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The minimality of g-non-autonomous discrete dynamical systems

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Abstract

In this study, we assume that there is that converges uniformly of a series (ℓ_n) that stimulates general non-autonomous discrete dynamical systems, then the transitivity and sensitivity can follow from g-nonautonomous discrete dynamical systems (g-NDS) to autonomous discrete dynamical systems (ADS). When the g-NDS and ADS are minimal, there are many differences. In this work, we introduce some of them.

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Keywords: *General non-autonomous dynamic, Minimal, Sensitive dependence, Equicontinuous, Topological transitive.*

1. Introduction

Ali, B. and Al-Shara 'a in [2] studied the g-non autonomous discrete dynamical systems (g-NDS for short). Assuming that (X, d) is the metric space, a g-NDS is a pair (X, ℓ_n^m) where (ℓ_n) be a sequence of a continuous map $\ell_n : X \longrightarrow X$, for all $n \in \mathbb{N}$, and the composition

$$\ell_n^m = \ell_m \circ \ell_{m-1} \circ \cdot \cdots \circ \ell_n, \forall\, 0 \leq n < m \in \mathbb{N}$$

Is said to be the mth-iterate of $\ell_{n,m}$ and ℓ_n^0 is the identity mapping on X , the set for all these iterations of x : $O_{\ell_{n,m}}(x) = \{x, \ell_n(x), \ell_n^2(x), \ldots, \ell_n^m(x)\}$, is

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called orbit for $x \in X$.The g-NDS generalize autonomous discrete systems (ADS for short). The ADS is represented by (X, ℓ), therefore (X, d) is the metric space $\&\ell: X \to X$ is continuous, we can consider these systems to be a special case of g-NDS by assuming that $\ell_n^m = \ell^{m-n}$ for all $0 \leq n < m \in \mathbb{N}$. In [1], they provided a minimal ADS is either sensitive or equicontinuous, in our work we study some chaotic properties of the sequence ℓ_n^m in g-NDS converged uniform are: transitive, sensitive, and equicontinuous. We provide when g-NDS is transitive, then the ADS is transitive, likewise for a sensitive. In [3] they explained the difference between (NDS) and (ADS) in the case of minimum. We show that when the(X, ℓ) is minimal, then it will be either sensitive or equicontinuous, and when the sequence ℓ_n^m is minimal, the third case in g-NDS its neither sensitive nor equicontinuous.

2. Some Chaotic Properties

A g-NDS $(X, \ell_{n,m})$ is considered topologically transitive if any of the $U, V \neq \emptyset$, open sets, $U \& V \subset X$ there exist, $0 \leq n < m \in \mathbb{N}, \ni \ell_n^m(U) \cap V \neq \emptyset$. It is described as having delicate initial condition dependence, ,$\forall x \in X$ a constant can be found $\delta > 0, \forall \varepsilon > 0, \exists y \in X$ where $d(x, y) < \varepsilon, \ni d\left(\ell_n^m(x), \ell_n^m(y)\right) > \delta$, for $0 \leq n < m \in \mathbb{N}$. And the point p is said to be a periodic point for g-NDS if $\exists\, 0 \leq n < m \in \mathbb{N}$, meaning that $\ell_n^{mk}(p) = p$ for any $k \in \mathbb{N}$.

2.1 *Proposition*

If the g-nonautonomous discrete dynamical system(g-NDS) is transitive, then the autonomous discrete dynamical systems (ADS) is transitive.

Proof: Let U, V be nonempty subsets of X; $s$ince The(g-NDS) is transitive, there exist $m, n \in \mathbb{N}, \ni 0 \leq n < m, \ni \ell_n^m(U) \cap V \neq \emptyset$, but $\ell_n: X \to X$ is converged uniform to ℓ, there exists $\ell_n^m = \ell^i, \ell^i\ (U) \cap V \neq \emptyset$, so the ADS is transitive. □

2.2 *Proposition*

If the g-nonautonomous discrete dynamical system (g-NDS) has sensitive, then the autonomous discrete dynamical system(ADS) is sensitive.

Proof: Let $x \epsilon X$, since the (g-NDS) is sensitive, $\exists\, \varepsilon > 0, \forall\, \delta > 0, \exists y \in B_{(X)}$, there exist $m, n \in \mathbb{N}, \ni 0 \leq n < m, d(x, y) < \delta, d\left(\ell_n^m(x), \ell_n^m(y)\right) > \varepsilon$, (since $\ell_n: X \to X$ converges uniformly to ℓ), There exists $\ell_n^m = \ell^i$, $d(\ell_n^m(x), \ell_n^m(y)) = d(\ell^i(x), \ell^i(y)) > \varepsilon$, then the ADS is sensitive.

