Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 8, pp. 1717–1731
DOI : 10.47974/JIM-1457

(11, 3)–arc configurations in PG(2, 5)

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Abstract

Taking a cue from two recent articles of S. Innamorati et al., [9] and [10], as part of promoting and improving pattern discovery skills among students, the construction of the (11, 3)–arc configurations of the projective plane of order five can be used as example of a problem solving task. We show that answering to the question of how to construct the maximum (k, 3)–arcs in PG(2, 5) opens the door to a wealth of interesting mathematics.

Subject Classification: *[2020] 97B50, 97D50, 97G10, 97K10.*

Keywords: *Teacher education, Prospective teacher, Mathematics communication, Problem solving.*

1. Introduction

Educating future school mathematics teachers poses specific challenges with regard to useful topics from the perspective of their future profession and learnable by teacher students. We argue the importance of maintaining high standard for mathematics education and suggest topics and inspiring investigations for those who are interested in pursuing the joy of learning mathematics. Wide access to mathematics should be encouraged by shaking off the boring tags and reinforcing passion. The topics adopted should be mathematically sound and provide the right

This paper is the extended version of an online lecture given by the author at a course for high school teachers at the University of Teramo, Italy, 17 March 2020, during italian lock-down.

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intellectual stimulation, cf. [19]. Finite geometry is one of the simplest and most expressive frameworks used to describe some aspects of the foundations of geometry, cf. [7], [12], [14] and [25]. It is a good candidate for Mathematics education: few axioms are needed and key properties can be formally stated and proved correct under specific assumptions, cf. [18] and [19]. In [17], pag. 171, the authors recommend that *the specialized preparation in geometry provided for prospective teachers of geometry should include the study of the following topics.*

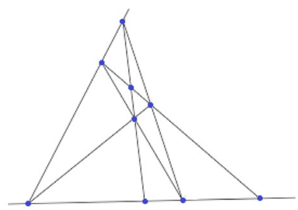
…Finite Geometries

To extend their concepts of geometry and to provide a source of theorems with relatively simple proofs, prospective teachers should have some experience with at least one finite geometry. A variety of representations, based upon different interpretations of the elements in the postulates, is particularly desiderable.

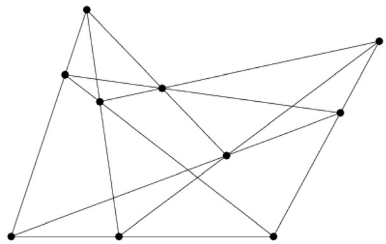
The Hilbert system of axioms for an incidence geometry deals with ideal elements such as points and lines, together with the notion of incidence relation. Configuration is one of the oldest combinatorial structure, since it appeared for the first time in 1876 in the second edition of Theodor Reye's book *Geometrie der Lage*, cf. [20]. A (v_r, b_k) *configuration* is a pair $C = (P, L)$ where P is a set of v elements, called *points*, and L is a family of b subsets, called *lines*, with k points on each line and r lines through each point. Two different lines intersect each other at most once and two different points are connected by a line at most once. By definition it easy follows that the parameters of a configuration (v_r, b_k) must satisfy $vr = bk$ and $v \geq r(k-1)+1$, cf. [8], [15] and [24]. If $v = b$, and, hence, $r = k$, the configuration is said to be *symmetric* and is denoted by v_k. A projective plane is a symmetric configuration such that any two lines meet in exactly one point, any two points are on exactly one common line, and there are four points no three of which are collinear. In the case of a finite projective plane, it can be shown that the number of points, and the number of lines, is m^2+m+1 for some integer $m>1$ which is called the order of the plane. Moreover, each line has $m+1$ points, and each point is on $m+1$ lines. For any prime number p we denote by $\mathbf{F}_p$ the field of residue classes of integers modulo p. And more generally, for any integer $h \geq 1$, we denote by $\mathbf{F}_q$ the finite field with $q = p^h$ elements. The classical example of projective plane of order q, PG$(2, q)$, can be defined as a set of q^2+q+1 points which are 3–vectors over the Galois field $\mathbf{F}_q$ where the zero vector is not allowed and two points are considered equal if the vector of one is a multiple of the vector of another. A line consists of all points $\boldsymbol{x} = (x_0, x_1, x_2)$ such that $a_0x_0+a_1x_1+a_2x_2 = 0$ for some coefficients (a_0, a_1, a_2) not all zero. A projectivity is a bijective map of the points of the plane onto itself which preserves collinear points. Two sets

