs, IvPFUPfs, IvPFiUPfs, IvPFcUPfs, IvPFsUPfs, IvPFUPis, and IvPFsUPis, give some instances, look into their characteristics, and show their generalizations.

Definition 3.1 : An *interval-valued picture fuzzy UP-subalgebra* (IvPFUPs) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following:

$$(\text{all}\, x, y \in \mathcal{U})(R_I(x * y) \succeq \text{rmin}\{R_I(x), R_I(y)\}), \tag{3.21}$$

$$(\text{all}\, x, y \in \mathcal{U})(G_I(x * y) \succeq \text{rmin}\{G_I(x), G_I(y)\}), \tag{3.22}$$

$$(\text{all}\, x, y \in \mathcal{U})(B_I(x * y) \preceq \text{rmax}\{B_I(x), B_I(y)\}). \tag{3.23}$$

Example 3.2 : As indicated in the Cayley table below, let $\mathcal{U} = \{0,1,2,3,4\}$ represent a UP-algebra $(\mathcal{U}, *, 0)$.

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	4
2	0	0	0	0	4
3	0	1	1	0	4
4	0	3	3	3	0

An IvPFS I is defined as follows:

$$R_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.2, 0.5] & [0.2, 0.4] & [0.1, 0.4] & [0.1, 0.3] & [0, 0.1] \end{pmatrix},$$

$$G_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.2, 0.4] & [0.1, 0.3] & [0, 0.2] & [0.1, 0.3] & [0, 0.1] \end{pmatrix},$$

$$B_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0, 0.1] & [0.1, 0.3] & [0.1, 0.3] & [0.2, 0.4] & [0.3, 0.6] \end{pmatrix}.$$

Therefore, I is an IvPFUPs of $\mathcal{U}$.

Definition 3.3 : An *interval-valued picture fuzzy near UP-filter* (IvPFnUPf) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following:

$$(\text{all}\, x, y \in \mathcal{U})(R_I(x * y) \succeq R_I(y)), \tag{3.24}$$
$$(\text{all}\, x, y \in \mathcal{U})(G_I(x * y) \succeq G_I(y)), \tag{3.25}$$
$$(\text{all}\, x, y \in \mathcal{U})(B_I(x * y) \preceq B_I(y)). \tag{3.26}$$

Example 3.4 : As indicated in the Cayley table below, let $\mathcal{U} = \{0,1,2,3,4\}$ represent a UP-algebra $(\mathcal{U}, *, 0)$.

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	0	4
3	0	0	2	0	4
4	0	0	0	0	0

An IvPFS I is defined as follows:

$$R_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 \\ [0.2,0.4] & [0.2,0.3] & [0.1,0.3] & [0.1,0.4] & [0,0.1] \end{pmatrix},$$
$$G_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 \\ [0.1,0.4] & [0.1,0.4] & [0,0.3] & [0.1,0.2] & [0,0.1] \end{pmatrix},$$
$$B_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 \\ [0,0.2] & [0.1,0.2] & [0.2,0.3] & [0,0.4] & [0.2,0.6] \end{pmatrix}.$$

Therefore, I is an IvPFnUPf of $\mathcal{U}$.

Definition 3.5 : An *interval-valued picture fuzzy UP-filter* (IvPFUPf) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following:

$$(\text{all}\, x \in \mathcal{U})(R_I(0) \succeq R_I(x)), \tag{3.27}$$
$$(\text{all}\, x \in \mathcal{U})(G_I(0) \succeq G_I(x)), \tag{3.28}$$
$$(\text{all}\, x \in \mathcal{U})(B_I(0) \preceq B_I(x)), \tag{3.29}$$
$$(\text{all}\, x, y \in \mathcal{U})(R_I(y) \succeq \text{rmin}\{R_I(x * y), R_I(x)\}), \tag{3.30}$$
$$(\text{all}\, x, y \in \mathcal{U})(G_I(y) \succeq \text{rmin}\{G_I(x * y), G_I(x)\}), \tag{3.31}$$
$$(\text{all}\, x, y \in \mathcal{U})(B_I(y) \preceq \text{rmax}\{B_I(x * y), B_I(x)\}). \tag{3.32}$$

Example 3.6 : [24] As indicated in the Cayley table below, let $\mathcal{U} = \{0,1,2,3,4\}$ represent a UP-algebra $(\mathcal{U}, *, 0)$.

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

An IvPFS I is defined as follows:

$$R_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.2, 0.4] & [0.2, 0.4] & [0.2, 0.4] & [0, 0.3] & [0, 0.3] \end{pmatrix},$$
$$G_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.3, 0.4] & [0.1, 0.3] & [0.1, 0.3] & [0, 0.2] & [0, 0.2] \end{pmatrix},$$
$$B_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0, 0.2] & [0, 0.2] & [0.1, 0.3] & [0.3, 0.5] & [0.3, 0.5] \end{pmatrix}.$$

Therefore, I is an IvPFUPf of $\mathcal{U}$.

Definition 3.7 : An *interval-valued picture fuzzy implicative UP-filter* (IvPFiUPf) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following: (3.27), (3.28), (3.29), and

$$(\text{all}\, x, y, z \in \mathcal{U})(R_I(x * z) \succeq \text{rmin}\{R_I(x * (y * z)), R_I(x * y)\}), \tag{3.33}$$
$$(\text{all}\, x, y, z \in \mathcal{U})(G_I(x * z) \succeq \text{rmin}\{G_I(x * (y * z)), G_I(x * y)\}), \tag{3.34}$$
$$(\text{all}\, x, y, z \in \mathcal{U})(B_I(x * z) \preceq \text{rmax}\{B_I(x * (y * z)), B_I(x * y)\}). \tag{3.35}$$

Example 3.8 : As indicated in the Cayley table below, let $\mathcal{U} = \{0, 1, 2, 3\}$ represent a UP-algebra $(\mathcal{U}, *, 0)$.

$*$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	3
3	0	1	2	0

An IvPFS I is defined as follows:

$$R_I = \begin{pmatrix} 0 & 2 & 3 & 3 \\ [0.2, 0.4] & [0, 0.3] & [0, 0.3] & [0, 0.3] \end{pmatrix},$$

$$G_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.1,0.4] & [0.1,0.3] & [0.1,0.3] & [0.1,0.3] \end{pmatrix},$$

$$B_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0,0.2] & [0.2,0.4] & [0.2,0.4] & [0.2,0.4] \end{pmatrix}.$$

Therefore, I is an IvPFiUPf of $\mathcal{U}$.

Definition 3.9 : An *interval-valued picture fuzzy comparative UP-filter* (IvPFcUPf) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following: (3.27), (3.28), (3.29), and

$$(\text{all}\, x,y,z \in \mathcal{U})(R_I(y) \succeq \text{rmin}\{R_I(x*((y*z)*y)), R_I(x)\}), \tag{3.36}$$

$$(\text{all}\, x,y,z \in \mathcal{U})(G_I(y) \succeq \text{rmin}\{G_I(x*((y*z)*y)), G_I(x)\}), \tag{3.37}$$

$$(\text{all}\, x,y,z \in \mathcal{U})(B_I(y) \preceq \text{rmax}\{B_I(x*((y*z)*y)), B_I(x)\}). \tag{3.38}$$

Example 3.10 : [24] As indicated in the Cayley table below, let $\mathcal{U} = \{0,1,2,3\}$ represent a UP-algebra $(\mathcal{U},*,0)$.

*	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	0	1	0

An IvPFS I is defined as follows:

$$R_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0,0.4] & [0,0.4] & [0,0.4] & [0.1,0.3] \end{pmatrix},$$

$$G_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.1,0.3] & [0.1,0.3] & [0.1,0.3] & [0,0.2] \end{pmatrix},$$

$$B_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.2,0.3] & [0,0.3] & [0,0.3] & [0,0.3] \end{pmatrix}.$$

Therefore, I is an IvPFcUPf of $\mathcal{U}$.

Definition 3.11 : An *interval-valued picture fuzzy shift UP-filter* (IvPFsUPf) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following: (3.27), (3.28), (3.29), and

$$(\text{all}\, x,y,z \in \mathcal{U})(R_I(((z*y)*y)*z) \succeq \text{rmin}\{R_I(x*(y*z)), R_I(x)\}), \tag{3.39}$$
$$(\text{all}\, x,y,z \in \mathcal{U})(G_I(((z*y)*y)*z) \succeq \text{rmin}\{G_I(x*(y*z)), G_I(x)\}), \tag{3.40}$$
$$(\text{all}\, x,y,z \in \mathcal{U})(B_I(((z*y)*y)*z) \preceq \text{rmax}\{B_I(x*(y*z)), B_I(x)\}). \tag{3.41}$$

Example 3.12 : As indicated in the Cayley table below, let $\mathcal{U} = \{0,1,2,3\}$ represent a UP-algebra $(\mathcal{U},*,0)$.

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	0	0	2
3	0	0	0	0

An IvPFS I is defined as follows:

$$R_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.1,0.4] & [0.1,0.4] & [0,0.3] & [0,0.3] \end{pmatrix},$$
$$G_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0.2,0.4] & [0.2,0.4] & [0.1,0.2] & [0.1,0.2] \end{pmatrix},$$
$$B_I = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0,0.2] & [0,0.2] & [0.1,0.4] & [0.1,0.4] \end{pmatrix}.$$

Therefore, I is an IvPFsUPf of $\mathcal{U}$.

Definition 3.13 : An *interval-valued picture fuzzy UP-ideal* (IvPFUPi) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following: (3.27), (3.28), (3.29), and

$$(\text{all}\, x,y,z \in \mathcal{U})(R_I(x*z) \succeq \text{rmin}\{R_I(x*(y*z)), R_I(y)\}), \tag{3.42}$$
$$(\text{all}\, x,y,z \in \mathcal{U})(G_I(x*z) \succeq \text{rmin}\{G_I(x*(y*z)), G_I(y)\}), \tag{3.43}$$
$$(\text{all}\, x,y,z \in \mathcal{U})(B_I(x*z) \preceq \text{rmax}\{B_I(x*(y*z)), B_I(y)\}). \tag{3.44}$$

Example 3.14 : As indicated in the Cayley table below, let $\mathcal{U} = \{0,1,2,3,4\}$ represent a UP-algebra $(\mathcal{U},*,0)$.

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	1	0	4
4	0	0	0	0	0

An IvPFS I is defined as follows:

$$R_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.3,0.5] & [0.2,0.4] & [0.2,0.3] & [0.1,0.2] & [0,0.1] \end{pmatrix},$$

$$G_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.3,0.4] & [0.2,0.4] & [0.2,0.3] & [0.1,0.2] & 0,0.1] \end{pmatrix},$$

$$B_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0,0.1] & [0.1,0.2] & [0.2,0.4] & [0.3,0.6] & [0.4,0.7] \end{pmatrix}.$$

Therefore, I is an IvPFUPi of $\mathcal{U}$.

