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Can you feel the algebraic structure? Quaternion and finite rings in the service of patterns and weaving

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Abstract

Artists frequently create symmetrical patterns, but disregard the mathematical structure that allows their formation. We here seek to expose this underlying construct and afford new ways of using mathematical structures to produce surprising, artistic symmetrical patterns. We introduce different multi-color symmetrical patterns whose generation entails only a transformation on an initial location matrix. By using different algebraic structures, we were able to obtain a range of surprising symmetric patterns. We demonstrate how planar patterns can be generated by generalizing a result of [15] to RGB and CMYK color formats using a Quaternion group. In addition, we show how finite rings can produces a perception of symmetrical weaving patterns. Lastly, we use the Quaternion group to produce spherical patterns in the RGB format. We believe these algebraic structures contain natural beauty that a variety of audiences will find interesting and applicable: digital artists, mathematics aficionados, teachers and college students.

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1. Introduction

From the mosaics of old to the Louvre's pyramid and digital art, artists throughout the decades have used repetitive patterns to expresses natural beauty and symmetry and attract the eye of the viewer ([11, 6, 8, 19, 21]). These repetitive patterns, in fact, represent a mathematical structure that defines a given symmetry by a combination of rotation, translation and reflection. The possibilities of artistic expression seem endless, but from a mathematics point of view, there are only 17 planar patterns. Surprisingly, artists occasionally create patterns (not necessarily planar patterns) that are hard to detect mathematically. This aids in the development of the mathematical theory of patterns.

Tiling is a mathematical concept that can be used to express natural beauty by constructing repetitive symmetrical patterns. While art boasts an endless array of patterns, mathematicians have encountered great difficulty in constructing them as a physical object and mostly imagine how to recreate the patterns. Thanks to advances in computer graphics, it is now possible to express the mathematical patterns, enabling mathematicians to contribute their perspective, which is what we aim to do in this work. Specifically, we provide the mathematical means to define algebraic structures that create unique colorful patterns of the sort typically constructed by artists like Anni Albert ([5, 12]), Damien Hirst ([10]) and Bridget Riley ([17]).

Our approach calls for viewing a canvas as a grid of points, each of which is assigned a color by some algebraic function, while taking advantage of known mathematical structures with inherent symmetries.

One of the first such works was done by Bosch and Herman. They showed how one can use an important problem in mathematics (travel salesman problem) to draw a continuous line using a computer. Advancing this idea further, Kaplan and Bosch [13] developed algorithms that can create artwork, thus demonstrating how mathematics can influence art.

Major progress in computer-guided pattern construction was achieved using coloring AA bitmaps [3, 1]. In these manuscripts, Abdalla et al. use this technique to define a color pattern by starting from two binary strings on a given point grid, with a given binary rule that leads to a black and white pattern. Their approach called for using the strings to divide the grid into clusters, and then color each cluster according to a given rule, which leads to exquisite patterns. An alternative approach is that of Puc and Škrekovski [16], who were able to construct a repetitive pattern using the concept of contour lines of maps of surfaces, generated by a periodic function of two variables. This work inspired us to attempt to generate repetitive patterns using algebraic structures rather than periodic functions.

The research presented herein can also be viewed as an attempt to generalize the idea proposed by Parberry [15] to the construction of colorful patterns using a mathematical structure. His main idea was to take a grid $(x, y) \in \mathbb{N} \times \mathbb{N}$, which represents points, and to define for each of its points a gray color. Since each color in gray is defined by a value between 0 and 255, the color of each pixel is defined by the function $f : \mathbb{N} \times \mathbb{N} \to k \bmod 256$, where $k \in N$, which leads to the common color scale format. Later on, Parberry showed how to move from gray scale to RGB by using a range of integer values. For example, if the color in gray scale is in the range [0, 10], the pixel will be the color yellow, if in the range [11, 20], it will be the color red, and so on.