3. Minimal on g-NDS

A g-NDS$(x, \ell_{n,m})$ is said to be equicontinuous when for every $\varepsilon > 0, \exists \delta > 0, \ni d(\ell_n^m(x), \ell_n^m(y)) < \varepsilon$ for all $m, n \in \mathbb{N}, \ni 0 \leq n < m$ when $d(x, y) < \delta$. The point $x \in X$ is said to be transitive if the orbit of *x* in X is dense. If All the elements in X are transitive, the g-NDS is termed minimal.

If an ADS (X,ℓ) is topologically transitive if it is equicontinuous at some point in X, then a set for transitive points corresponds with a set for equicontinuous points. We show that a minimum ADS can be characterized as delicate or equicontinuous see[1].

Definition 3.1: Assume that (X,ℓ) be an autonomous discrete dynamical system (ADS for short), Σ_ε is a set of equicontinuous, $\Sigma_\varepsilon \subseteq X, \ni \ell_i^{-1}(\Sigma_\varepsilon) \subseteq \Sigma_\varepsilon$, then Σ_ε is said to be inversely invariant.If $\Sigma = X$ hence the autonomous discrete dynamical system (X,ℓ) is equicontinuous ……….(1)

If $\Sigma_\varepsilon \neq \emptyset$ hence $\Sigma_\varepsilon = \{x \in X, \forall\, z, y \in B_\delta(x), \forall i \geq 0, d(\ell^i(z), \ell^i(y)) < \varepsilon\}$ hence Σ_ε is inversely invariant open & $\Sigma = \cap\ \Sigma_{1/m}, m > 0$, then (X,ℓ) is almost equicontinuous. ……………(2)

If $\Sigma_\varepsilon = \emptyset\ we\ get\ \exists x \in X, \ni \forall z, y \in B_\delta(x), \exists n \in \mathbb{N}\ , d\left(\ell^i(z), \ell^i(y)\right) > \varepsilon$ hence either $d(\ell^i(x), \ell^i(y)) > \frac{\varepsilon}{2}$ or $d(\ell^i(x), \ell^i(z)) > \frac{\varepsilon}{2}$

Then (X,ℓ) is sensitive with a sensitive constant. $\frac{\varepsilon}{2}$. ………(3)

For more details, see [4]. If the autonomous discrete dynamical system(ADS) is minimal, then it's transitive but conversely not necessary and the (ADS) if is transitive, it is either nearly equicontinuous or sensitive.

Theorem 3.2: *Assume that (X,ℓ) be a dynamical system such that for each $\varepsilon > 0, \Sigma_\varepsilon$ is inversely invariant; if the (ADS) is minimal, then it will be either sensitive or almost equicontinuous.*

***Proof**:* Since the (ADS) is minimal, then it's transitive; hence the system is either Almost equicontinuous or sensitive. Since Σ_ε is inversely invariant since (ADS) is transitive; hence there exists

$$U = \frac{X}{\Sigma_\varepsilon}, \exists\ i > 0, i \in \mathbb{N}, \ni\ \ell^i(\Sigma_\varepsilon) \cap U \neq \emptyset, \text{or } \ell_i^{-1}(\Sigma_\varepsilon) \cap U\ \neq \emptyset,$$

But $\ell_i^{-1}(\Sigma_\varepsilon)\ \subseteq \Sigma_\varepsilon$ (by definition)

Hence $\ell_i^{-1}(\Sigma_\varepsilon) \cap U\ \subseteq\ \Sigma_\varepsilon \cap U$, but $\Sigma_\varepsilon \cap U = \emptyset\ \rightarrow contradiction!$

$\therefore$ either $\Sigma_\varepsilon = \emptyset\ or\ \Sigma_\varepsilon \neq \emptyset$, hence If $\Sigma_\varepsilon = \emptyset$ by (3)

$\Sigma_\varepsilon = \{x \in X, \exists\ z, y \in\ B_\delta(x), \exists\ \varepsilon > 0\ \&\ \varepsilon \in \mathbb{N}\ \ni d(\ell^i(z), \ell^i(y)) > \varepsilon\ , \forall\ i \in\ \mathbb{N}\}$ hence $d(\ell^i(x), \ell^i(y)) > \frac{\varepsilon}{2}\ or\ d(\ell^i(x), \ell^i(z)) > \frac{\varepsilon}{2}$

$\therefore$ The (ADS) is sensitive with sensitive constant $\frac{\varepsilon}{2}$.