Definition 3.15 : An *interval-valued picture fuzzy strong UP-ideal* (IvPFsUPi) of $\mathcal{U}$ is an IvPFS I in $\mathcal{U}$ that meets the following: (3.27), (3.28), (3.29), and

$$(\text{all}\, x,y,z \in \mathcal{U})(R_I(x) \succeq \text{rmin}\{R_I((z*y)*(z*x)), R_I(y)\}), \tag{3.45}$$

$$(\text{all}\, x,y,z \in \mathcal{U})(G_I(x) \succeq \text{rmin}\{G_I((z*y)*(z*x)), G_I(y)\}), \tag{3.46}$$

$$(\text{all}\, x,y,z \in \mathcal{U})(B_I(x) \preceq \text{rmax}\{B_I((z*y)*(z*x)), B_I(y)\}). \tag{3.47}$$

Example 3.16 : As indicated in the Cayley table below, let $\mathcal{U} = \{0,1,2,3\}$ represent a UP-algebra $(\mathcal{U},*,0)$.

$*$	0	1	2	3
0	0	1	2	3
1	0	0	1	1
2	0	0	0	1
3	0	0	1	0

An IvPFS I is defined as follows:

$$(\text{all}\, x \in \mathcal{U})\begin{pmatrix} R_I(x) = [0.4,0.5] \\ G_I(x) = [0.2,0.3] \\ B_I(x) = [0.1,0.2] \end{pmatrix}.$$

Therefore, I is an IvPFsUPi of $\mathcal{U}$.

Definiton 3.17 : If an IvPFS I in a nonempty set $\mathcal{U}$ is a constant function from $\mathcal{U}$ to $\text{int}[0,1]^3$, it is said to be *constant*.

Theorem 3.18 : *Every IvPFUPs of $\mathcal{U}$ fulfills the requirements (3.27), (3.28), and (3.29).*

Proof : Let I be an IvPFUPs of $\mathcal{U}$. Then for each $x \in \mathcal{U}$, $R_I(0) = R_I(x*x) \succeq \text{rmin}\{R_I(x), R_I(x)\} = R_I(x)$ (by (2.6), (3.21), (3.3)), $G_I(0) = G_I(x*x) \succeq \text{rmin}\{G_I(x), G_I(x)\} = G_I(x)$ (by (2.6), (3.22), (3.3)), and $B_I(0) = B_I(x*x) \preceq \text{rmax}\{B_I(x), B_I(x)\} = B_I(x)$ (by (2.6), (3.23), (3.2)). Therefore, I fulfills the requirements (3.27), (3.28), and (3.29). $\square$

Theorem 3.19 : *An IvPFS I in $\mathcal{U}$ is constant if and only if it is an IvPFsUPi of $\mathcal{U}$.*

Proof : Suppose that an IvPFS I is constant in $\mathcal{U}$. Then $R_I(x) = R_I(0), G_I(x) = G_I(0),$ and $B_I(x) = B_I(0)$ for all $x \in \mathcal{U}$. Then for each $x \in, R_I(0) \succeq R_I(x), G_I(0) \succeq G_I(x),$ and $B_I(0) \preceq B_I(x)$. Let $x, y, z \in \mathcal{U}$. Then $R_I(x) \succeq R_I(0) = \text{rmin}\{R_I(0), R_I(0)\} = \text{rmin}\{R_I((z*y)*(z*x)), R_I(y)\}$ (by (3.3)), $G_I(x) \succeq G_I(0) = \text{rmin}\{G_I(0), G_I(0)\} = \text{rmin}\{G_I((z*y)*(z*x)), G_I(y)\}$ (by (3.3)), and $B_I(x) \preceq B_I(0) = \text{rmax}\{B_I(0), R_I(0)\} = \text{rmax}\{B_I((z*y)*(z*x)), B_I(y)\}$ (by (3.2)). Therefore, I is an IvPFsUPi of $\mathcal{U}$.

Conversely, suppose that I is an IvPFsUPi of $\mathcal{U}$. For each $x \in \mathcal{U}$, $R_I(x) \succeq \text{rmin}\{R_I((x*0)*(x*x)), R_I(0)\} = \text{rmin}\{R_I(0*(x*x)), R_I(0)\} = \text{rmin}\{R_I(x*x), R_I(0)\} = \text{rmin}\{R_I(0), R_I(0)\} = R_I(0) \succeq R_I(x)$ (by (3.45), (2.3), (2.2), (2.6), (3.3), (3.27)), $G_I(x) \succeq \text{rmin}\{G_I((x*0)*(x*x)), G_I(0)\} = \text{rmin}\{G_I(0*(x*x)), G_I(0)\} = \text{rmin}\{G_I(x*x), G_I(0)\} = \text{rmin}\{G_I(0), G_I(0)\} \succeq G_I(0) = G_I(x)$ (by (3.46), (2.3), (2.2), (2.6), (3.3), (3.28)), and $B_I(x) \preceq \text{rmax}\{B_I((x*0)*(x*x)), B_I(0)\} = \text{rmax}\{B_I(0*(x*x)), B_I(0)\} = \text{rmax}\{B_I(x*x), B_I(0)\} = \text{rmax}\{B_I(0), B_I(0)\} = B_I(0) \preceq B_I(x)$ (by (3.47), (2.3), (2.2), (2.6), (3.2), (3.29)). Thus $R_I(x) = R_I(0), G_I(x) = G_I(0),$ and $B_I(x) = B_I(0)$ for all $x \in \mathcal{U}$. Thus $R_I, G_I,$ and B_I are constant. Therefore, I is constant. $\square$

Theorem 3.20 : *Every IvPFnUPf of $\mathcal{U}$ is an IvPFUPs.*

Proof : Suppose that I is an IvPFnUPf of $\mathcal{U}$. Let $x, y \in \mathcal{U}$. Then $R_I(x*y) \succeq R_I(y) \succeq \text{rmin}\{R_I(x), R_I(y)\}$ (by (3.25), (3.7)), $G_I(x*y) \succeq G_I(y) \succeq \text{rmin}\{G_I(x), G_I(y)\}$ (by (3.25), (3.7)), and $B_I(x*y) \preceq$

$B_I(y) \preceq \mathrm{rmax}\{B_I(x), B_I(y)\}$ (by (3.26), (3.6)). Therefore, I is an IvPFUPs of $\mathcal{U}$. □

Example 3.21 : We know that I is an IvPFUPs of $\mathcal{U}$ from Example 3.2. Since $R_I(4*1) = [0.1, 0.3] \not\succeq [0.2, 0.4] = R_I(1)$, we obtain I is not an IvPFnUPf of $\mathcal{U}$. This demonstrates that the converse of Theorem 3.20 is incorrect.

Theorem 3.22 : *Every IvPFnUPf of $\mathcal{U}$ fulfills the requirements (3.27), (3.28), and (3.29).*

Proof : By Theorems 3.20 and 3.18, it's straightforward. □

Theorem 3.23 : *Every IvPFUPf of $\mathcal{U}$ is an IvPFnUPf.*

Proof : Suppose that I is an IvPFUPf of $\mathcal{U}$. Let $x, y \in \mathcal{U}$. Then $R_I(x*y) \succeq \mathrm{rmin}\{R_I(y*(x*y)), R_I(y)\} = \mathrm{rmin}\{R_I(0), R_I(y)\} = R_I(y)$ (by (3.30), (2.10), (3.27), (3.14)), $G_I(x*y) \succeq \mathrm{rmin}\{G_I(y*(x*y)), G_I(y)\} = \mathrm{rmin}\{G_I(0), G_I(y)\} = G_I(y)$ (by (3.31), (2.10), (3.28), (3.14)), and $B_I(x*y) \preceq \mathrm{rmax}\{B_I(y*(x*y)), B_I(y)\} = \mathrm{rmax}\{B_I(0), B_I(y)\} = B_I(y)$ (by (3.32), (2.10), (3.29), (3.13)). Therefore, I is an IvPFnUPf of $\mathcal{U}$. □

Example 3.24 : We know that I is an IvPFnUPf of $\mathcal{U}$ from Example 3.4. Since $R_I(1) = [0.2, 0.3] \not\succeq [0.1, 0.4] = \mathrm{rmin}\{R_I(3*1), R_I(3)\}$, we obtain I is not an IvPFUPf of $\mathcal{U}$. This demonstrates that the converse of Theorem 3.23 is incorrect.

Theorem 3.25 : *Every IvPFUPi of $\mathcal{U}$ is an IvPFUPf.*

Proof : Suppose that I is an IvPFUPi of $\mathcal{U}$. Then I fulfills the requirements (3.27), (3.28), and (3.29). Let $x, y \in \mathcal{U}$. Then $R_I(y) = R_I(0*y) \succeq \mathrm{rmin}\{R_I(0*(x*y)), R_I(x)\} = \mathrm{rmin}\{R_I(x*y), R_I(x)\}$ (by (2.2), (3.42)), $G_I(y) = G_I(0*y) \succeq \mathrm{rmin}\{G_I(0*(x*y)), G_I(x)\} = \mathrm{rmin}\{G_I(x*y), G_I(x)\}$ (by (2.2), (3.43)), and $B_I(y) = B_I(0*y) \preceq \mathrm{rmax}\{B_I(0*(x*y)), B_I(x)\} = \mathrm{rmax}\{B_I(x*y), B_I(x)\}$ (by (2.2), (3.44)). Therefore, I is an IvPFUPf of $\mathcal{U}$. □

Example 3.26 : We know that I is an IvPFUPf of $\mathcal{U}$ from Example 3.6. Since $B_I(3*2) = [0.3, 0.5] \not\preceq 0.1, 0.3] = \mathrm{rmax}\{B_I(3*(2*4)), B_I(2)\}$, we obtain I is not an IvPFUPi of $\mathcal{U}$. This demonstrates that the converse of Theorem 3.25 is incorrect.

We omit the proof of Theorems 3.27, 3.29, 3.31, 3.33, 3.35, 3.36, and 3.37 because the proof is similar to Theorem 3.25.

Theorem 3.27 : *Every IvPFiUPf of $\mathcal{U}$ is an IvPFUPf.*

Example 3.28 : We know that I is an IvPFUPf of $\mathcal{U}$ from Example 3.6. Since $R_I(3*4) = [0, 0.3] \not\succeq [0.2, 0.4] = \text{rmin}\{R_I(3*(2*4)), R_I(3*2)\}$, we obtain I is not an IvPFiUPf of $\mathcal{U}$. This demonstrates that the converse of Theorem 3.27 is incorrect.

Theorem 3.29 : *Every IvPFiUPf of $\mathcal{U}$ is an IvPFUPi.*

Example 3.30 : We know that I is an IvPFUPi of $\mathcal{U}$ from Example 3.14. Since $B_I(3*2) = [0.1, 0.2] \not\preceq [0, 0.1] = \text{rmax}\{B_I(3*(3*2)), B_I(3*3)\}$, we obtain I is not an IvPFiUPf of $\mathcal{U}$. This demonstrates that the converse of Theorem 3.29 is incorrect.