We here describe how we were able to generalize [15] to multi-color formats using only the given function, i.e., the function decided the color. This required that a color be assigned to each location in a 2-dimensional image. The RGB format is defined by 3 channels (R, G, B), with each channel located within the integer range [0, 255]. We, therefore, had to define a transformation $f : \mathbb{N}^2 \to (i \bmod 256,\ j \bmod 256,\ k \bmod 256)$ that represents the full color range. Of note, our method can be adapted to other color formats such as CMYK- cyan, magenta, yellow and black, i.e., it is defined by 4 parameters, one for each color.

Great thought was directed to selecting the algebraic structure suitable for creating multi-color symmetry. The basic algebraic structure is the natural numbers, $\mathbb{N}$. This can be extended to $\mathbb{Z}$, which, in turn, can be extended to $\mathbb{Q}$ ($\frac{m}{n}$ and $m \in \mathbb{Z}, n \in \mathbb{N}$) and so on, all the way to $\mathbb{C}$ ($a + ib$, where $a, b \in \mathbb{R}$). The complex numbers represent transformations from $\mathbb{R}^2 \to \mathbb{R}^2$, so in order to achieve our goal of defining multi-color symmetry, we need to extend $\mathbb{C}$ (which corresponds to $\mathbb{R}^2$) to a higher dimensional algebraic structure. The Quaternion algebra $\mathbb{H}$ (see Mathematical Background section) contains the complex numbers $\mathbb{C}$. It is a set of transformations $\mathbb{R}^4 \to \mathbb{R}^4$ with specific relations. It turns out that this structure can be applied to various fields. In [14] unit graphs of symmetric, quaternion and heisenberg groups have been studied. In [20] public key authentication scheme over quaternions have been introduced. Quaternions were already applied to art by Hart and Segerman [9], who showed how they can be transformed into physical sculptures.

Our algorithm works as follow: Given a location matrix, we chose an algebraic function, $\mathbb{H} \in \mathbb{N}^2 \to \mathbb{H} \in \mathbb{N}^3$, which will chose the colors (herein *coloring function*). Each location, defined by a set of two parameters, is embedded into the quaternions and the coloring function is applied to get a value, defined by four parameters. We then project this value onto a finite 3-dimensional object of the order 256^3 (representing the different colors in RGB). This process yields an image with a clear repetitive colorful pattern, which visualizes this abstract process.

We further introduce the concept of using algebraic structures to construct planar repetitive patterns. Surprisingly, the outcome is reminiscent of the weaving patterns produce by looms, a handcraft that already in ancient times was used to form decorative patterns. Of note, it is difficult to even produce black-and-white checkerboard woven fabrics as a symmetric pattern using a computer. [2, 4] proposed to use simple looms to obtain black-and-white weaving patterns. When our model limits the number of colors, the weaving patterns formed are similar to those achieved in [2]. However, when the color space (in gray scale) increases, the patterns become more intricate. Choosing an infinite ring as the algebraic structure (see section 2), which is represented in the digital image, leads to an 'unlimited' number of patterns -- ones defined not by human consideration but by a mathematical structure. Our procedure can be thought of as a **generator** that produces not only planar patterns but also a kind of visually rough texture that is usually hard to create on a plane.

In section 2, we provide the necessary mathematical background. In section 3, we show how quaternion algebra can be easily adapted to obtain and explore planar patterns in the RGB and CMYK color formats. In section 4, we change the standard finite object from $\mathbb{Z} \bmod 256$ to a finite field and different finite rings for a given function, obtaining new symmetries that are derived from the finite object. Lastly in section 5, we produce and explore symmetries on the unit sphere $(\mathbb{S}^2)$ and show how the quaternion and its respective projection can lead us to spherical patterns.

2. Mathematical Background

In this section, we provide the necessary definitions for the basic objects we use herein.

Definition 2.1: Ring is a set R equipped with two binary operations, $(+)$ (addition) and $(\cdot)$ (multiplication), that satisfy the following axioms for all $a, b, c \in R$:

(1) $(a+b)+c=a+(b+c)$ (associative).

(2) $a+b=b+a$ (commutative).