K_1 and K_2 in PG(2, q) are projectively equivalent if there exists a projectivity τ such that $K_2 = \tau(K_1)$. A (k, n)–arc in a projective plane is a set of k points such that some n, but no n+1 of them, are collinear. A (k, 2)–arc is simply called a k–arc. The main application of (k, n)–arc is to the study of linear codes. Arcs and (k, 3)–arcs in PG(2, q) correspond to respectively MDS and NMDS codes of dimension 3, cf. [6]. These types of linear codes are the best in term of minimum distance, among the linear codes with the same length and dimension. In general, (k, n)–arcs in PG(2, q) correspond to linear codes with Singleton defect equal to n–2. Define $m_n(2, q)$ to be the maximum size of a (k, n)–arc in PG(2, q). In 1947 R. C. Bose [4] proved that $m_2(2, q) = q+1$ for q odd, and $m_2(2, q) = q+2$ for q even. In 1955 B. Segre [21] proved that for q odd every (q+1)–arc is a conic and for $q = 2$, $q = 4$, $q = 8$ every (q+2)–arc is a conic plus its nucleus, but for $q \geq 16$, even, there exists a (q+2)–arc other than the conic plus its nucleus. In 1975 J. Thas [23] proved that if $q>3$, then $m_3(2, q) \leq 2q+1$.

A fundamental configuration is the complete quadrangle, which consists of seven points and six lines obtained by taking four points, no three of which are collinear, that are its vertices, and drawing all six lines, that are its sides, connecting them. Opposite sides of the quadrangle are those sides that do not meet in a vertex. The three intersections of opposite sides of the quadrangle are called its diagonal points.



Another important point-line incidence structure is the Desargues 10_3 symmetric configuration. Among the ten 10_3 symmetric configurations the Desargues configuration is the only one where for each of its points the three points that are not collinear with it lie on a line.



The projective plane of order five is the dominant focus of this work. The aim is to find the two fundamental geometric configurations by analyzing maximal 3–arcs in this projective plane. Our reason for deciding to conduct a detailed investigation of this special case is that PG(2, 5) is a charming small projective plane, cf. [1], [2], [3], [5], [11], [13] and [16]. We show how the request to have in PG(2, 5) the maximum point-set such that at most three points are collinear is linked to the two configurations. More precisely, we will show the following theorem.

Main Theorem : *A* (11, 3)–*arc in* PG(2, 5) *has character vector either* (4, 4, 7, 16) *or* (5, 1, 10, 15). *The complete quadrangle identifies the* (11, 3)–arc *of* PG(2, 5) *with character vector* (4, 4, 7, 16). *The Desargues configuration identifies the* (11, 3)–arc *of* PG(2, 5) *with character vector* (5, 1, 10, 15).

To prove this result, the possible character vectors of a (11, 3)–arc in PG(2, 5) will be analyzed. Firstly, in Section 2, we found that the character vectors a (11, 3)–arc in PG(2, 5) are either (4, 4, 7, 16) or (5, 1, 10, 15). In Section 3, we will suppose that a (11, 3)–arc K in PG(2, 5) has character vector (4, 4, 7, 16) and we will conclude that K is identified by a complete quadrangle. In Section 4, we will investigate a (11, 3)–arc K in PG(2, 5) with character vector (5, 1, 10, 15) and we will obtain that K is identified by a Desargues configuration. Finally, in Section 5, concrete constructions are presented by using Singer representation.