Theorem 3.31 : *Every IvPFcUPf of $\mathcal{U}$ is an IvPFUPf.*

Example 3.32 : We know that I is an IvPFUPf of $\mathcal{U}$ from Example 3.6. Since $G_I(3) = [0, 0.2] \not\succeq [0.3, 0.4] = \text{rmin}\{G_I(0*((3*4)*3)), G_I(0)\}$, we obtain I is not an IvPFcUPf of $\mathcal{U}$. This demonstrates that the converse of Theorem 3.33 is incorrect.

Theorem 3.33 : *Every IvPFsUPf of $\mathcal{U}$ is an IvPFUPf.*

Example 3.34 : We know that I is an IvPFUPf of $\mathcal{U}$ from Example 3.6. Since $G_I(((1*0)*2)*1) = [0.1, 0.3] \not\succeq [0.3, 0.4] = \text{rmin}\{G_I(0*(2*1)), G_I(0)\}$, we obtain I is not an IvPFsUPf of $\mathcal{U}$. This demonstrates that the converse of Theorem 3.33 is incorrect.

Theorem 3.35 : *Every IvPFsUPi of $\mathcal{U}$ is an IvPFiUPf.*

Theorem 3.36 : *Every IvPFsUPi of $\mathcal{U}$ is an IvPFcUPf.*

Theorem 3.37 : *Every IvPFsUPi of $\mathcal{U}$ is an IvPFsUPf.*

According to Theorem Theorems 3.20, 3.23, 3.25, 3.27, 3.29, 3.31, 3.33, 3.35, 3.36, and 3.37 and Examples 3.21, 3.24, 3.26, 3.28, 3.30, 3.32, and 3.34, the notion of IvPFUPss is a generalization of IvPFnUPfs, IvPFnUPfs is a generalization of IvPFUPfs, IvPFUPfs is a generalization of IvPFcUPfs, IvPFUPfs is a generalization of IvPFsUPfs, IvPFUPfs is a generalization of IvPFUPis, IvPFUPis is a generalization of IvPFiUPfs, and IvPFiUPfs, IvPFcUPfs, and IvPFsUPfs is a generalization of IvPFsUPis. Moreover, by Theorem 3.19, we obtain that IvPFsUPis and constant IvPFS coincide.

Theorem 3.38 : *If I is an IvPFUPs of $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x, y \in \mathcal{U})\left(x * y \neq 0 \Rightarrow \begin{cases} R_I(x) \succeq R_I(y) \\ G_I(x) \succeq G_I(y) \\ B_I(x) \preceq B_I(y) \end{cases} \right), \tag{3.48}$$

then I is an IvPFnUPf of $\mathcal{U}$.

Proof : Suppose that *I* is an IvPFUPs of $\mathcal{U}$ that meets (3.48). Let $x, y \in \mathcal{U}$. If $x * y = 0$, then $R_I(x * y) = R_I(0) \succeq R_I(y)$ (by (3.27)), $G_I(x * y) = G_I(0) \succeq G_I(y)$ (by (3.28)), and $B_I(x * y) = B_I(0) \preceq B_I(y)$ (by (3.29)). If $x * y \neq 0$, then $R_I(x * y) \succeq \text{rmin}\{R_I(x), R_I(y)\} = R_I(y)$ (by (3.21), (3.48) for R_I, (3.14)), $G_I(x * y) \succeq \text{rmin}\{G_I(x), G_I(y)\} = G_I(y)$ (by (3.22), (3.48) for G_I, (3.14)), and $B_I(x * y) \preceq \text{rmax}\{B_I(x), B_I(y)\} = B_I(y)$ (by (3.23), (3.48) for B_I, (3.13)). Therefore, *I* is an IvPFnUPf of $\mathcal{U}$. □

Theorem 3.39 : *If I is an IvPFnUPf of $\mathcal{U}$ that meets the following:*

$$\begin{pmatrix} R_I(0) = G_I(0) = B_I(0) \\ R_I \succeq B_I \\ G_I \succeq B_I \end{pmatrix}, \tag{3.49}$$

then I is an IvPFsUPi of $\mathcal{U}$.

Proof : Suppose that *I* is an IvPFnUPf of $\mathcal{U}$ that meets (3.49). By Theorem 3.22, we obtain *I* fulfills the requirements (3.27), (3.28), and (3.29). Let $x \in \mathcal{U}$. Then $R_I(0) \succeq R_I(x) \succeq B_I(x) \succeq B_I(0) = R_I(0)$ (by (3.27), (3.49), (3.29)), $G_I(0) \succeq G_I(x) \succeq B_I(x) \succeq B_I(0) = G_I(0)$ (by (3.28), (3.49), (3.29)), and $B_I(0) \preceq B_I(x) \preceq R_I(x) \preceq R_I(0) = B_I(0)$ (by (3.29), (3.49), (3.27)). Thus $R_I(0) = R_I(x)$, $G_I(0) = G_I(x)$, and $B_I(0) = B_I(x)$, so *I* is constant. By Theorem 3.19, we obtain *I* is an IvPFsUPi of $\mathcal{U}$. □

Theorem 3.40 : *If I is an IvPFUPf of $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x, y, z \in \mathcal{U})\begin{pmatrix} R_I(y * (x * z)) \succeq R_I(y) \\ G_I(y * (x * z)) \succeq G_I(y) \\ B_I(y * (x * z)) \preceq B_I(y) \end{pmatrix}, \tag{3.50}$$

then I is an IvPFUPi of $\mathcal{U}$.

Proof : Suppose that *I* is an IvPFUPf of $\mathcal{U}$ that meets (3.50). Then *I* fulfills the requirements (3.27), (3.28), and (3.29). Let $x, y, z \in \mathcal{U}$. Then $R_I(x * z) \succeq$

$\text{rmin}\{R_I(y*(x*z)),\ R_I(y)\} = R_I(y) \succeq \text{rmin}\{R_I(x*(y*z)), R_I(y)\}$ (by (3.20), (3.50) for R_I, (3.14), (3.7)), $G_I(x*z) \succeq \text{rmin}\{G_I(y*(x*z)),\ G_I(y)\} = G_I(y) \succeq \text{rmin}\{G_I(x*(y*z)),\ G_I(y)\}$ (by (3.31), (3.50) for G_I, (3.14), (3.7)), and $B_I(x*z) \preceq \text{rmax}\{B_I(y*(x*z)),\ B_I(y)\} = B_I(y) \preceq \text{rmax}\{B_I(x*(y*z)),\ B_I(y)\}$ (by (3.32), (3.50) for B_I, (3.13), (3.6)). Therefore, I is an IvPFUPi of $\mathcal{U}$. □

Theorem 3.41 : *If I is an IvPFS in $\mathcal{U}$ that meets (3.30), (3.31), and (3.32) and the following:*

$$(\text{all}\, x,y,z \in \mathcal{U})\begin{pmatrix} R_I((x*(y*z))*(x*z)) \succeq R_I(x*(y*z)) \\ G_I((x*(y*z))*(x*z)) \succeq G_I(x*(y*z)) \\ B_I((x*(y*z))*(x*z)) \preceq B_I(x*(y*z)) \end{pmatrix}, \tag{3.51}$$

then I is an IvPFUPi of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.30), (3.31), (3.32), and (3.51). If $x=0, y=0$, and by (3.51), we obtain I fulfills the requirements (3.27), (3.28), and (3.29). Let $x,y,z \in \mathcal{U}$. Then $R_I(x*z) \succeq \text{rmin}\{R_I((x*(y*z))*(x*z)),\ R_I(x*(y*z))\} = R_I(x*(y*z)) \succeq \text{rmin}\{R_I(x*(y*z)),\ R_I(y)\}$ (by (3.30), (3.51) for R_I, (3.14), (3.7)), $G_I(x*z) \succeq \text{rmin}\{G_I((x*(y*z))*(x*z)),\ G_I(x*(y*z))\} = G_I(x*(y*z)) \succeq \text{rmin}\{G_I(x*(y*z)),\ G_I(y)\}$ (by (3.31), (3.51) for G_I, (3.14), (3.7)), and $B_I(x*z) \preceq \text{rmax}\{B_I((x*(y*z))*(x*z)),\ B_I(x*(y*z))\} = B_I(x*(y*z)) \preceq \text{rmax}\{B_I(x*(y*z)),\ B_I(y)\}$ (by (3.32), (3.51) for B_I, (3.13), (3.6)). Therefore, I is an IvPFUPi of $\mathcal{U}$. □

Theorem 3.42 : *If I is an IvPFUPi of $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x,y,z \in \mathcal{U})\begin{pmatrix} R_I(y) \succeq R_I(x*(y*z)) \\ G_I(y) \succeq G_I(x*(y*z)) \\ B_I(y) \preceq B_I(x*(y*z)) \end{pmatrix}, \tag{3.52}$$

then I is an IvPFiUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFUPi of $\mathcal{U}$ that meets (3.52). Then I fulfills the requirements (3.27), (3.28), and (3.29). Let $x,y,z \in \mathcal{U}$. Then $R_I(x*z) \succeq \text{rmin}\{R_I(x*(y*z)),\ R_I(y)\} = R_I(x*(y*z)) \succeq \text{rmin}\{R_I(x*(y*z)),\ R_I(x*y)\}$ (by (3.42), (3.52) for R_I, (3.14), (3.7)), $G_I(x*z) \succeq \text{rmin}\{G_I(x*(y*z)),\ G_I(y)\} = G_I(x*(y*z)) \succeq \text{rmin}\{G_I(x*(y*z)),\ G_I(x*y)\}$ (by (3.43), (3.52) for G_I, (3.14), (3.7)), and $B_I(x*z) \preceq \text{rmax}\{B_I(x*(y*z)),\ B_I(y)\} = B_I(x*(y*z)) \preceq \text{rmax}\{B_I(x*(y*z)),\ B_I(x*y)\}$ (by (3.44), (3.52) for B_I, (3.13), (3.6)). Therefore, I is an IvPFiUPf of $\mathcal{U}$. □