(3) There is an element 0 in R such that $a+0=a$ for all $a\in R$ (additive identity).

(4) For each $a\in R$, there exists $-a\in R$ such that $a+(-a)=0$.

(5) R is a monoid under multiplication, meaning that:

(6) $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ (associative).

(7) There is an element $1\in R$ such that $a\cdot 1=a$ and $1\cdot a=a$ for all $a\in R$ (that is, 1 is the multiplicative identity).

(8) Multiplication is distributive with respect to addition, meaning that:
$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$
(left distributivity).

(9) $(b+c)\cdot a=(b\cdot a)+(c\cdot a)$ (right distributivity).

For more details, see [18].

So a ring is basically a set with two operations, addition and multiplication, that satisfies a certain set of axioms.

In this work, we use the following $R=\mathbb{Z}_{256}$, which is a set of the residues modulo 256, that is, $R=\{0,\ldots,255\}$, where the addition and multiplication are defined as follows

$$\forall a,b\in R: a+b=(a+b)\mod 256,\quad a\cdot b=(a\cdot b)\mod 256. \tag{1}$$

By this definition (for this given ring), $254+3=1$ and $2\cdot 128=0$. Obviously, this number $n=256$ can be replaced with any other finite number n, which leads to $\mathbb{Z}_n$. We chose $n=256$ because the spectrum of color in gray scale has 256 options.

Quaternion ring. We use the quaternion ring to define our RGB patterns. The quaternion ring is a generalization of $\mathbb{C}$ and can be thought of as a specific set of transformations from $\mathbb{R}^4\to\mathbb{R}^4$.

Definition 2.2: The Quaternion ring is defined as follows:

$$\mathbb{H}:=\{q_0+q_1\cdot i+q_2\cdot j+q_3\cdot k : q_0,q_1,q_2,q_3\in\mathbb{R}\}, \tag{2}$$

satisfying the relations $i^2=j^2=k^2=-1$ and $i\cdot j\cdot k=-1$. These relations imply $i\cdot j=-j\cdot i=k$, $j\cdot k=-k\cdot j=-i$ and $k\cdot i=-i\cdot k=j$.

We can think of the elements of the ring as vectors (x,y,z,w), where each entry represents a color channel. For example, in CMYK, each color is defined by four channels, so we will use all four entries. In RGB, we only need three channels, so we will ignore one of the entries.

This new structure is not commutative in the sense that if you take two elements, x, y, in $\mathbb{H}$, it is not true that $x \cdot y \neq y \cdot x$. For example, in the CMYK format, if we take $x = i$, which is represented by the vector $(0,1,0,0)$ (magenta color), and $y = j$, which is represented by the vector $(0,0,1,0)$ (yellow color), we will get that $x \cdot y$ is the vector $(0,0,0,1)$ (color key-black), while $y \cdot x$ is the vector $(0,0,0,-1)$, whose value of black is different. This phenomena leads to new symmetries that do not occur in more standard number systems, which is precisely why we have chosen to use this structure.

Remark 1: Note that in Definition 2.2, we take the scalars q_0, q_1, q_2, q_3 from $\mathbb{R}$, but we can replace them with scalars from $\mathbb{Z}$ or any other ring. This definition enables us to define a function on the quaternions. The squaring function, for example, takes a quaternion $h \in \mathbb{H}$ to its square, leading to surprising new symmetries.

3. Planar Patterns and the Quaternion Ring

Symmetry on a plane is defined by three transformations: translation, rotation and reflection. All symmetries reflect a combination of these transformations.

We will show how the quaternion ring generates images with colorful symmetric patterns, thus generalizing the work of [15]. Choosing an arbitrary transformation on the quaternion can lead to digital colorful artistic symmetrical patterns. This should impress upon artists that a symmetrical pattern, even if the underlying rule is not understood, always 'contains' a mathematical basis.