2. The character vectors of a (11, 3)–arc of PG(2, 5).

In this Section, the character vectors of a (11, 3)–arc K in PG(2, 5) are calculated. For each integer h such that $0 \le h \le 3$, let us denote by $t_h = t_h(K)$ the number of lines of PG(2, 5) meeting K in exactly h points. The numbers t_h are called the *characters* of K with respect to the lines, see [8]. By double counting the number of lines, the number of pairs (P, r), where $P \in K$ and r is a line through P, and the number of pairs $((P, Q), r)$, where $\{P, Q\} \subset K$ and r is the line through P and Q, we get the following equations:

$$\begin{cases} t_0 + t_1 + t_2 + t_3 = 31 \\ t_1 + 2t_2 + 3t_3 = 66 \\ 2t_2 + 6t_3 = 110 \end{cases}$$

Solving these equations, we get:

$$\begin{cases} t_0 = 20 - t_3 \\ t_1 = 3t_3 - 44, \\ t_2 = 55 - 3t_3 \end{cases}$$

It follows that $15 \le t_3 \le 18$.

Therefore, we get the set of character vectors:

$(t_0, t_1, t_2, t_3) \in \{(2, 10, 1, 18), (3, 7, 4, 17), (4, 4, 7, 16), (5, 1, 10, 15)\}$.

Let P be a point of K. For each integer h such that $0 \le h \le 3$, let us denote by $v_h = v_h(P)$ the number of lines through P meeting K in exactly h points. The numbers v_h are called the *characters* of P with respect to the lines, see [8]. By double counting the number of lines through P, the number of pairs (Q, r), where $Q \in K-P$ and r is a line through P and Q, we get the following equations:

$$\begin{cases} v_1 + v_2 + v_3 = 6 \\ v_2 + 2v_3 = 10 \end{cases}$$

Solving these equations, we get:

$$\begin{cases} v_1 = v_3 - 4 \\ v_2 = 10 - 2v_3 \end{cases}$$

Since $v_1 \ge 0$ and $v_2 \ge 0$ we get $v_3 \in \{4, 5\}$. Therefore, we get the set of inner points types:

$$(v_1, v_2, v_3) \in \{(0, 2, 4), (1, 0, 5)\}.$$

Let Q be a point nonbelonging to K. For each integer h such that $0 \le h \le 3$, let us denote by $u_h = u_h(Q)$ the number of lines through Q meeting K in exactly h points. The numbers u_h are called the *characters* of Q with respect to the lines, see [8]. By double counting the number of lines through Q, we get

$$\begin{cases} u_0 + u_1 + u_2 + u_3 = 6 \\ u_1 + 2u_2 + 3u_3 = 11 \end{cases}$$

Solving these equations, we get:

$$\begin{cases} u_1 = 1 + u_3 - 2u_0 \\ u_2 = 5 - 2u_3 + u_0 \end{cases}$$

Therefore, we get the set of outer points types:

$(u_0, u_1, u_2, u_3) \in \{(0, 1, 5, 0), (0, 2, 3, 1), (0, 3, 1, 2), (1, 0, 4, 1), (1, 1, 2, 2), (1, 2, 0, 3), (2, 0, 1, 3)\}$.

We prove

Proposition 2.1 : The character vector of a (11, 3)–arc in PG(2, 5) is either (4, 4, 7, 16) or (5, 1, 10, 15).

Proof : Let K be a (11, 3)–arc in PG(2, 5) with character vector (t_0, t_1, t_2, t_3) = (2, 10, 1, 18). Since $t_2 = 1$, let l denote the 2–line. An inner point $P \in l$ has character $v_2(P) = 1$, a contradiction.

Now, let K be a (11, 3)–arc in PG(2, 5) with character vector (t_0, t_1, t_2, t_3) = (3, 7, 4, 17). Since $t_2 = 4$, we get that the four 2–lines form a quadrangle with vertices four inner points of type (0, 2, 4) and exactly two diagonal points of type (1, 1, 2, 2). Thus, outer points of type (0, 1, 5, 0), (0, 2, 3, 1), (1, 0, 4, 1) do not exist. The outer points are of types $(u_0, u_1, u_2, u_3) \in \{(0, 3, 1, 2), (1, 1, 2, 2), (1, 2, 0, 3), (2, 0, 1, 3)\}$. Let x, y and z denote the number of outer points of type (0, 3, 1, 2), (1, 1, 2, 2) and (1, 2, 0, 3), respectively, of a 1–line l_1. By double counting the number of pairs (Q, r), where $Q \in l_1$ and r is a i–line, $i = 0, 2, 3$, through Q, we get:

$$\begin{cases} y + z = 3 \\ x + 2y = 4 \\ 2x + 2y + 3z = 12 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1. \\ z = 2 \end{cases}$$

So, there are exactly 7 outer points of type (1, 1, 2, 2), a contradiction.