Theorem 3.43 : *If I is an IvPFUPi of $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x,y,z \in \mathcal{U})\begin{pmatrix} R_I(x*((x*y)*z)) \succeq R_I(x*y) \\ G_I(x*((x*y)*z)) \succeq G_I(x*y) \\ B_I(x*((x*y)*z)) \preceq B_I(x*y) \end{pmatrix}, \tag{3.53}$$

then I is an IvPFiUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFUPi of $\mathcal{U}$ that meets (3.53). Then I fulfills the requirements (3.27), (3.28), and (3.29). Let $x,y,z \in \mathcal{U}$. Then $R_I(x*z) \succeq \text{rmin}\{R_I(x*((x*y)*z)), R_I(x*y)\} = R_I(x*y) \succeq \text{rmin}\{R_I(x*(y*z)), R_I(x*y)\}$ (by (3.42), (3.53) for R_I, (3.14), (3.7)), $G_I(x*z) \succeq \text{rmin}\{G_I(x*((x*y)*z)), G_I(x*y)\} = G_I(x*y) \succeq \text{rmin}\{G_I(x*(y*z)), G_I(x*y)\}$ (by (3.43), (3.53) for G_I, (3.14), (3.7)), and $B_I(x*z) \preceq \text{rmax}\{B_I(x*((x*y)*z)), B_I(x*y)\} = B_I(x*y) \preceq \text{rmax}\{B_I(x*(y*z)), B_I(x*y)\}$ (by (3.44), (3.53) for B_I, (3.13), (3.6)). Therefore, I is an IvPFiUPf of $\mathcal{U}$. □

Theorem 3.44 : *If I is an IvPFUPf of $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x,y,z \in \mathcal{U})\begin{pmatrix} R_I(x*y) \succeq R_I(x) \\ G_I(x*y) \succeq G_I(x) \\ B_I(x*y) \preceq B_I(x) \end{pmatrix}, \tag{3.54}$$

then I is an IvPFcUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFUPf of $\mathcal{U}$ that meets (3.54). Then I fulfills the requirements (3.27), (3.28), and (3.29). Let $x,y,z \in \mathcal{U}$. Then $R_I(y) \succeq \text{rmin}\{R_I(x*y), R_I(x)\} = R_I(x) \succeq \text{rmin}\{R_I(x*((y*z)*y)), R_I(x)\}$ (by (3.30), (3.54) for R_I, (3.14), (3.7)), $G_I(y) \succeq \text{rmin}\{G_I(x*y), G_I(x)\} = G_I(x) \succeq \text{rmin}\{G_I(x*((y*z)*y)), G_I(x)\}$ (by (3.31), (3.54) for G_I, (3.14), (3.7)), and $B_I(y) \preceq \text{rmax}\{B_I(x*y), B_I(x)\} = B_I(x) \preceq \text{rmax}\{B_I(x*((y*z)*y)), B_I(x)\}$ (by (3.32), (3.54) for B_I, (3.13), (3.6)). Therefore, I is an IvPFcUPf of $\mathcal{U}$. □

Theorem 3.45 : *If I is an IvPFS in $\mathcal{U}$ that meets (3.30), (3.31), and (3.32) and the following:*

$$(\text{all}\, x,y,z \in \mathcal{U})\begin{pmatrix} R_I((x*((y*z)*y))*y) \succeq R_I(x*((y*z)*y)) \\ G_I((x*((y*z)*y))*y) \succeq G_I(x*((y*z)*y)) \\ B_I((x*((y*z)*y))*y) \preceq B_I(x*((y*z)*y)) \end{pmatrix}, \tag{3.55}$$

then I is an IvPFcUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.30), (3.31), (3.32), and (3.55). If $x = 0, y = z,$ and by (3.55), we obtain I fulfills the requirements (3.27), (3.28), and (3.29). Let $x, y, z \in \mathcal{U}$. Then $R_I(y) \succeq \text{rmin}\{R_I((x * ((y*z)*y))*y),\ R_I(x*\ ((y*z)*y))\} = R_I(x*\ ((y*z)*y)) \succeq \text{rmin}\{R_I(x* ((y*z)*y)),\ R_I(x)\}$ (by (3.30), (3.55) for R_I, (3.14), (3.7)), $G_I(y) \succeq \text{rmin}\{G_I((x*\ ((y*z)*y))*y),\ G_I(x*\ ((y*z)*y))\} = G_I(x*\ ((y*z)*y)) \succeq \text{rmin}\{G_I(x*\ ((y*z)*y)),\ G_I(x)\}$ (by (3.31), (3.55) for G_I, (3.14), (3.7)), and $B_I(y) \preceq \text{rmax}\{B_I((x*\ ((y*z)*y))*y),\ B_I(x*\ ((y*z)*y))\} = B_I(x* ((y*z)*y)) \preceq \text{rmax}\{B_I(x*\ ((y*z)*y)),\ B_I(x)\}$ (by (3.32), (3.55) for B_I, (3.13), (3.6)). Therefore, I is an IvPFcUPf of $\mathcal{U}$. □

Theorem 3.46 : *If I is an IvPFUPf of $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x, y, z \in \mathcal{U}) \begin{pmatrix} R_I(x*(((z*y)*y)*z)) \succeq R_I(x) \\ G_I(x*(((z*y)*y)*z)) \succeq G_I(x) \\ B_I(x*(((z*y)*y)*z)) \preceq B_I(x) \end{pmatrix}, \tag{3.56}$$

then I is an IvPFsUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFUPf of $\mathcal{U}$ that meets (3.56). Then I fulfills the requirements (3.27), (3.28), and (3.29). Let $x, y, z \in \mathcal{U}$. Then $R_I(((z*y)*y)*z) \succeq \text{rmin}\{R_I(x*\ (((z*y)*y)*z),\ R_I(x)\} = R_I(x) \succeq \text{rmin}\{R_I(x*\ (y*z)), R_I(x)\}$ (by (3.30), (3.56) for R_I, (3.14), (3.7)), $G_I(((z*y)*y)*z) \succeq \text{rmin}\{G_I(x*\ (((z*y)*y)*z)),\ G_I(x)\} = G_I(x) \succeq \text{rmin}\{G_I(x*(y*z)),\ G_I(x)\}$ (by (3.31), (3.56) for G_I, (3.14), (3.7)), and $B_I(((z*y)*y)*z) \preceq \text{rmax}\{B_I(x*\ (((z*y)*y)*z),\ B_I(x)\} = B_I(x) \preceq \text{rmax}\{B_I(x*\ (y*z)), B_I(x)\}$ (by (3.32), (3.56) for B_I, (3.13), (3.6)). Therefore, I is an IvPFsUPf of $\mathcal{U}$. □

Theorem 3.47 : *If I is an IvPFS in $\mathcal{U}$ that meets (3.30), (3.31), and (3.32) and the following:*

$$(\text{all}\, x, y, z \in \mathcal{U}) \begin{pmatrix} R_I((x*(y*z))*(((z*y)*y)*z) \succeq R_I(x*(y*z)) \\ G_I((x*(y*z))*(((z*y)*y)*z) \succeq G_I(x*(y*z)) \\ B_I((x*(y*z))*(((z*y)*y)*z) \preceq B_I(x*(y*z)) \end{pmatrix}, \tag{3.57}$$

then I is an IvPFsUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.30), (3.31), (3.32), and (3.57). If $x = 0, y = 0,$ and by (3.57), we obtain I fulfills the requirements (3.27), (3.28), and (3.29). Let $x, y, z \in \mathcal{U}$. Then $R_I(((z*y)*y)*z) \succeq$

$\text{rmin}\{R_I((x*(y*z))* (((z*y)*y)*z), R_I(x*(y*z))\} = R_I(x*(y*z)) \succeq \text{rmin}\{R_I(x* (y*z)), R_I(x)\}$ (by (3.30), (3.57) for R_I, (3.14), (3.17)), $G_I(((z*y)*y)*z) \succeq \text{rmin}\{G_I((x*(y*z))* (((z*y)*y)*z)), G_I(x*(y*z))\} = G_I(x*(y*z)) \succeq \text{rmin}\{G_I(x*(y*z)), G_I(x)\}$ (by (3.31), (3.57) for G_I, (3.14), (3.7)), and $B_I(((z*y)*y)*z) \preceq \text{rmax}\{B_I((x*(y*z))* (((z*y)*y)*z), B_I(x*(y*z))\} = B_I(x*(y*z)) \preceq \text{rmax}\{B_I(x*(y*z)), B_I(x)\}$ (by (3.32), (3.57) for B_I, (3.13), (3.6)). Therefore, I is an IvPFsUPf of $\mathcal{U}$. □

Theorem 3.48 : *If I is an IvPFS in $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x, y, z \in \mathcal{U})\left(z \leqslant x*y \Rightarrow \begin{cases} R_I(z) \succeq \text{rmin}\{R_I(x), R_I(y)\} \\ G_I(z) \succeq \text{rmin}\{G_I(x), G_I(y)\} \\ B_I(z) \preceq \text{rmax}\{B_I(x), B_I(y)\} \end{cases} \right), \tag{3.58}$$

then I is an IvPFUPs of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.58). Let $x, y \in \mathcal{U}$. By (2.6), we obtain $(x*y)*(x*y) = 0$. Thus $x*y \leqslant x*y$. From (3.58), $R_I(x*y) \succeq \text{rmin}\{R_I(x), R_I(y)\}$, $G_I(x*y) \succeq \text{rmin}\{G_I(x), G_I(y)\}$, and $B_I(x*y) \preceq \text{rmax}\{B_I(x), B_I(y)\}$. Therefore, I is an IvPFUPs of $\mathcal{U}$. □

Theorem 3.49 : *If I is an IvPFS in $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x, y, z \in \mathcal{U})\left(z \leqslant x*y \Rightarrow \begin{cases} R_I(y) \succeq \text{rmin}\{R_I(z), R_I(x)\} \\ G_I(y) \succeq \text{rmin}\{G_I(z), G_I(x)\} \\ B_I(y) \preceq \text{rmax}\{B_I(z), B_I(x)\} \end{cases} \right), \tag{3.59}$$

then I is an IvPFUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.59). Let $x, y \in \mathcal{U}$. By (2.3), we obtain $x*(x*0) = 0$. Thus $x \leqslant x*0$. From (3.59), $R_I(0) \succeq \text{rmin}\{R_I(x), R_I(x)\} = R_I(x)$ (by (3.3)), $G_I(0) \succeq \text{rmin}\{G_I(x), G_I(x)\} = G_I(x)$ (by (3.3)), and $B_I(0) \preceq \text{rmax}\{B_I(x), B_I(x)\} = B_I(x)$ (by (3.2)). Let $x, y \in \mathcal{U}$. By (2.6), we obtain $(x*y)*(x*y) = 0$. Thus $x*y \leqslant x*y$. From (3.59), $R_I(y) \succeq \text{rmin}\{R_I(x*y), R_I(x)\}$, $G_I(y) \succeq \text{rmin}\{G_I(x*y), G_I(x)\}$, and $B_I(y) \preceq \text{rmax}\{B_I(x*y), B_I(x)\}$. Therefore, I is an IvPFUPf of $\mathcal{U}$. □