Our algorithm takes a given location, $(x, y) \in \mathbb{N} \times \mathbb{N}$, and returns a vector, $pixel(x, y) \in [0, 255]^3$ (for RGB format), which represents the color of the pixel. A similar process occurs with the CMYK format. Generally, we define the process as follows: Given a location matrix in $(i, j) \in \mathbb{N} \times \mathbb{N}$, a function f is chosen according to the quaternion rule. Since f is defined by $f(x, y, z, w) \in \mathbb{R}^4$, we reduce the image dimension by choosing $x = i, y = j, w = 0$, $z = const$ or $z = g(x, y)$, which leads to $f \in \mathbb{N}^3$. Then, taking modulo 256 to each channel produces an image in RGB format.

Next, we generate the final image, e.g., Figure 1. This is achieved by defining a pattern for a specific function (each function will define a specific pattern). Given the quaternion transformation

$$f(x,y,z,w)=(x+yi+zj+wk)^2 ,$$

the simplification of f leads to

$$f(x,y,z,w)=x^2-y^2-z^2-w^2+2xy\cdot i+2xz\cdot j+2xw\cdot k . \tag{3}$$

Without losing generality, (x,y) represents the location of each pixel, so the other two scalars can be chosen arbitrary. In Figure 1, we chose $z=1$ and $w=0$, which leads to

$$f(x,y,1,0)=x^2-y^2-1+2xy\cdot i+2x\cdot j+0\cdot k , \tag{4}$$

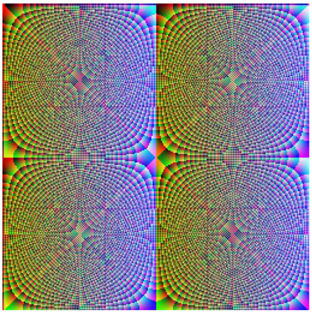
which is represented by the vector

$$(x^2-y^2-1,2xy,2x), \tag{5}$$

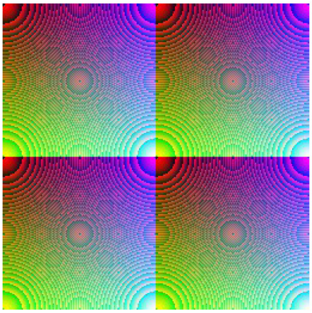
which determines the RGB colors.

As we can immediately see, the first two elements in (5) define an hyperbola, and, indeed, in Figure 1a, we can see the hyperbolic patterns constructed by the red and green channels after taking module 256 (by the color definition). The third channel is just a line. When combined, these three channels lead to Figure 1a, which features colorful symmetry and tiles the plane.

Figure 1b shows the beauty of the non-commutative structure chosen. Changing the location matrix (x,y) to (y,z) defines the transformation $f(1,y,z,0)$. In this case, the respective channels are $(1-y^2-z^2,2y,2z)$. The first channel, $1-y^2-z^2$, defines a family of circles centered at $(0,0)$ (red channel). So we define a location $(y,z)\in\mathbb{N}^2$ as the origin (the location



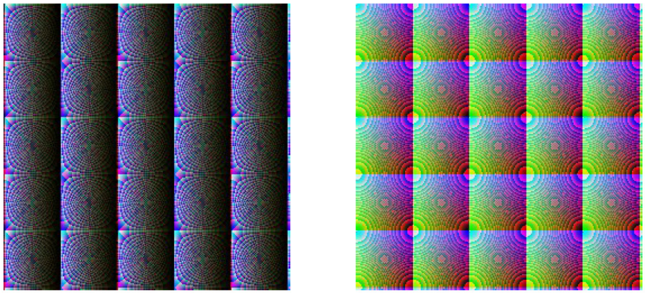
(a) $f(x,y,1,0)$.



(b) $f(1,y,z,0)$.

Figure 1

Images in the RGB format. In the left image, the location is placed in the first two entries. In the right image, the location is placed in the second and third entries. As can be seen, each option leads to different symmetries.



(a) $f(1,x,y,1)$. (b) $f(x,1,1,y)$.