□

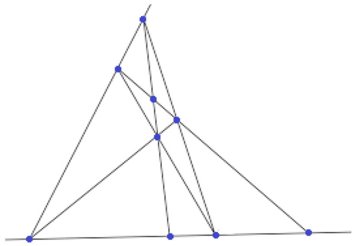
3. The geometric structure of a (11, 3)–arc of PG(2, 5) with character vector (4, 4, 7, 16).

In this Section, a (11, 3)–arc K in PG(2, 5) with character vector (4, 4, 7, 16) is analyzed and it is proved that K is closely related to the complete quadrangle. We prove

Proposition 3.1 : The (11, 3)–arc of PG(2, 5) with character vector (4, 4, 7, 16) is identified by the complete quadrangle.

Proof: Let K be a (11, 3)–arc in PG(2, 5) with character vector (t_0, t_1, t_2, t_3) = (4, 4, 7, 16). Since a 2–line contains inner points of type (0, 2, 4) and $t_2 = 7$, outer points of type (0, 1, 5, 0) and (1, 0, 4, 1) do not exist.

So, the outer points are of types $(u_0, u_1, u_2, u_3) \in \{(0, 2, 3, 1), (0, 3, 1, 2), (1, 1, 2, 2), (1, 2, 0, 3), (2, 0, 1, 3)\}$. Since $u_0 \leq 2$, it follows that the four 0–lines ℓ_i, $i = 1, 2, 3, 4$ form a quadrangle.



There are exactly two points not on $K \cup \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$.

Let x, y and z denote the number of outer points of type (1, 1, 2, 2), (1, 2, 0, 3), (2, 0, 1, 3), respectively, of a 0–line l_0. By double counting the number of pairs (Q, r), where $Q \in l_0 - K$ and $r \neq l_0$ is a i–line, i = 0, 1, 2, 3, through Q, we get:

$$\begin{cases} y + z = 3 \\ x + 2y = 4 \\ 2x + 2y + 3z = 12 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1. \\ z = 2 \end{cases}$$

Thus, there are exactly 8 points of type (1, 1, 2, 2), 4 points of type (1, 2, 0, 3), 6 points of type (2, 0, 1, 3).

Therefore, the 0–lines are of type (x, y, z) = (2, 1, 3).

Let x, y, z and t denote the number of outer points of type (0, 2, 3, 1), (0, 3, 1, 2), (1, 1, 2, 2), (1, 2, 0, 3), respectively, of a 1–line l_1. By double counting the number of pairs (Q, r), where $Q \in l_1 - K$ and $r \neq l_1$ is a i–line, i = 0, 1, 2, 3, through Q, we get:

$$\begin{cases} z + t = 4 \\ x + 2y + t = 3 \\ 3x + y + 2z = 7 \\ x + 2y + 2z + 3t = 11 \end{cases} \Rightarrow \begin{cases} x = t - 1 \\ y = 2 - t. \\ z = 4 - t \end{cases}$$

We get $1 \leq t \leq 2$. Thus, the 1–lines can be of type (x, y, z, t) $\in$ {(0, 1, 3, 1), (1, 0, 2, 2)}. Let x and y denote the number of 1–lines of type (0, 1, 3, 1) and (1, 0, 2, 2), respectively. By double counting the number of 1–lines and the number of pairs (Q, r), where $Q \in r - K$ is a point of type (1, 1, 2, 2) and r is a 1–line, we get: $\begin{cases} x+y=4 \\ 3x+2y=8 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=4 \end{cases}$. Therefore, the 1–lines are of type

$(x,y,z,t)=(1,0,2,2)$. Thus, outer points of type (0, 3, 1, 2) do not exist and the two points not on $K\cup\ell_1\cup\ell_2\cup\ell_3\cup\ell_4$ are of type (0, 2, 3, 1).