Theorem 3.50 : *If I is an IvPFS in $\mathcal{U}$ that meets the following:*

$$(\text{all}\, a,x,y,z \in \mathcal{U})\left(a \leqslant x*(y*z) \Rightarrow \begin{cases} R_I(x*z) \succeq \text{rmin}\{R_I(a), R_I(y)\} \\ G_I(x*z) \succeq \text{rmin}\{G_I(a), G_I(y)\} \\ B_I(x*z) \preceq \text{rmax}\{B_I(a), B_I(y)\} \end{cases} \right), \tag{3.60}$$

then I is an IvPFUPi of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.60). Let $x \in \mathcal{U}$. By (2.3), we obtain $x*(0*(x*0) = 0$. Thus $x \leqslant 0*(x*0)$. From (3.60), $R_I(0) = R_I(0*0) \succeq \text{rmin}\{R_I(x), R_I(x)\} = R_I(x)$ (by (2.2), (3.3)), $G_I(0) = G_I(0*0) \succeq \text{rmin}\{G_I(x), G_I(x)\} = G_I(x)$ (by (2.2), (3.3)), and $B_I(0) = B_I(0*0) \preceq \text{rmax}\{B_I(x), B_I(x)\} = B_I(x)$ (by (2.2), (3.2)). Let $x,y,z \in \mathcal{U}$. By (2.6), we obtain $x*(0*(x*0)) = 0$. Thus $x*(y*z) \leqslant x*(y*z)$. From (3.60), $R_I(x*z) \succeq \text{rmin}\{R_I(a), R_I(y)\}$, $G_I(x*z) \succeq \text{rmin}\{G_I(a), G_I(y)\}$, and $B_I(x*z) \preceq \text{rmax}\{B_I(a), B_I(y)\}$. Therefore, I is an IvPFUPi of $\mathcal{U}$. □

Theorem 3.51 : *If I is an IvPFS in* $\mathcal{U}$ *that meets the following:*

$$(\text{all}\, a,x,y,z \in \mathcal{U})\left(a \leqslant x*(y*z) \Rightarrow \begin{cases} R_I(x*z) \succeq \text{rmin}\{R_I(a), R_I(x*y)\} \\ G_I(x*z) \succeq \text{rmin}\{G_I(a), G_I(x*y)\} \\ B_I(x*z) \preceq \text{rmax}\{B_I(a), B_I(x*y)\} \end{cases} \right), \tag{3.61}$$

then I is an IvPFiUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.61). Let $x \in \mathcal{U}$. By (2.3), we obtain $x*(0*(x*0)) = 0$. Thus $x \leqslant 0*(x*0)$. From (3.61), $R_I(0) = R_I(0*0) \succeq \text{rmin}\{R_I(x), R_I(0*x)\} = \text{rmin}\{R_I(x), R_I(x)\} = R_I(x)$ (by (2.2), (3.3)), $G_I(0) = G_I(0*0) \succeq \text{rmin}\{G_I(x), G_I(0*x)\} = \text{rmin}\{G_I(x), G_I(x)\} = G_I(x)$ (by (2.2), (3.3)), and $B_I(0) = B_I(0*0) \preceq \text{rmax}\{B_I(x), B_I(0*x)\} = \text{rmax}\{B_I(x), B_I(x)\} = B_I(x)$ (by (2.2), (3.2)). Let $x,y,z \in \mathcal{U}$. By (2.6), we obtain $(x*(y*z))*(x*(y*z)) = 0$. Thus $x*(y*z) \leqslant x*(y*z)$. From (3.61), $R_I(x*z) \succeq \text{rmin}\{R_I(x*(y*z)), R_I(x*y)\}$, $G_I(x*z) \succeq \text{rmin}\{G_I(x*(y*z)), G_I(x*y)\}$, and $B_I(x*z) \preceq \text{rmax}\{B_I(x*(y*z)), B_I(x*y)\}$. Therefore, I is an IvPFiUPf of $\mathcal{U}$. □

Theorem 3.52 : *If I is an IvPFS in* $\mathcal{U}$ *that meets the following:*

$$(\text{all}\, a,x,y,z \in \mathcal{U})\left(a \leqslant x*((y*z)*y) \Rightarrow \begin{cases} R_I(y) \succeq \text{rmin}\{R_I(a), R_I(x)\} \\ G_I(y) \succeq \text{rmin}\{G_I(a), G_I(x)\} \\ B_I(y) \preceq \text{rmax}\{B_I(a), B_I(x)\} \end{cases} \right), \tag{3.60}$$

then I is an IvPFcUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.62). Let $x \in \mathcal{U}$. By (2.3), we obtain $x*(x*((0*x)*0)) = 0$. Thus $x \leqslant x*((0*x)*0)$. From (3.62), $R_I(0) \succeq \text{rmin}\{R_I(x), R_I(x)\} = R_I(x)$ (by (3.3)), $G_I(0) \succeq \text{rmin}\{G_I(x), G_I(x)\} = G_I(x)$ (by (3.3)), and $B_I(0) \preceq \text{rmax}\{B_I(x), B_I(x)\} = B_I(x)$ (by (3.2)). Let $x, y, z \in \mathcal{U}$. By (2.6), we obtain $(x*((y*z)*y))*(x*((y*z)*y)) = 0$. Thus $x*((y*z)*y) \leqslant x*((y*z)*y)$. From (3.62), $R_I(y) \succeq \text{rmin}\{R_I(x*((y*z)*y)), R_I(x)\}$, $G_I(y) \succeq \text{rmin}\{G_I(x*((y*z)*y)), G_I(x)\}$, and $B_I(y) \preceq \text{rmax}\{B_I(x*((y*z)*y)), B_I(x)\}$. Therefore, I is an IvPFcUPf of $\mathcal{U}$. □

Theorem 3.53 : *If P is an IvPFS in $\mathcal{U}$ that meets the following:*

$$(\text{all}\, a, x, y, z \in \mathcal{U})\left(a \leqslant x*(y*z) \Rightarrow \begin{cases} R_I(((z*y)*y)*z) \succeq \text{rmin}\{R_I(a), R_I(x)\} \\ G_I(((z*y)*y)*z) \succeq \text{rmin}\{G_I(a), G_I(x)\} \\ B_I(((z*y)*y)*z) \preceq \text{rmax}\{B_I(a), B_I(x)\} \end{cases} \right), \tag{3.63}$$

then I is an IvPFsUPf of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.63). Let $x \in \mathcal{U}$. By (2.3), we obtain $x*(x*(x*0)) = 0$. Thus $x \leqslant x*(x*0)$. From (3.63), $R_I(0) = R_I(((0*x)*x)*0) \succeq \text{rmin}\{R_I(x), R_I(x)\} = R_I(x)$ (by (2.3), (3.3)), $G_I(0) = G_I(((0*x)*x)*0) \succeq \text{rmin}\{G_I(x), G_I(x)\} = G_I(x)$ (by (2.3), (3.3)), and $B_I(0) = B_I(((0*x)*x)*0) \preceq \text{rmax}\{B_I(x), B_I(x)\} = B_I(x)$ (by (2.3), (3.2)). Let $x, y, z \in \mathcal{U}$. By (2.6), we obtain $(x*(y*z))*(x*(y*z)) = 0$. Thus $x*(y*z) \leqslant x*(y*z)$. From (3.63), $R_I(((z*y)*y)*z) \succeq \text{rmin}\{R_I(x*(y*z)), R_I(x)\}$, $G_I(((z*y)*y)*z) \succeq \text{rmin}\{G_I(x*(y*z)), G_I(x)\}$, and $B_I(((z*y)*y)*z) \preceq \text{rmax}\{B_I(x*(y*z)), B_I(x)\}$. Therefore, I is an IvPFsUPf of $\mathcal{U}$. □

Theorem 3.54 : *If I is an IvPFS in $\mathcal{U}$ that meets the following:*

$$(\text{all}\, x, y, z \in \mathcal{U})\left(z \leqslant x*y \Rightarrow \begin{cases} R_I(z) \succeq R_I(y) \\ G_I(z) \succeq G_I(y) \\ B_I(z) \preceq B_I(y) \end{cases} \right), \tag{3.64}$$

then I is an IvPFsUPi of $\mathcal{U}$.

Proof : Suppose that I is an IvPFS in $\mathcal{U}$ that meets (3.64). Let $x, y \in \mathcal{U}$. By (2.3) and (2.6), we obtain $x*(y*y) = 0$. Thus $x \leqslant y*y$. From (3.64), $R_I(x) \succeq R_I(y)$, $G_I(x) \succeq G_I(y)$, and $B_I(x) \preceq B_I(y)$. Similarly, $R_I(y) \succeq R_I(x)$, $G_I(y) \succeq G_I(x)$, and $B_I(y) \preceq B_I(x)$. Thus $R_I(x) = R_I(y)$, $G_I(x) = G_I(y)$, and

$B_I(x) = B_I(y)$, so I is constant. According to Theorem 3.19, I is an IvPFsUPi of $\mathcal{U}$. $\square$

In this section, we define an IvPFS similarly to a characteristic function and explore its characterizations with the help of the associated subset.

Let $\tilde{r}^+, \tilde{r}^-, \tilde{g}^+, \tilde{g}^-, \tilde{b}^+, \tilde{b}^- \in \mathrm{int}[0,1]$ be such that $\tilde{r}^+ \succ \tilde{r}^-, \tilde{g}^+ \succ \tilde{g}^-, \tilde{b}^+ \succ \tilde{b}^-$ and $\varnothing \neq S \subseteq \mathcal{U}$, an IvPFS $I^S[{}^{\tilde{r}^+,\tilde{g}^+,\tilde{b}^-}_{\tilde{r}^-,\tilde{g}^-,\tilde{b}^+}] = (\mathcal{U}, R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}], G_I^S[{}^{\tilde{g}^+}_{\tilde{g}^-}], B_I^S[{}^{\tilde{b}^-}_{\tilde{b}^+}])$ in $\mathcal{U}$, where $R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}], G_I^S[{}^{\tilde{g}^+}_{\tilde{g}^-}]$, and $B_I^S[{}^{\tilde{b}^-}_{\tilde{b}^+}]$ are functions on $\mathcal{U}$ which are listed below:

$$R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](x) = \begin{cases} \tilde{r}^+ & \text{if } x \in S, \\ \tilde{r}^- & \text{otherwise,} \end{cases}, \quad G_I^S[{}^{\tilde{g}^+}_{\tilde{g}^-}](x) = \begin{cases} \tilde{g}^+ & \text{if } x \in S, \\ \tilde{g}^- & \text{otherwise,} \end{cases},$$

$$B_I^S[{}^{\tilde{b}^-}_{\tilde{b}^+}](x) = \begin{cases} \tilde{b}^- & \text{if } x \in S, \\ \tilde{b}+ & \text{otherwise.} \end{cases}$$

Lemma 3.55 : *If $0 \in S \subseteq \mathcal{U}$, then the IvPFS $I^S[{}^{\tilde{r}^+,\tilde{g}^+,\tilde{b}^-}_{\tilde{r}^-,\tilde{g}^-,\tilde{b}^+}]$ in $\mathcal{U}$ fulfills the requirements (3.27), (3.28), and (3.29).*