Figure 2

Notice that *f* in the CMYK format leads to different patterns from those in the RGB format.

matrix). As expected, the respective symmetry pattern is a combination of circles, which correspond to the red channel, and blue and green blocks, which corresponds to the second and third channels, respectively.

Remark 2: Since color symmetry is defined by layers, exploring it according to the colors can lead to different symmetry types. For example, in Figure 1b, the pattern is merely a translation of a square that tiles the plane, since the green color is always at the bottom of the square and the red color at its top. If we omit the channels' colors, the pattern can be defined by the circles at different radii around a given point in the plane, which leads to a symmetry determined by reflection and translation, i.e., the pattern stores different symmetries, depending on the color channel exploration.

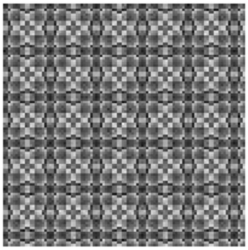
Figure 2 captures the function f in the CMYK format; notice that in this case, $w \neq 0$ adapting to the 4-channel color format.

4. Planar Patterns over Finite Rings

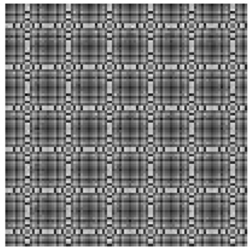
In previous works such as [15], the process of obtaining planar patterns using the ring $\mathbb{Z}_{256}$ was described. The basic process involves attaching a color to each number in the finite ring $\mathbb{Z}_{256} = \{0, \cdots, 255\}$ and using a function whose range is $\mathbb{Z}_{256}$ in order to obtain planar patterns.

In this section, we define another structure of a finite ring, one not of the form $\mathbb{Z}_n$ (in our specific case, $n = 256$ or less). The ring we constructed generalizes the ring $\mathbb{Z}_{256}$ by defining the polynomial ring, $R = [x]$. The members of R are polynomial expressions defined by

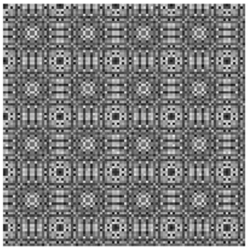
$$\forall a_i \in \mathbb{Z} : p(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_0 . \tag{6}$$



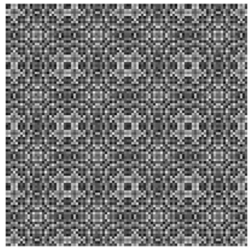
(a) Modulo 38, output 256 unique colors.



(b) Modulo 17, output 124 unique colors.



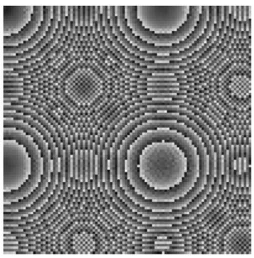
(c) Modulo 23, output 217 unique colors.



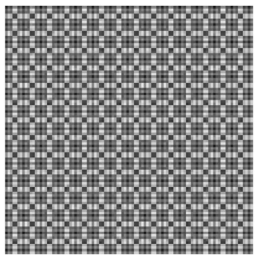
(d) Modulo 25, output 161 unique colors.

Figure 3

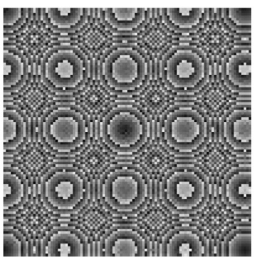
Gray scale images for different modulo, $a = i, b = i^2, c = j, d = j^2$.



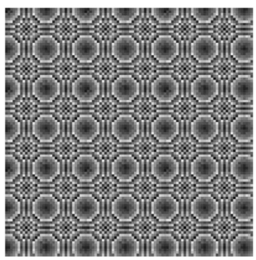
(a) Modulo 255, output 255 unique colors.



(b) Modulo 7, output 7 unique colors.



(c) Modulo 97, output 97 unique colors.



(d) Modulo 32, output 17 unique colors.

Figure 4

All the images in gray scale for the different modulo. The choice $a = b = i$ and $c = d = j$.