Let x, y and z denote the number of outer points of type (0, 2, 3, 1), (1, 1, 2, 2), (2, 0, 1, 3), respectively, of a 2–line l_2. By double counting the number of pairs (Q, r), where $Q\in l_2-K$ and r is a i–line, i = 0, 1, 3, through Q, we get:

$$\begin{cases} y+2z=4 \\ 2x+y=4 \\ x+2y+3z=8 \end{cases} \Rightarrow \begin{cases} x=z \\ y=4-2z \end{cases}.$$

We get $0\le z\le 2$. Thus, the 2–lines can be of type (x,y,z) $\in$\{(0, 4, 0), (1, 2, 1), (2, 0, 2)\}. Let x, y and z denote the number of 2–lines of type (0, 4, 0), (1, 2, 1) and (2, 0, 2), respectively. By double counting the number of 2–lines, the number of pairs (Q, r), where $Q\in r-K$ is a point of type (0, 2, 3, 1) and r is a 2–line, the number of pairs (Q, r), where $Q\in r-K$ is a point of type (1, 1, 2, 2) and r is a 2–line, we get: $\begin{cases} x+y+z=7 \\ y+z=6 \end{cases} \Rightarrow \begin{cases} x=z+1 \\ y=6-2z \end{cases}$. Since there are exactly 2 outer points of type (0, 2, 3, 1), $z\le 1$. Therefore, $1\le x\le 2$. So, we get either $\begin{cases} x=1 \\ y=6 \\ z=0 \end{cases}$ or $\begin{cases} x=2 \\ y=4 \\ z=1 \end{cases}$ and the 2–lines are of type (x,y,z) $\in$\{(0, 4, 0), (1, 2, 1), (2, 0, 2)\}.

Let x, y, z and t denote the number of outer points of type (0, 2, 3, 1), (1, 1, 2, 2), (1, 2, 0, 3), (2, 0, 1, 3), respectively, of a 3–line l_3. By double counting the number of pairs (Q, r), where $Q\in l_3-K$ and $r\neq l_3$ is a i–line, i = 0, 1, 2, 3, through Q, we get:

$$\begin{cases} y+z+2t=4 \\ \begin{pmatrix} 2x+y+2z \\ 3x+2y+t \\ y+2z+2t \end{pmatrix} \in \left\{ \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix} \right\} \end{cases} \Rightarrow \left\{ \begin{pmatrix} 2x+y \\ x+y+t \\ z \end{pmatrix} \in \left\{ \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} \right\}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Thus, the 3–lines can be of type $(x,y,z,t) \in\{(0, 0, 2, 1), (0, 1, 1, 1), (0, 2, 0, 1), (1, 0, 0, 2)\}$. Since there are exactly 2 outer points of type (0, 2, 3, 1), there are exactly two 3–lines of type (x,y,z,t) = (1, 0, 0, 2). Moreover, since $x \leq 1$, the line joining the two outer points of type (0, 2, 3, 1) is a 2–line of type (2, 0, 2) and so it is a diagonal line, d_1, of the quadrangle of 0–lines. The other two diagonal lines, d_2 and d_3, are the two 3–lines of type (x,y,z,t) = (1, 0, 0, 2). Therefore, a (11, 3)–arc in PG(2, 5) with character vector $(t_0, t_1, t_2, t_3) = (4, 4, 7, 16)$ is the complement of a complete quadrangle minus one diagonal point. □

4. The geometric structure of a (11, 3)–arc of PG(2, 5) with character vector (5, 1, 10, 15).

In this Section, a (11, 3)–arc K in PG(2, 5) with character vector (5, 1, 10, 15) is analyzed. We prove

Proposition 4.1 : The (11, 3)–arc of PG(2, 5) with character vector (5, 1, 10, 15) is identified by the Desargues configuration.