Proof : If $0 \in S$, then $R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](0) = \tilde{r}^+$, $G_I^S[{}^{\tilde{g}^+}_{\tilde{g}^-}](0) = \tilde{g}^+$, $B_I^S[{}^{\tilde{b}^-}_{\tilde{b}^+}](0) = \tilde{b}^-$. Thus for all $x \in \mathcal{U}$, $R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](0) = \tilde{r}^+ \succeq R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](x)$, $G_I^S[{}^{\tilde{g}^+}_{\tilde{g}^-}](0) = \tilde{g}^+ \succeq G_I^S[{}^{\tilde{g}^+}_{\tilde{g}^-}](x)$, and $B_I^S[{}^{\tilde{b}^-}_{\tilde{b}^+}](0) = \tilde{b}^- \preceq B_I^S[{}^{\tilde{b}^-}_{\tilde{b}^+}](x)$. Therefore, $I^S[{}^{\tilde{r}^+,\tilde{g}^+,\tilde{b}^-}_{\tilde{r}^-,\tilde{g}^-,\tilde{b}^+}]$ fulfills the requirements (3.27), (3.28), and (3.29). $\square$

Lemma 3.56 : *If the IvPFS $I^S[{}^{\tilde{r}^+,\tilde{g}^+,\tilde{b}^-}_{\tilde{r}^-,\tilde{g}^-,\tilde{b}^+}]$ in $\mathcal{U}$ satisfies (3.27) (resp., (3.28), (3.29)), then $0 \in S \subseteq \mathcal{U}$.*

Proof : Suppose that the IvPFS $I^S[{}^{\tilde{r}^+,\tilde{g}^+,\tilde{b}^-}_{\tilde{r}^-,\tilde{g}^-,\tilde{b}^+}]$ in $\mathcal{U}$ satisfies (3.27). Then $R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](0) \succeq R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](x)$ for all $x \in \mathcal{U}$. Since S is nonempty, there exists $g \in S$. Thus $R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](g) = \tilde{r}^+$ and so $R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](0) \succeq R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](g) = \tilde{r}^+ \succeq R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](0)$. Thus $R_I^S[{}^{\tilde{r}^+}_{\tilde{r}^-}](0) = \tilde{r}^+$. Therefore, $0 \in S$. $\square$

Theorem 3.57 : *The IvPFS $I^S[{}^{\tilde{r}^+,\tilde{g}^+,\tilde{b}^-}_{\tilde{r}^-,\tilde{g}^-,\tilde{b}^+}]$ in $\mathcal{U}$ is an IvPFUPs of $\mathcal{U}$ if and only if $\varnothing \neq S \subseteq \mathcal{U}$ is a UPs of $\mathcal{U}$.*

Proof : Suppose that $I^S[^{\tilde{r}^+, \tilde{g}^+, \tilde{b}^-}_{\tilde{r}^-, \tilde{g}^-, \tilde{b}^+}]$ is an IvPFUPs of $\mathcal{U}$. Let $x, y \in S$. Then $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x) = \tilde{r}^+ = R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y)$. Thus $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) \succeq \mathrm{rmin}\{R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x), R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y)\} = \mathrm{rmin}\{\tilde{r}^+, \tilde{r}^+\} = \tilde{r}^+ \succeq R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x*y)$ (by (3.21), (3.3)) and so $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) = \tilde{r}^+$. Thus $x*y \in S$. Therefore, S is a UPs of $\mathcal{U}$.

Conversely, suppose that S is a UPs of $\mathcal{U}$. Let $x, y \in \mathcal{U}$. If $x, y \in S$, then $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x) = \tilde{r}^+ = R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y)$, $G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x) = \tilde{g}^+ = G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](y)$, and $B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x) = \tilde{b}^- = B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](y)$. Thus $\mathrm{rmin}\{R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x), R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y)\} = \mathrm{rmin}\{\tilde{r}^+, \tilde{r}^+\} = \tilde{r}^+$ (by (3.3)), $\mathrm{rmin}\{G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x), G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](y)\} = \mathrm{rmin}\{\tilde{g}^+, \tilde{g}^+\} = \tilde{g}^+$ (by (3.3)), and $\mathrm{rmax}\{B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x), B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](y)\} = \mathrm{rmax}\{\tilde{b}^-, \tilde{b}^-\} = \tilde{b}^-$ (by (3.2)). Since S is a UPs of $\mathcal{U}$, we obtain $x*y \in S$ and so $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) = \tilde{r}^+$, $G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x*y) = \tilde{g}^+$, and $B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x*y) = \tilde{b}^-$. Therefore, $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) = \tilde{r}^+ \succeq \tilde{r}^+ = \mathrm{rmin}\{R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x), R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y)\}$, $G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x*y) = \tilde{g}^+ \succeq \tilde{g}^+ = \mathrm{rmin}\{G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x), G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](y)\}$, and $B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x*y) = \tilde{b}^- \preceq \tilde{b}^- = \mathrm{rmax}\{B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x), B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](y)\}$. If $x \notin S$ or $y \notin S$, then $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x) = \tilde{r}^-$ or $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y) = \tilde{r}^-$, $G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x) = \tilde{g}^-$ or $G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](y) = \tilde{g}^-$, and $B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x) = \tilde{b}^+$ or $B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](y) = \tilde{b}^+$. Thus $\mathrm{rmin}\{R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x), R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y)\} = \tilde{r}^-$ (by (3.14)), $\mathrm{rmin}\{G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x), G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](y)\} = \tilde{g}^-$ (by (3.14)), and $\mathrm{rmax}\{B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x), B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](y)\} = \tilde{b}^+$ (by (3.13)). Therefore, $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) \succeq \tilde{r}^- = \mathrm{rmin}\{R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x), R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y)\}$, $G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x*y) \succeq \tilde{g}^- = \mathrm{rmin}\{G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](x), G_I^S[^{\tilde{g}^+}_{\tilde{g}^-}](y)\}$, and $B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x*y) \preceq \tilde{b}^+ = \mathrm{rmax}\{B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](x), B_I^S[^{\tilde{b}^-}_{\tilde{b}^+}](y)\}$. Therefore, $I^S[^{\tilde{r}^+, \tilde{g}^+, \tilde{b}^-}_{\tilde{r}^-, \tilde{g}^-, \tilde{b}^+}]$ is an IvPFUPs of $\mathcal{U}$. $\square$

We omit the proof of Theorem 3.58 because the proof is similar to Theorem 3.57.

Theorem 3.58 : *The IvPFS $I^S[^{\tilde{r}^+, \tilde{g}^+, \tilde{b}^-}_{\tilde{r}^-, \tilde{g}^-, \tilde{b}^+}]$ in $\mathcal{U}$ is an IvPFnUPf of $\mathcal{U}$ if and only if $\varnothing \neq S \subseteq \mathcal{U}$ is a nUPf of $\mathcal{U}$.*

Theorem 3.59 : *The IvPFS $I^S[^{\tilde{r}^+, \tilde{g}^+, \tilde{b}^-}_{\tilde{r}^-, \tilde{g}^-, \tilde{b}^+}]$ in $\mathcal{U}$ is an IvPFUPf of $\mathcal{U}$ if and only if $\varnothing \neq S \subseteq \mathcal{U}$ is a UPf of $\mathcal{U}$.*

Proof : Suppose that $I^S[^{\tilde{r}^+, \tilde{g}^+, \tilde{b}^-}_{\tilde{r}^-, \tilde{g}^-, \tilde{b}^+}]$ is an IvPFUPf of $\mathcal{U}$. Since $I^S[^{\tilde{r}^+, \tilde{g}^+, \tilde{b}^-}_{\tilde{r}^-, \tilde{g}^-, \tilde{b}^+}]$ satisfies (3.27) and Lemma 3.56, we get $0 \in S$. Let $x, y \in \mathcal{U}$ be such that $x*y \in S$ and $x \in S$. Then $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) = \tilde{r}^+ = R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](x)$. Thus $R_I^S[^{\tilde{r}^+}_{\tilde{r}^-}](y) \succeq$

$\mathrm{rmin}\{R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x*y),\ R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x)\} = \mathrm{rmin}\{\tilde{r}^+, \tilde{r}^+\} = \tilde{r}^+ \succeq R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](y)$ (by (3.30), (3.3)) and so $R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](y) = \tilde{r}^+$. Thus $y \in S$. Therefore, S is a UPf of $\mathcal{U}$.

Conversely, suppose that S is a UPf of $\mathcal{U}$. Since $0 \in S$ and Lemma 3.55, we get $I^S[^{\tilde{r}^+,\ \tilde{g}^+,\ \tilde{b}^-}_{\tilde{r}^-,\ \tilde{g}^-,\ \tilde{b}^+}]$ fulfills the requirements (3.27), (3.28), and (3.29). Let $x, y \in \mathcal{U}$. If $x*y \in S$ and $x \in S$, then $R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) = \tilde{r}^+ = R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x)$, $G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x*y) = \tilde{g}^+ = G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x)$, and $B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x*y) = \tilde{b}^- = B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x)$. Thus $\mathrm{rmin}\{R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x*y),\ R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x)\} = \mathrm{rmin}\{\tilde{r}^+, \tilde{r}^+\} = \tilde{r}^+$ (by (3.3)), $\mathrm{rmin}\{G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x*y),\ G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x)\} = \mathrm{rmin}\{\tilde{g}^+, \tilde{g}^+\} = \tilde{g}^+$ (by (3.3)), and $\mathrm{rmax}\{B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x*y),\ B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x)\} = \mathrm{rmax}\{\tilde{b}^-, \tilde{b}^-\} = \tilde{b}^-$ (by (3.2)). Since S is a UPf of $\mathcal{U}$, we obtain $y \in S$ and so $R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](y) = \tilde{r}^+$, $G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](y) = \tilde{g}^+$, and $B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](y) = \tilde{b}^-$. Thus $R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](y) = \tilde{r}^+ \succeq \tilde{r}^+ = \mathrm{rmin}\{R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x*y),\ R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x)\}$, $G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](y) = \tilde{g}^+ \succeq \tilde{g}^+ = \mathrm{rmin}\{G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x*y),\ G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x)\}$, and $B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](y) = \tilde{b}^- \preceq \tilde{b}^- = \mathrm{rmax}\{B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x*y),\ B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x)\}$. If $x*y \notin S$ or $x \notin S$, then $R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x*y) = \tilde{r}^-$ or $R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x) = \tilde{r}^-$, $G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x*y) = \tilde{g}^-$ or $G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x) = \tilde{g}^-$, and $B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x*y) = \tilde{b}^+$ or $B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x) = \tilde{b}^+$. Thus $\mathrm{rmin}\{R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x*y),\ R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x)\} = \tilde{r}^-$ (by (3.14)), $\mathrm{rmin}\{G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x*y),\ G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x)\} = \tilde{g}^-$ (by (3.14)), and $\mathrm{rmax}\{B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x*y),\ B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x)\} = \tilde{b}^+$ (by (3.13)). Therefore, $R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](y) \succeq \tilde{r}^- = \mathrm{rmin}\{R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x*y),\ R^S_I[^{\tilde{r}^+}_{\tilde{r}^-}](x)\}$, $G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](y) \succeq \tilde{g}^- = \mathrm{rmin}\{G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x*y),\ G^S_I[^{\tilde{g}^+}_{\tilde{g}^-}](x)\}$, and $B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](y) \preceq \tilde{b}^+ = \mathrm{rmax}\{B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x*y),\ B^S_I[^{\tilde{b}^-}_{\tilde{b}^+}](x)\}$. Therefore, $I^S[^{\tilde{r}^+,\ \tilde{g}^+,\ \tilde{b}^-}_{\tilde{r}^-,\ \tilde{g}^-,\ \tilde{b}^+}]$ is an IvPFUPf of $\mathcal{U}$. $\square$

We omit the proof of Theorem 3.60 because the proof is similar to Theorem 3.59.