To define a ring, one need to define a proper addition and multiplication. Given the two polynomials

$$p_1(x) = \sum_{i=n}^{0} a_i \cdot x^i \quad \text{and} \quad p_2(x) = \sum_{i=n}^{0} b_i \cdot x^i \ ,$$

each of a degree n, the addition is defined by

$$p_1(x) + p_2(x) = \sum_{i=n}^{0} (a_i + b_i) \cdot x^i \tag{7}$$

and the multiplication is defined by

$$p_1(x) \cdot p_2(x) = \sum_{i=2n}^{0} \Big(\sum_{t+r=i} a_t b_r \Big) \cdot x^i \ . \tag{8}$$

As an example, in the case of $p_1(x) = x^2 + 2x + 1$ and $p_2(x) = 3x^2 + 4x - 7$, then $p_1(x) + p_2(x) = 4x^2 + 6x - 6$ and $p_1(x) \cdot p_2(x) = 3x^4 + 10x^3 + 4x^2 - 10x - 7$, which preserve the polynomial properties and define a ring.

Notice that the degrees of the polynomials are not bounded. Since we would like to use this idea to construct a planar pattern (in gray scale), the color output needs to be a transformation to $\{0 \ldots 255\}$, so we need to limit the polynomial degree and the coefficients in a way that will lead to a finite set and then define a bijection function, which will determine the colors (a finite structure that is more 'controllable'). To obtain such a finite set, we use the idea of a quotient ring: If $p(x) = \sum a_i \cdot x^i$ (from an arbitrary degree), then the quotient ring is defined by:

$$R := \mathbb{Z}[x] / \langle p(x) \rangle = \left\{ t(x) = \sum_{i=0}^{n-1} a_i \cdot x^i \right\}. \tag{9}$$

This defines R to be the set of polynomials of degree $\leq (n-1)$. To define addition, we use the definition as in (7), For multiplication, we use the following rule, which imitates the modulo operator. The problem in using the multiplication in (8) is that multiplying two polynomials from our set might result in a polynomial $\geq n$, which is outside our set. To solve this problem, we define the following operation: Given the two polynomials $t_1(x), t_2(x)$, multiplying them as in (8) leads to a polynomial $q(x) = t_1(x) \cdot t_2(x)$. This is followed by long division of q by p to obtain $q(x) = h \cdot p(x) + r(x)$, where the degree of p in n and the degree of r is, at most, $n-1$. Lastly, $t_1(x) \cdot t_2(x) = r(x)$ is defined. One way to think about it is to declare that x is a root of $p(x)$, thus $p(x) = 0$ and the degree is reduced to, at most, $(n-1)$. For example, in the case of $p(x) = x^2 - x - 1$, since the degree is two, we get that R is a set of polynomials with a degree of at most one. To illustrate the multiplication choice, multiplying, for example, $t_1(x)=2x+3$ and $t_2(x)=x+1$, will lead to

$$q(x)=t_1(x)\cdot t_2(x)=2x^2+5x+3\,. \tag{10}$$

Now, we use long division of $q(x)$ by $p(x)$ to obtain $q(x)=h\cdot p(x)+r(x)=2\cdot p(x)+(7x+5).$ Finally, we get that $q(x)=t_1(x)\cdot t_2(x)=7x+5,$ as we defined. Another way to think about is to consider x to be a root of $p(x)$, so that

$$p(x)=0\Rightarrow x^2=x+1\,. \tag{11}$$

Now, the polynomial $q(x)=2x^2+5x+3$ becomes (after replacing x^2 by $x+1$) $q(x)=2(x+1)+(5x+3)=7x+5.$

Notice that even now the ring is infinite (because of the coefficients, $\mathbb{Z}$ is infinite). Hence, we choose a finite number m that will be zero and get the quotient ring $R=\mathbb{Z}_m[x]p(x)$, which is

$$R=\left\{\sum_{i=0}^{n-1}a_i\cdot x^i \mid a_i\in\mathbb{Z}_m\right\},$$

where $\mathbb{Z}_m=\{0,\ldots,m-1\}$ with modulo m operations. This construction defines the ring R above. Notice that $|R|=m^n$, i.e., the ring is finite. For example, $n=2$ and $m=p$, where p is a prime number, is the ring R with at most p^2 elements.