Proof: Let K be a (11, 3)–arc in PG(2, 5) with character vector (t_0, t_1, t_2, t_3) = (5, 1, 10, 15). Since $t_1 = 1$, outer points of type (0, 2, 3, 1), (0, 3, 1, 2) and (1, 2, 0, 3) do not exist. So, the outer points are of types $(u_0, u_1, u_2, u_3)\in\{(0, 1, 5, 0), (1, 0, 4, 1), (1, 1, 2, 2), (2, 0, 1, 3)\}$. There are exactly one inner point P of type (1, 0, 5) and 10 inner points of type (0, 2, 4).

Let x, y and z denote the number of outer points of type (1, 0, 4, 1), (1, 1, 2, 2), (2, 0, 1, 3), respectively, of a 0–line l_0. By double counting the number of pairs (Q, r), where $Q\in l_0-K$ and $r\neq l_0$ is a i–line, $i = 0, 1, 2$, through Q, we get:

$$\begin{cases} z = 4 \\ y = 1 \\ 4x + 2y + z = 10 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1. \\ z = 4 \end{cases}$$

Thus, the 0–lines are of type (x,y,z) = (1, 1, 4), there are exactly 5 points of type (1, 0, 4, 1), 5 points of type (1, 1, 2, 2), 10 points of type (2, 0, 1, 3), and consequently 0 points of type (0, 1, 5, 0).

So, the outer points are of types $(u_0, u_1, u_2, u_3)\in\{(1, 0, 4, 1), (1, 1, 2, 2), (2, 0, 1, 3)\}$.

Let x, y and z denote the number of outer points of type (1, 0, 4, 1), (1, 1, 2, 2), (2, 0, 1, 3), respectively, of a 2–line l_2. By double counting the number of pairs (Q, r), where $Q \in l_2 - K$ and $r \neq l_2$ is a i–line, $i = 0, 1, 2$, through Q, we get:

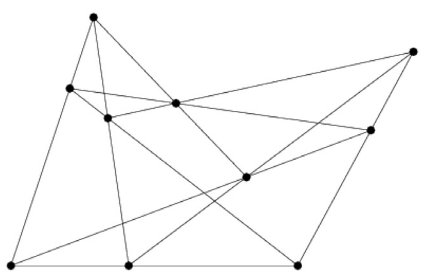
$$\begin{cases} x + y + 2z = 5 \\ y = 1 \\ 3x + y = 7 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1. \\ z = 1 \end{cases}$$

Thus, the 2–lines are of type $(x, y, z) = (2, 1, 1)$.

Let x, y and z denote the number of outer points of type (1, 0, 4, 1), (1, 1, 2, 2), (2, 0, 1, 3), respectively, of a 3–line l_3. Since there is exactly one inner point P of type (1, 0, 5), there are exactly 5 3–lines with P and two inner points of type (0, 2, 4) and exactly 10 3–lines with three inner points of type (0, 2, 4). By double counting the number of pairs (Q, r), where $Q \in l_3 - K$ and $r \neq l_3$ is a i–line, $i = 0, 1, 2, 3$, through Q, we get:

$$\begin{cases} x + y + 2z = 5 \\ \begin{pmatrix} y \\ 4x + 2y + z \\ y + 2z \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \right\} \end{cases} \Rightarrow \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\} \right\}.$$

Thus, the 3–lines are of type $(x, y, z) \in \{(0, 1, 2), (1, 0, 2)\}$. There are 5 3–lines of type (1, 0, 2) and 10 of type (0, 1, 2). Every 3–line of type (1, 0, 2) contains the inner point P of type (1, 0, 5) and one point of type (1, 0, 4, 1). So, the line joining two points of type (1, 0, 4, 1) is a 2–line. Therefore, the 6–set union of the point P and the five points of type (1, 0, 4, 1) is a conic C. The 10–set of points of type (0, 2, 4) is the set of internal points of C. The 10 points of type (0, 2, 4) with the 10 3–lines of type (0, 1, 2) form a symmetric (10_3) configuration. Up to isomorphism, there exist exactly ten symmetric (10_3) configurations. The Desargues configuration is, unlike the others, the only one such that for every point V the three points that are not collinear with V lie on a line.