Theorem 3.60 : *The IvPFS $I^S[^{\tilde{r}^+,\ \tilde{g}^+,\ \tilde{b}^-}_{\tilde{r}^-,\ \tilde{g}^-,\ \tilde{b}^+}]$ in $\mathcal{U}$ is an IvPFiUPf (resp., IvPFcUPf, IvPFsUPf, IvPFUPi, IvPFsUPi) of $\mathcal{U}$ if and only if $\varnothing \neq S \subseteq \mathcal{U}$ is an iUPf (resp., cUPf, sUPf, UPi, sUPi) of $\mathcal{U}$.*

4. Level subsets of an IvPFS

The connections between IvPFUPss (resp., IvPFnUPfs, IvPFUPfs, IvPFiUPfs, IvPFcUPfs, IvPFsUPfs, IvPFUPis, and IvPFsUPis) of UP-algebras and their level subsets are discussed in this section.

Definition 4.1 : Let $f :\to \text{int}[0,1]$ and $\tilde{t} \in \text{int}[0,1]$. Level sets are defined as follows: $U(f;\tilde{t}) = \{x \in \mathcal{U} \mid f(x) \succeq \tilde{t}\}$, $U^{+}(f;\tilde{t}) = \{x \in \mathcal{U} \mid f(x) \succ \tilde{t}\}$, $L(f;\tilde{t}) = \{x \in \mathcal{U} \mid f(x) \preceq \tilde{t}\}$, $L^{-}(f;\tilde{t}) = \{x \in \mathcal{U} \mid f(x) \prec \tilde{t}\}$, and $E(f;\tilde{t}) = \{x \in \mathcal{U} \mid f(x) = \tilde{t}\}$.

Theorem 4.2 : *An IvPFS I in $\mathcal{U}$ is an IvPFUPs of $\mathcal{U}$ if and only if for each $\tilde{t} \in \text{int}[0,1]$, nonempty subsets $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, , and $L(B_I;\tilde{t})$ are UPss of $\mathcal{U}$.*

Proof : Suppose that I is an IvPFUPs of $\mathcal{U}$. Let $\tilde{t} \in \text{int}[0,1]$ be such that $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are nonempty. Let $x, y \in U(R_I;\tilde{t})$. Then $R_I(x) \succeq \tilde{t}$ and $R_I(y) \succeq \tilde{t}$. We get $R_I(x*y) \succeq \text{rmin}\{R_I(x), R_I(y)\} \succeq \tilde{t}$ by using (3.21) and (3.12). Thus $x*y \in U(R_I;\tilde{t})$. The case of G_I can be proved in the same way as R_I. Let $x, y \in L(B_I;\tilde{t})$. Then $B_I(x) \preceq \tilde{t}$ and $B_I(y) \preceq \tilde{t}$. We get $B_I(x*y) \preceq \text{rmax}\{B_I(x), B_I(y)\} \preceq \tilde{t}$ by using (3.23) and (3.11). Thus $x*y \in L(B_I;\tilde{t})$. Therefore, $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are UPss of $\mathcal{U}$.

Conversely, suppose that for all $\tilde{t} \in \text{int}[0,1]$, nonempty subsets $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are UPss of $\mathcal{U}$. Let $x, y \in \mathcal{U}$. Then $R_I(x)$, $R_I(y) \in \text{int}[0,1]$. Pick $\tilde{t} = \text{rmin}\{R_I(x), R_I(y)\}$. Thus $R_I(x) \succeq \tilde{t}$ and $R_I(y) \succeq \tilde{t}$, so $x, y \in U(R_I;\tilde{t}) \neq \varnothing$. We may deduce that $U(R_I;\tilde{t})$ is a UPs of $\mathcal{U}$, and hence $x*y \in U(R_I;\tilde{t})$. Thus $R_I(x*y) \succeq \tilde{t} = \text{rmin}\{R_I(x), R_I(y)\}$. The case of G_I can be proved in the same way as R_I. Let $x, y \in \mathcal{U}$. Then $B_I(x)$, $B_I(y) \in \text{int}[0,1]$. Pick $\tilde{t} = \text{rmax}\{B_I(x), B_I(y)\}$. Thus $B_I(x) \preceq \tilde{t}$ and $B_I(y) \preceq \tilde{t}$, so $x, y \in L(B_I;\tilde{t}) \neq \varnothing$. We may deduce that $L(B_I;\tilde{t})$ is a UPs of $\mathcal{U}$, and hence $x*y \in L(B_I;\tilde{t})$. Thus $B_I(x*y) \preceq \tilde{t} = \text{rmax}\{B_I(x), B_I(y)\}$. Therefore, I is an IvPFUPs of $\mathcal{U}$. □

We omit the proof of Theorems 4.3 and 4.4 because the proof is similar to Theorem 4.2.

Theorem 4.3 : *An IvPFS I in $\mathcal{U}$ is an IvPFnUPf of $\mathcal{U}$ if and only if for each $\tilde{t} \in \text{int}[0,1]$, nonempty subsets $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are nUPfs of $\mathcal{U}$.*

Theorem 4.4 : *If I is an IvPFnUPf of $\mathcal{U}$, then for each $\tilde{t} \in \text{int}[0,1]$, nonempty subsets $U^{+}(R_I;\tilde{t})$, $U^{+}(G_I;\tilde{t})$, and $L^{-}(B_I;\tilde{t})$ are nUPfs of $\mathcal{U}$.*

Theorem 4.5 : *An IvPFS I in $\mathcal{U}$ is an IvPFUPf of $\mathcal{U}$ if and only if for each $\tilde{t} \in \text{int}[0,1]$, nonempty subsets $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are UPfs of $\mathcal{U}$.*

Proof : Suppose that I is an IvPFUPf of $\mathcal{U}$. Let $\tilde{t} \in \text{int}[0,1]$ be such that $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are nonempty. Let $x \in U(R_I;\tilde{t})$. Then $R_I(x) \succeq \tilde{t}$. We get $R_I(0) \succeq R_I(x) \succeq \tilde{t}$ by using (3.27). Thus $0 \in U(R_I;\tilde{t})$. Let $x, y \in \mathcal{U}$ be such that $x * y \in U(R_I;\tilde{t})$ and $x \in U(R_I;\tilde{t})$. Then $R_I(x*y) \succeq \tilde{t}$ and $R_I(x) \succeq \tilde{t}$. We get $R_I(y) \succeq \text{rmin}\,\{R_I(x*y), R_I(x)\} \succeq \tilde{t}$ by using (3.30) and (3.12). Thus $y \in U(R_I;\tilde{t})$. The case of G_I can be proved in the same way as R_I. Let $x \in L(B_I;\tilde{t})$. Then $B_I(x) \preceq \tilde{t}$. We get $B_I(0) \preceq B_I(x) \preceq \tilde{t}$ by using (3.29). Thus $0 \in L(B_I;\tilde{t})$. Let $x, y \in \mathcal{U}$ be such that $x * y \in L(B_I;\tilde{t})$ and $x \in L(B_I;\tilde{t})$. Then $B_I(x*y) \preceq \tilde{t}$ and $B_I(x) \preceq \tilde{t}$. We get $B_I(y) \preceq \text{rmax}\,\{B_I(x*y), B_I(x)\} \preceq \tilde{t}$ by using (3.32) and (3.11). Thus $y \in L(B_I;\tilde{t})$. Therefore, $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are UPfs of $\mathcal{U}$.

Conversely, suppose that for all $\tilde{t} \in \text{int}[0,1]$, nonempty subsets $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are UPfs of $\mathcal{U}$. Let $x \in \mathcal{U}$. Then $R_I(x) \in \text{int}[0,1]$. Pick $\tilde{t} = R_I(x)$. Thus $R_I(x) \succeq \tilde{t}$, so $x \in U(R_I;\tilde{t}) \neq \emptyset$. We may deduce that $U(R_I;\tilde{t})$ is a UPf of $\mathcal{U}$, and hence $0 \in U(R_I;\tilde{t})$. Thus $R_I(0) \succeq \tilde{t} = R_I(x)$. Let $x, y \in \mathcal{U}$. Then $R_I(x*y), R_I(x) \in \text{int}[0,1]$. Pick $\tilde{t} = \text{rmin}\{R_I(x*y), R_I(x)\}$. Thus $R_I(x*y) \succeq \tilde{t}$ and $R_I(x) \succeq \tilde{t}$, so $x*y$, $x \in U(R_I;\tilde{t}) \neq \emptyset$. We may deduce that $U(R_I;\tilde{t})$ is a UPf of $\mathcal{U}$, and hence $y \in U(R_I;\tilde{t})$. Thus $R_I(y) \succeq \tilde{t} = \text{rmin}\,\{R_I(x*y), R_I(x)\}$. The case of G_I can be proved in the same way as R_I. Let $x \in \mathcal{U}$. Then $B_I(x) \in \text{int}[0,1]$. Pick $\tilde{t} = B_I(x)$. Thus $B_I(x) \preceq \tilde{t}$, so $x \in L(B_I;\tilde{t}) \neq \emptyset$. We may deduce that $L(B_I;\tilde{t})$ is a UPf of $\mathcal{U}$, and hence $0 \in L(B_I;\tilde{t})$. Thus $B_I(0) \preceq \tilde{t} = B_I(x)$. Let $x, y \in \mathcal{U}$. Then $B_I(x*y), B_I(x) \in \text{int}[0,1]$. Pick $\tilde{t} = \text{rmax}\{B_I(x*y), B_I(x)\}$. Thus $B_I(x*y) \preceq \tilde{t}$ and $B_I(x) \preceq \tilde{t}$, so $x*y$, $x \in L(B_I;\tilde{t}) \neq \emptyset$. We may deduce that $L(B_I;\tilde{t})$ is a UPf of $\mathcal{U}$, and hence $y \in L(B_I;\tilde{t})$. Thus $B_I(y) \preceq \tilde{t} = \text{rmax}\{B_I(x*y), B_I(x)\}$. Therefore, I is an IvPFUPf of $\mathcal{U}$. □

We omit the proof of Theorem 4.6 because the proof is similar to Theorem 4.5.