Next, we define our image: We choose the ring

$$R=\mathbb{Z}_p[x]/\langle x^2-x-1\rangle\,. \tag{12}$$

This ring is defined by the solution to the equation $x^2-x-1=0$, which is $x_{1,2}=\frac{1\pm\sqrt{5}}{2}$. Take the positive root $\beta:=x_1=\frac{1+\sqrt{5}}{2}$, so that $R=\{a+b\beta\}$. Due to equation (11), all our polynomials do not exceed degree one, and each polynomial can be represented by

$$R=\mathbb{Z}_p[\beta]=\{a_0+a_1\cdot\beta \mid \quad a_0,a_1\in\mathbb{Z}_p\}\,. \tag{13}$$

Take the function $\pi:\mathbb{Z}[x]/\langle p(x)\rangle\to R$, defined by

$$\pi(\sum a_i\cdot x^i)=\sum a_i(\operatorname{mod} p)\cdot x^i\ , \tag{14}$$

This process folds $\mathbb{Z}$ to a finite set. Now we are ready to construct our image in gray scale, which will lead to planar patterns.

Given the location (i, j) in our grid:

(1) Define a coloring function. A natural choice (since we would like to obtain planar patterns) is

$$f(u,v) = u^2 + v^2 , \tag{15}$$

where $u, v \in R$.

(2) We can write

$$u = a + b\beta, \quad v = c + d\beta . \tag{16}$$

Notice that β is a solution for the polynomial we chose, i.e., we have $\beta^2 - \beta - 1 = 0$. Substituting *u* and *v* for *f* leads to

$$a^2 + b^2 + c^2 + d^2 + (2ab + 2cd + b^2 + d^2)\beta , \tag{17}$$

(3) by the rule of π (modulo):

$$(a^2 + b^2 + c^2 + d^2) \bmod p + (2ab + 2cd + b^2 + d^2) \bmod p \cdot \beta \tag{18}$$

(4) We now need to define a transformation $(i, j) \to (u, v)$ that connects the coordinate (i, j) of the grid of our image to the two elements of our ring. Note that different transformations will lead to different patterns. Figure 4 shows the outcome of the transformation where we chose $a = c = i, b = d = j$, that is, $(i, j) \to (i + j\beta, i + j\beta)$.

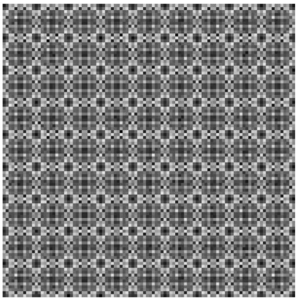
(5) This pair of elements will now correspond to a color by the rule we choose. For example, if we pick modulo $p = 2$, the maximal number of couples (depending on the transformation) is $\{(0,0), (0,1), (1,0), (1,1)\}$. Each of these couples corresponds to a color in gray scale $\{0, 63, 127, 191, 255\}$, by dividing 256 into 4 segments (uniformly).

Figure 3 explores different values of *p*. If *p* is a prime number, then the number of unique colors increases (compare Figure 3c to Figure 3d).

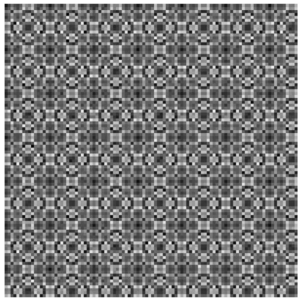
Figures 3, 4 and 5 show different $(i, j) \to (u, v)$ transformations, which lead to different patterns. This construction was applied to $f(u,v) = u^2 + v^2$ but can be done to any function.