5. The Singer representation of PG(2, 5).

In this Section, concrete constructions are provided by using Singer representation. Among the automorphism of PG(2, q) there is an element of order q^2+q+1 that permutes the points of the geometry in a Singer cycle of this length. The existence of Singer cycles is derived in [22] from the fact that the multiplicative group of the field $\mathbf{F}_{q^3}$ is cyclic. In order to construct this element, recall that the finite field $\mathbf{F}_{q^3}$ can be viewed as a vector space of three dimension over $\mathbf{F}_q$, with basis 1, ω, ω^2, where ω is a primitive element of $\mathbf{F}_{q^3}$. The multiplication by ω induces a linear transformation of the vector space. Recalling that the field $\mathbf{F}_{q^3}$ has ω^{q^2+q+1} as primitive element, this linear transformation induces an automorphism of PG(2, q) that acts as a cycle of length q^2+q+1 on the q^2+q+1 points of the geometry. We recall that a polynomial is called irreducible over a finite field if it cannot be factorized into nontrivial polynomials over the same field. In other case, polynomial is reducible; e.g. x^3+x+1 is irreducible over $\mathbf{F}_5$, while x^3+x+2 is reducible because $x^3+x+2 = (x+1)(x^2+4x+2)$. A primitive polynomial of degree n with coefficients over the field $\mathbf{F}_q$ is a polynomial such that the simplest monomial x generates all the elements of the extension field $\mathbf{F}_{q^n}$. Let's us consider polynomial $f(x) = x^3+x+1$ over $\mathbf{F}_5$. Then, ω = x is a root of the polynomial $f(x)$. Thus, $\omega^3 = 4+4\omega$.

$\omega^0 = (1, 0, 0)$					
$\omega^1 = (0, 1, 0)$	$\omega^2 = (0, 0, 1)$	$\omega^3 = (1, 1, 0)$	$\omega^4 = (0, 1, 1)$	$\omega^5 = (1, 1, 4)$	$\omega^6 = (1, 2, 1)$
$\omega^7 = (1, 0, 3)$	$\omega^8 = (1, 4, 0)$	$\omega^9 = (0, 1, 4)$	$\omega^{10} = (1, 1, 1)$	$\omega^{11} = (1, 0, 4)$	$\omega^{12} = (1, 2, 0)$
$\omega^{13} = (0, 1, 2)$	$\omega^{14} = (1, 1, 2)$	$\omega^{15} = (1, 3, 2)$	$\omega^{16} = (1, 3, 1)$	$\omega^{17} = (1, 0, 2)$	$\omega^{18} = (1, 3, 0)$
$\omega^{19} = (0, 1, 3)$	$\omega^{20} = (1, 1, 3)$	$\omega^{21} = (1, 4, 3)$	$\omega^{22} = (1, 4, 2)$	$\omega^{23} = (1, 3, 3)$	$\omega^{24} = (1, 4, 4)$
$\omega^{25} = (1, 2, 4)$	$\omega^{26} = (1, 2, 2)$	$\omega^{27} = (1, 3, 4)$	$\omega^{28} = (1, 2, 3)$	$\omega^{29} = (1, 4, 1)$	$\omega^{30} = (1, 0, 1)$

Let us denote the points represented by ω^i simply by i. So, the Singer group is isomorphic to the additive group Z_{31}, the integers modulo 31. Now select any line: for example, we choose the line $x_1 = x_2$, which contains the points: ℓ_0 = {0, 4, 10, 23, 24, 26}. The remaining lines of the plane are found by adding 1 to each point of the preceding line beginning with ℓ_0 and using addition modulo 31. For convenience, we represent the projective plane of order 5 as a set of orthogonal arrays of the affine plane of order 5 with the intersection point of the member of each parallel class indicated to the right of the row array and at the bottom of the column array. We do this by using the Singer difference set defining PG(2, 5) as the line at infinity, designated by ℓ_∞. Thus, let $\ell_\infty = \{0, 4, 10, 23, 24, 26\}$. The remaining lines of the plane are found by adding 1 to each point of the preceding line beginning with ℓ_∞ as ℓ_0 and using addition modulo 31. The pencil of lines on the point 4 is then intersected by the pencil of lines on the point 0 to form the first array. Thus, each row (column) plus its point at infinity represents a line of the plane. Now, let us take into account the Singer representation and the description of Proposition 3.1. By considering the reference quadrangle with vertices 0, 1, 2, 10 and diagonal points 3, 4, 30, if we choose as diagonal line d_1 the line joining the vertices 3 and 30, we get the (11, 3)–arc of PG(2, 5) with character vector (4, 4, 7, 16).