Theorem 4.6 : *An IvPFS I in $\mathcal{U}$ is an IvPFiUPf (resp., IvPFcUPf, IvPFsUPf, IvPFUPi) of $\mathcal{U}$ if and only if for each $\tilde{t} \in \text{int}[0,1]$, nonempty subsets $U(R_I;\tilde{t})$, $U(G_I;\tilde{t})$, and $L(B_I;\tilde{t})$ are iUPfs (resp., cUPfs, sUPfs, UPis) of $\mathcal{U}$.*

Theorem 4.7 : *An IvPFS I in $\mathcal{U}$ is an IvPFsUPi of $\mathcal{U}$ if and only if $E(R_I;R_I(0))$, $E(G_I;G_I(0))$, and $E(B_I;B_I(0))$ are sUPis of $\mathcal{U}$.*

Proof : The proof is a direct result of Theorem 3.19. □

Definiton 4.8 : Let I be an IvPFS in $\mathcal{U}$ and $\tilde{r}, \tilde{g}, \tilde{b} \in \text{int}[0,1]$. Level sets are defined as follows: $UUL_I(\tilde{r}, \tilde{g}, \tilde{b}) = \{x \in \mathcal{U} \mid R_I(x) \succeq \tilde{r}, G_I(x) \succeq \tilde{g}, B_I(x) \preceq \tilde{b}\}$, $LLU_I(\tilde{r}, \tilde{g}, \tilde{b}) = \{x \in \mathcal{U} \mid R_I(x) \preceq \tilde{r}, G_I(x) \preceq \tilde{g}, B_I(x) \succeq \tilde{b}\}$, and $E_I(\tilde{r}, \tilde{g}, \tilde{b}) = \{x \in \mathcal{U} \mid R_I(x) = \tilde{r}, G_I(x) = \tilde{g}, B_I(x) = \tilde{b}\}$. Then $UUL_I(\tilde{r}, \tilde{g}, \tilde{b}) = U(R_I; \tilde{r}) \cap U(G_I; \tilde{g}) \cap L(B_I; \tilde{b})$, $LLU_I(\tilde{r}, \tilde{g}, \tilde{b}) = L(R_I; \tilde{r}) \cap L(G_I; \tilde{g}) \cap U(B_I; \tilde{b})$, and $E_I(\tilde{r}, \tilde{g}, \tilde{b}) = E(R_I; \tilde{r}) \cap E(G_I; \tilde{g}) \cap E(B_I; \tilde{b})$.

By Theorems 4.2, 4.3, 4.5, 4.6, 4.7, and 2.3, we obtain the following corollary.

An IvPFS I in $\mathcal{U}$ is an IvPFUPs (resp., IvPFnUPf, IvPFUPf, IvPFiUPf, IvPFcUPf, IvPFsUPf, IvPFUPi, IvPFsUPi) of $\mathcal{U}$ if and only if for each $\tilde{r}, \tilde{g}, \tilde{b} \in \text{int}[0,1]$, the nonempty subset $UUL_I(\tilde{r}, \tilde{g}, \tilde{b})$ is a UPs (resp., nUPf, UPf, iUPf, cUPf, sUPf, UPi, sUPi) of $\mathcal{U}$.

5. Conclusions

We have proposed eight new notions of IvPFSs in UP-algebras in this paper: IvPFUPss, IvPFnUPfs, IvPFUPfs, IvPFiUPfs, IvPFcUPfs, IvPFsUPfs, IvPFUPis, and IvPFsUPis of UP-algebras and looked into a few of their key characteristics. The diagram of generalization of IvPFSs in UP-algebras (Figure 1) and sufficient conditions of IvPFSs (Figure 2) are then obtained.

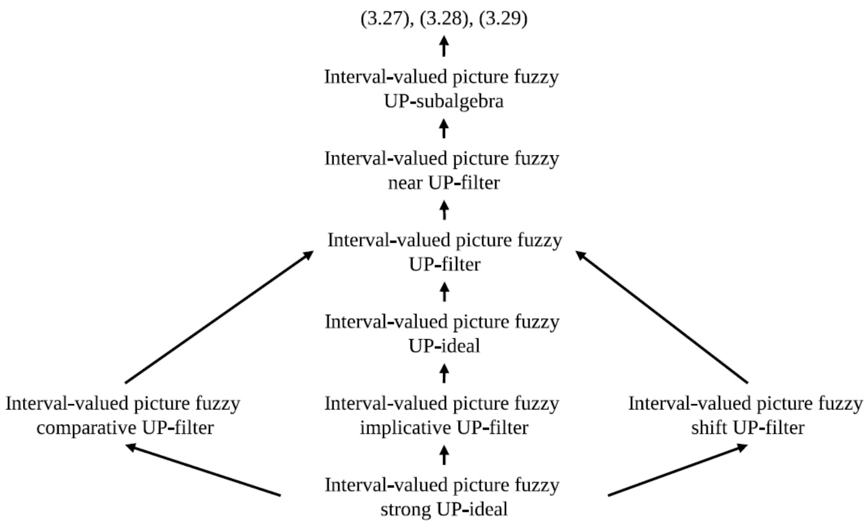

Figure 1

IvPFSs in UP-algebras

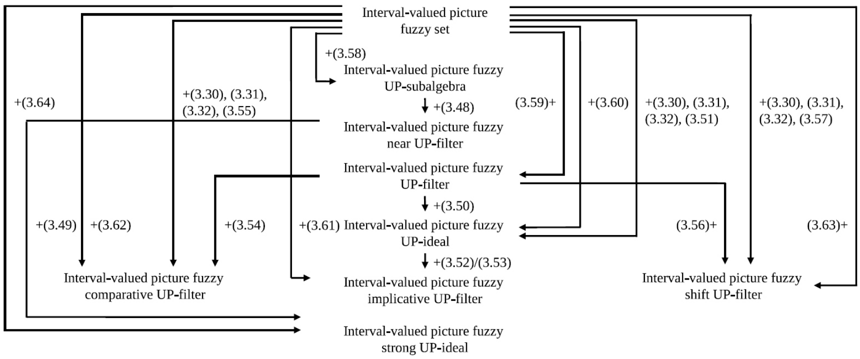

Figure 2

IvPFSs under sufficient conditions in UP-algebras

In the future, we'll extend the ideas/results to a wider range of IvPFSs in UP-algebras. We'll also look at soft sets, cubic sets, and rough sets of IvPFSs in UP-algebras.

References

[1] M. A. Ansari, A. N. A. Koam, and A. Haider, Rough set theory applied to UP-algebras, *Ital. J. Pure Appl. Math.* 42 (2019), 388–402.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.* 20 (1986), 87–96.

[3] B. C. Cuong, Picture fuzzy sets, *J. Comput. Sci. Cybern.* 30 (2014), no. 4, 409–420.

[4] B. C. Cuong and V. Kreinovich, Picture fuzzy sets-a new concept for computational intelligence problems, Proceedings of the Third World Congress on Information and Communication Technologies WIICT, 2013, pp. 1–6.

[5] A. H. Ganie, S. Singh, and P. K. Bhatia, Some new correlation coefficients of picture fuzzy sets with applications, *Neural Computing and Applications* 32 (2020), 12609–12625.

[6] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, Fuzzy translations of a fuzzy set in UP-algebras, *J. Indones. Math. Soc.* 23 (2017), no. 2, 1–19.

[7] A. Iampan, A new branch of the logical algebra: UP-algebras, *J. Algebra Relat. Top.* 5 (2017), no. 1, 35–54.

[8] A. Iampan, Introducing fully UP-semigroups, *Discuss. Math., Gen. Algebra Appl.* 38 (2018), no. 2, 297–306.

[9] A. Iampan, Multipliers and near UP-filters of UP-algebras, *J. Discrete Math. Sci. Cryptography* 24 (2021), no. 3, 667–680.

[10] A. Iampan, M. Songsaeng, and G. Muhiuddin, Fuzzy duplex UP-algebras, *Eur. J. Pure Appl. Math.* 13 (2020), no. 3, 459–471.

[11] T. Klinseesook, S. Bukok, and A. Iampan, Rough set theory applied to UP-algebras, *J. Inf. Optim. Sci.* 41 (2020), no. 3, 705–722.

[12] P. Liu, M. Munir, T. Mahmood, and K. Ullah, Some similarity mesasures for interval-valued picture fuzzy sets and their applications in decision making, Information 10 (2019), 369.

[13] A. Satirad, P. Mosrijai, and A. Iampan, Formulas for finding UP-algebras, *Int. J. Math. Comput. Sci.* 14 (2019), no. 2, 403–409.

[14] A. Satirad, P. Mosrijai, and A. Iampan, Generalized power UP-algebras, *Int. J. Math. Comput. Sci.* 14 (2019), no. 1, 17–25.

[15] T. Senapati, G. Muhiuddin, and K. P. Shum, Representation of UP-algebras in interval-valued intuitionistic fuzzy environment, *Ital. J. Pure Appl. Math.* 38 (2017), 497–517.

[16] P. Singh, Correlation coefficients for picture fuzzy sets, *J. Intell. Fuzzy Syst*. 28 (2015), 591–604.

[17] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, Fuzzy sets in UP-algebras, *Ann. Fuzzy Math. Inform*. 12 (2016), no. 6, 739–756.

[18] M. Songsaeng and A. Iampan, Neutrosophic sets in UP-algebras by means of interval-valued fuzzy sets, *J. Int. Math. Virtual Inst*. 10 (2020), no. 1, 93–122.

[19] S. Sripaeng, K. Tanamoon, and A. Iampan, On anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, *J. Inf. Optim. Sci.* 39 (2018), no. 5, 1095–1127.

[20] K. Taboon, P. Butsri, and A. Iampan, A cubic set theory approach to UP-algebras, *J. Interdiscip. Math.* 23 (2020), no. 8, 1449–1486.

[21] G. Wei, Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making, *Informatica* 28 (2017), no. 3, 547–564.

[22] G. Wei, Some similarity measures for picture fuzzy sets and their applcations, *Iran. J. Fuzzy Syst.* 15 (2018), no. 1, 77–89.

[23] G. Wei and H. Gao, The generalized dice similarity measures for picture fuzzy sets and their applications, *Informatica* 29 (2018), no. 1, 107–124.

[24] S. Yuphaphin, P. Kankaew, N. Lapo, R. Chinram, and A. Iampan, Picture fuzzy sets in UP-algebras by means of special type, J. *Math. Computer Sci.* 25 (2022), no. 1, 37–72.

[25] L. A. Zadeh, Fuzzy sets, Inf. Cont. 8 (1965), 338–353.

[26] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, *Inf. Sci.* 8 (1975), 199–249.

Received March, 2021

Revised June, 2021