Figure 3 is defined by another rule, $a = i, \ b = i^2, \ c = j, \ d = j^2$, and leads to different patterns, since it assigns different properties to the algebraic structure. For example, in Fig. 3c, modulo 23 leads to 217 unique colors, while in Fig. 3d, modulo 25 leads to 161 unique colors.

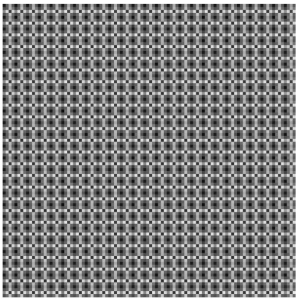
This process can be applied to different functions. Small changes in the function *f*, such as $f(u,v) = (u + v)^2$, which is defined by



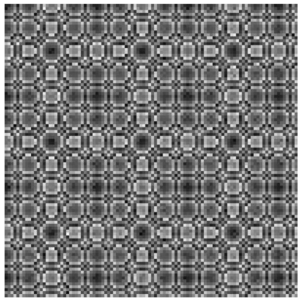
(a) Modulo 11, output 36 unique colors.



(b) Modulo 15, output 55 unique colors.



(c) Modulo 16, output 10 unique colors.



(d) Modulo 31, output 256 unique colors.

Figure 5

All the images in gray scale for the different modulo. The choice $a = c = i$ and $b = d = j$ decreases the number of colors in the image due to the existence of a zero derivation.

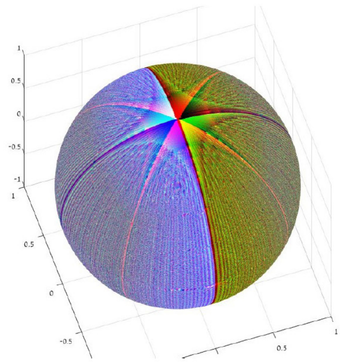
$$a^2 + b^2 + c^2 + d^2 + 2ac + 2bd + (2ab + 2ad + 2bc + 2bd + 2cd + b^2 + d^2)\beta , \qquad (19)$$

open a new window into new planar patterns. Thus, the ability to define different patterns is almost unlimited.

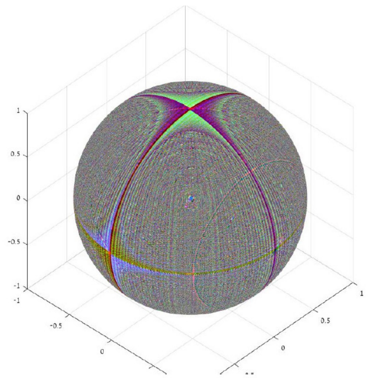
5. A Glance at Spherical Patterns

The symmetries obtained on the sphere are totally different from those of planner patterns and are represented by a different symmetry group (there exists only 17 for planar patterns and only 14 for spherical patterns; for more details, see [7]).

The sphere $\mathbb{S}^2$ is defined by the equation $x^2 + y^2 + z^2 = 1$. We tile the sphere by choosing points of the type $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and projecting them onto the sphere by normalizing each point (since all the points are at a distance of 1 from the origin (0, 0, 0). Next, for each point $(x, y, z) \in \mathbb{S}^2$, we assign a



(a) $f(x, y, 1, 0)$.



(b) $f(x, y, z, 0)$.

Figure 6

Projecting the quaternion ring *f* on a sphere

color (RGB format) by the respective location based on the properties of the quaternion ring.

Figure 6a presents the projection of $f(x,y,1,0)$ on a sphere. The respective channels are given in expression (5). Choosing $z=1$ causes the blue channel to be constant.

Figure 6b shows $f(x,y,z,0)$. The blue color has a significant contribution and reduces the red color, which leads to a pattern similar to that in Figure 6a but, surprisingly, a less colorful one.

6. Summary and Conclusions

We have shown different symmetries that offer a glimpse into algebraic structures. These algebraic structures can convey a sense of the mathematics, something we would never have imagined -- texture and equations can be related. We believe that if digital artists, designers and other professionals become familiar with these algebraic ideas, it can provide them with new ways of thinking and modes of explorations.

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