1	2	9	**13**	**19**	
3	11	**15**	**21**	6	
8	30	**28**	14	**27**	**4**
12	7	5	16	**22**	
18	17	29	**25**	20	
		0			

1	6	16	29	30	
7	8	**15**	**19**	25	
20	5	2	14	3	10
21	17	**27**	12	9	
28	**13**	18	11	**22**	
		23			

1	5	11	**25**	**27**	
14	18	9	6	7	
15	30	20	12	**13**	**24**
17	**19**	16	**28**	3	
22	21	8	2	29	
		26			

Moreover, by the Singer representation and the description of Proposition 4.1, since $-\ell_\infty = \{-0, -4, -10, -23, -24, -26\} = \{0, 5, 7, 8, 21, 27\}$ is a conic, by deleting the points on the tangent lines, we get the 10–set of internal points of the conic {1, 12, 13, 17, 19, 22, 25, 28, 29, 30}, cf. [8]. By taking one point of the conic, say 0, we get the (11, 3)–arc of PG(2, 5) with character vector (5, 1, 10, 15).

1	2	9	13	19	
3	11	15	**21**	6	
8	30	28	14	**27**	4
12	**7**	**5**	16	22	
18	17	29	25	20	
		0			

1	6	16	29	30	
7	**8**	15	19	25	
20	**5**	2	14	3	10
21	17	**27**	12	9	
28	13	18	11	22	
		23			

1	**5**	11	25	**27**	
14	18	9	6	**7**	
15	30	20	12	13	24
17	19	16	28	3	
22	**21**	**8**	2	29	
		26			

1	2	9	13	19	
~~3~~	~~11~~	~~15~~	**21**	~~6~~	
8	30	28	14	**27**	~~4~~
12	**7**	**5**	16	22	
18	17	29	25	20	
		0			

1	6	~~16~~	29	30	
7	**8**	~~15~~	19	25	
~~20~~	**5**	~~2~~	~~14~~	~~3~~	~~10~~
21	17	**27**	12	9	
28	13	~~18~~	11	22	
		23			

1	**5**	~~11~~	25	**27**	
~~14~~	~~18~~	~~9~~	~~6~~	**7**	
15	30	~~20~~	12	13	~~24~~
17	19	~~16~~	28	3	
22	**21**	**8**	2	29	
		26			

1	2	9	**13**	**19**		**1**	6	16	**29**	**30**		**1**	5	11	**25**	27	
3	11	15	21	6		7	8	15	**19**	25		14	18	9	6	7	
8	**30**	**28**	14	27	4	20	5	2	14	3	10	15	**30**	20	**12**	**13**	24
12	7	5	16	**22**		21	**17**	27	**12**	9		**17**	**19**	16	**28**	3	
18	**17**	**29**	**25**	20		**28**	**13**	18	11	**22**		**22**	21	8	2	**29**	
		0						23						26			

Conclusion

Experience reveals that the investigation of the (11, 3)-arc configurations in PG(2, 5) can be a very productive learning activities. As a feedback most of our teacher students handed a list points of the (11, 3)-arcs in PG(2, 5) by using Singer representation. The coherent, useful, and logical subject makes use of their ability to make sense of problem situations. Since most students tend to enjoy games, to look for remarkable hidden sets may help raising student interests, enhance their problem solving skills and be a gateway into mathematical thinking generally. They enthusiastically spent hours to find these examples and we was very impressed by them.

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Received July, 2021