Or $\Sigma_\varepsilon \neq \emptyset\ by(2)\ hence\ \Sigma = \cap\ \Sigma_{1/m}$ hence the (ADS) is almost equicontinuous, but in g-NDS, that is not necessary.

Theorem 3: *There is a minimal g-non autonomous discrete dynamical system (g-NDS) that lacks neither equicontinuous nor sensitivity.*

***Proof**:* Suppose that $\{M_1, M_2\}$ be a partition for a clopen subgroup M_i with the cantor set for each i = 1, 2, In order to build a g-NDS that is not equicontinuous, we must first define a series of homeomorphisms $\{g_n, n \in \mathbb{N}\}$ on $\mathbb{C}$ select a point x_1 in C_2 and conceive of a series $\{\mathcal{L}_n, n \in \mathbb{N}\}$ that, for each $n \in \mathbb{N}$, fulfills the following:

$\mathcal{L}_n$ is a clopen subset of C whose diameter is $1/n$, $\mathcal{L}_{n+1} \subsetneq \mathcal{L}_n$,$x_1 \in \mathcal{L}_{n+1}$

Just now, select the clopen subset S_1, S_2 of $C_2 \ni d(S_1, S_2) = \varepsilon_0 > 0$, select for any $n \in \mathbb{N}$ homeomorphisms, $w_n = C_2 \backslash \mathcal{L}_N \to C_2 \setminus (S_1 \cup S_2)$, $h_n = \mathcal{L}_n \backslash \mathcal{L}_{n+1} \to S_1$, $q_n = \mathcal{L}_{n+1} \to S_2$. We define G_n as follow

$$G_n = \begin{cases} x & if \quad x \in C_1, \\ w_n(x) & if \quad x \in C_2 \setminus \mathcal{L}_n, \\ h_n(x) & if \quad x \in \mathcal{L}_n \setminus \mathcal{L}_{n+1}, \\ q_n(x) & if \quad x \in \mathcal{L}_{n+1}. \end{cases}$$

Now let $(C, \ell_{n,m})$ be a (g-NDS) defined by

$$\{T, \ell_1, \ell_1^{-1}, T, T^{-1}, \ell_2, \ell_2^{-1}, T^2, T^{-2}, \ell_3, \ell_3^{-1}, T^3, T^{-3}, \ldots..\}$$

It is simple to confirm that

$$\ell_n^n = T \text{ , } \ell_n^{n+1} = \ell_n + T \text{ , } \ell_n^{n+2} = \ell_n^{-1} \circ \ell_n \circ T = T$$
$$\ell_n^{n+3} = T^n \circ \ell_n^{-1} \circ \ell_n \circ T = T^{n+1} \text{ , } \ell_n^{n+4} = T^{-n} \circ T^n \circ \ell_n^{-1} \circ \ell_n \circ T = T$$

Note that the sequence $\{T^m, m \in \mathbb{N}\}$ is subsequence for $\{\ell^m m \in \mathbb{N}\}$ consequently, Minimality and transitivity of $(C, \ell_{n,m})$, resulting from the Minimality and transitivity of (C, T). Additionally, since the prohibition of $G_n to\ C1$ is the identity for each $n \in \mathbb{N}$ the g-NDS $(C, \ell_{n,m})$ is equal at every point $x \in C_1$ hence $(C, \ell_{n,m})$ is not sensitive. We will prove that $(C, \ell_{n,m})$ is non-equicontinuous for fixed $\delta > 0$. We can select$n \in \mathbb{N} \ni 1/n < \delta, if\ x \in \mathcal{L}_n \backslash \mathcal{L}_{n+1}$, then $G_n(x) = h_n(x) \in S_1$, since $G_n(x_1) = q_n(x) \in S_2$ we have $d((\ell_m \circ T^n)\, T^{-n}(x), (\ell_m \circ T^n)\, T^{-n}(x_1)) \geq \varepsilon_0$ since T^{-n} is an isometric, such that proving that$(C, \ell_{n,m})$ is not equicontinuous at $T^{-n}(x_1)$.

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